Last time:

1. The Lie algebra \( \mathfrak{g} \) of a Lie group \( G \) is the algebra of left-invariant vector fields.
2. There is a canonical isomorphism \( \mathfrak{g} \cong T_1 G \)
   \[ \mathfrak{g} X \longmapsto \mathfrak{g} x(1) \]
   \[ \gamma(a) = (dla)(x) \hookrightarrow \mathfrak{g} \]

3. Maurer-Cartan form \( \omega \in \Omega^1(M, \mathfrak{g}) \):
   \[ \forall a \in G, \forall \omega \in \mathfrak{g}, \]
   \[ \omega(a) = (dl_a)^* a \omega \]
   - \( \omega \) is left invariant:
     \[ \forall a \in G : \omega \mathfrak{l}_a = \omega \]
   - \( \forall X \in \mathfrak{g}, \omega(X) = X(1) \)
   - \( R^* a \omega = \text{Ad}(a^{-1}) \omega \)
     where \( \text{Ad}(g) : T_1 G \to T_1 G, \text{Ad}(g)v = (dl_a dR^* a^{-1})v \)
   - If \( G = \text{GL}(n, \mathbb{R}) \) or \( G = \text{GL}(n, \mathbb{C}) \)
     \[ \omega_a = \text{Ad}^* dA \]

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Connections on fiber bundles.

If \( F \to Q \) is a fiber bundle with typical fiber \( M \), \( F \), a connection on \( Q \), whatever it is, should give us parallel transport:

If \( \gamma: [a,b] \to M \) is a curve, it should give

\[ \gamma_b = \text{way to get smoothly from } Q_{\gamma(a)} = \Pi^{-1}(\gamma(a)) \text{ to } Q_{\gamma(b)} \]

One can show:

\[ \gamma^* \Omega_1 Q \cong [a,b] \times \Omega_1 F \]

\[ (l_a)_* \gamma(b) \sim (l_b)_* \gamma(a) \text{ not canonically} \]

So if we can find a vector field \( X \) on \( \gamma^* \Omega_1 Q \) so that

\[ d\pi(X(\gamma)) = \frac{\partial}{\partial t} \gamma(t) \times \gamma^* \Omega_1 Q \]

we can take the flow if \( X \) and we have parallel transport.
If \( G \to P \to M \) is a principal \( G \)-bundle, one requires more: we want parallel transport to be \( G \)-equivariant.

This also guarantees its existence.

For any \( g \in G \) we have \( R_g : P \to P \), \( R_g(p) = pg \).

**Def.** A **connection** on a principal \( G \)-bundle \( G \to P \to M \) in a subbundle \( \mathcal{U} \subset TP \) such that

1. \( \forall p \in P \) \( \mathcal{U}_p \otimes (\nu P)_p = T_p P \)
2. \( \forall g \in G \) \( \forall p \in P \) \( \mathcal{U}_{pg} = (dR_g)_p (\mathcal{U}_p) \)

We'll see choice of \( \omega \):

① A connection on a principal \( G \)-bundle \( P \) is equivalent to a choice of \( \omega \in \Omega^1(P, \mathfrak{g}) \), \( \omega_g = \text{Lie}(g) \)

with \( R_g^* \omega = \text{Ad}(g^{-1}) \omega + 1 \) more property

② A connection on the frame bundle \( Fr(E) \to M \) of a vector bundle \( E \to M \) defines a covariant derivative on \( E \) (and conversely).

③ If \( E \to M \) has a metric \( h \) and \( OFr(E) \to M \) is the bundle of orthonormal frames (on \( O(k) \)-prin bundle), then a connection on \( OFr(E) \to M \) is a metric connection on \( E \to M \).
If \( Q \to M \) is a vector bundle, we want the flow to be linear in some sense. If \( Q \to M \) is a principal \( G \)-bundle we want the flow to be \( G \)-equivariant.

Now let \( x \in T_x Q \), \( (d\Pi)_x : T_x (T^* Q) \to T_{\pi(x)} \{a, b\} \) be onto, so the equation

\[
(*) \quad \frac{d}{dt} \pi(x) = \frac{d}{dt} \pi(\alpha(t)) \quad \text{for all } \alpha \in \mathcal{F} \subseteq T_{\pi(x)} \{a, b\}
\]

has lots of solutions. We want one that depends smoothly on \( x \).

\[
dim(\ker d\pi_x) = \dim \mathbb{F} = \frac{1}{x \in T_x Q} \quad (\text{ker} d\pi_x) \subseteq T_x (T^* Q)
\]

is a subbundle of \( T_x (T^* Q) \), called the vertical bundle, \( V_x (T^* Q) \).

So what we want in a complementary subbundle \( N \subseteq T_x (T^* Q) \)

Thus \( \forall x \in T_x Q \) is \( d\pi_x : N_x \to T_{\pi(x)} \{a, b\} \) in an \( \mathfrak{g} \) and any section \( x \in T(\mathcal{H} \to T^* Q) \) with \( \mathfrak{g} \text{-flow } x = \frac{d}{dt} \).

Such \( x \) is called the horizontal lift of \( \mathfrak{g} \).

We can always choose such \( \mathcal{H} \) (put an inner product on \( T_x (T^* Q) \to T^* Q \) and let \( \mathcal{H} = \mathcal{H} \perp V_x \)).

Warning: This doesn't quite guarantee existence of parallel transport.

\[
(0, x) \mapsto \mathbb{R} \times (0, x)
\]

\[
\mathbb{R} \times (0, x) : \quad x(y) = \frac{2}{\partial x} + y \frac{2}{\partial y}
\]
ODE
\[ \begin{align*}
\frac{dx}{dt} &= 1 \\
\frac{dy}{dt} &= y
\end{align*} \]
\( (x(0), y(0)) = (x_0, y_0) \)

More generally, we don't want to do horizontal lift one curve at a time, so we don't really want to choose splittings of \( T(\pi^* Q) \) for each \( r \) separately.

**Better:** A fiber bundle \( F \to Q \to M \)

we have the canonical vertical bundle \( VQ \in TQ \)

\[ (VQ)_q := \ker (d\pi_q : T_q Q \to T_{\pi(q)} M) \). \]

**Def:** A connection on a fiber bundle \( Q \to M \)
in a choice of a subbundle \( HQ \subset TQ \) such that \( VQ \oplus HQ = TQ \).

By construction, since \( \ker (d\pi_q : T_q Q \to T_{\pi(q)} M) = VQ_q \)

\[ d\pi_q \mid HQ : HQ \to T_{\pi(q)} M \]

is an isomorphism.

So \( V \subset T_{\pi(q)} M \) \( \exists! \) \( \tilde{V}^h \in \mathfrak{h}(Q) \) at

\[ d\pi_q (\tilde{V}^h) = V. \]

Given a curve \( x : (a, b) \to M \) we then get a vector field \( X \in \mathfrak{t}(T\pi^* Q) \).