Last time: A principal $G$-bundle $P \to M$ is trivializable $\iff$ there is a global section.

Hence, local trivializations of $P \to M$ are local sections.

$$
\varepsilon_i : C^n \to \mathbb{CP}^{n-1}, \quad \varepsilon_i : \{z \in \mathbb{CP}^{n-1} \mid z \neq 0\} \to C^n \setminus \{0\}
$$

are local sections.

Note: if $P|U_i : U_i \times G \to P|U_j : U_j \times G$ are two local trivializations, then we have transition maps

$$
\psi_{ij} : (U_i \cap U_j) \times G \to (U_i \cap U_j) \times G,
$$

where $U_{ij} := U_i \cap U_j$.

HW: If $\phi_i : U_i \times G \to U_i \times G$ is $G$-equivariant and commutes with projections to $U_i$, then $\exists \ h : U \to G$ such that $\phi_i(x, a) = (x, h(x) a)$.

So, open cover $\{U_i\}$ of $M$ and local trivializations $P|U_i : U_i \times G$

give rise to $\{\psi_{ij} : U_{ij} \to G\}$, with

$$
\psi_{ij}^{-1}(x, a) = (x, \phi_{ij}(x) a).
$$

Not hard to show $\{\psi_{ij}\}$ satisfies the cocycle conditions.

Now, if $\{U_i\}$ is a cover of $M$ and $\{\varepsilon_i : U_i \to P|U_i\}$ are local sections, how do we read off the corresponding cocycle $\{\psi_{ij} : U_{ij} \to G\}$?

Answer in two steps.

"Recall" if $P \to M$ is a fiber bundle, the fiber product

$$
P_{\pi} \left\{ \begin{array}{c}
P \times_M P \leftarrow \left( P \times_P P \right) \left\{ \pi_1 \pi_2 \right\} \left\{ \pi_1(\pi_2) = \pi(\pi_2) \right\} \end{array} \right.
$$


$P_{\pi} \left\{ \begin{array}{c}
P \leftarrow \left( P \times_P P \right) \left\{ \pi_1 \pi_2 \right\} \left\{ \pi_1(\pi_2) = \pi(\pi_2) \right\} \end{array} \right.$

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$P_{\pi} \left\{ \begin{array}{c}
P \leftarrow \left( P \times_P P \right) \left\{ \pi_1 \pi_2 \right\} \left\{ \pi_1(\pi_2) = \pi(\pi_2) \right\} \end{array} \right.$
is a fiber bundle over $M$ with typical fiber $F \times F$. 

If $P \xrightarrow{\phi} U \times F$ is a local trivialization, then $U \times F \ni (x, f) \mapsto (U \times F) \times (U \times F)$ induces a diffeo $U \times F \ni (x, f) \mapsto (x, f) \times (x, f)$. So $U \times F \times F = (U \times F) \times (U \times F)$, and composing with $\phi^{-1}$ gives us $P \xrightarrow{\phi} (U \times F) \times (U \times F) \cong U \times F \times F$.

**Proposition** If $P \xrightarrow{\pi} M$ is a principal $G$-bundle, then $A : P \times G \rightarrow P \times_M P$, $A(p, g) = (p, p g)$ is an isomorphism of fiber bundles over $M$.

(Here $\pi : P \rightarrow M$ is given by $\pi(p, g) = \pi(p)$.)

**Proof** Since $P \times G \xrightarrow{\nu_1} P \times_M P$ commutes,

$$\xrightarrow{\phi_1} \xrightarrow{\phi_2} \xrightarrow{\phi_3}$$

$A$ is a map of fiber bundles. To prove that $A$ is a diffeo, it is enough to prove: if $P \xrightarrow{\phi_1}$ is trivial then $A|_{P \times G} : P \times G \rightarrow P \times_M P \xrightarrow{\phi_3}$ is a diffeo.

Let $\psi : P \xrightarrow{\phi_1}$ be a trivialization. Then

$$P \xrightarrow{\nu_1} P \xrightarrow{A} P \xrightarrow{\phi_3} U \times G \times G$$

commutes, where $\xrightarrow{A} \xrightarrow{A}$ is $A : (x, a, b) \mapsto (x, a, a b)$.

It has an inverse: $(A^{-1})_*(x, a, c) = (x, a, a^{-1} c)$.

**Corollary** If $P \xrightarrow{\pi} M$ is a principal $G$-bundle, then $\exists$ (unique) $C^\infty$ map $g : P \times_M P \rightarrow G$ with $P \xrightarrow{\nu_1} P \xrightarrow{\psi} P \xrightarrow{\pi \circ \psi} P \xrightarrow{g \circ \psi} P$.

$p_1, p_2 : P \xrightarrow{\phi_1} P \xrightarrow{\pi} P \xrightarrow{\pi} P \xrightarrow{p_2} P \xrightarrow{p_1 \circ g \circ \psi} P$. 

$p_2 = p_1 \circ g \circ \psi(f, p_2)$.
**Proof**  \[ q \circ \text{the composite} \]

\[ P \times M \xrightarrow{A^{-1}} P \times G \xrightarrow{P \circ \pi} G \]

**Consequence:** If \( P \to M \) prin. \( G \)-bundle, \( \{ U_i \} \) open cover of \( M \) and \( s_i : U_i \to P \) local sections, then the transition maps corresponding to the trivializations \( P|_{U_i} \to U_i \times G \)

are \( q_{ij}(x) = q(s_i(x), s_j(x)) \)

\[ q(s_i(x), s_j(x)) \to \]

**Def** A Lie algebra in a vector space \( V \) (over \( \mathbb{R} \)) together with a bilinear map \( [\cdot, \cdot] : V \times V \to V \) (bracket)

So that 1) \( [v, w] = -[w, v] \)

2) \( [v, [w, u]] = [[v, w], u] + [w, [v, u]] \)

for all \( v, w, u \in V \).

**Ex** \( M \) manifold \( V = \Gamma(TM) \) vector space of vector fields.

\( [\cdot, \cdot] = \text{Lie bracket} \)

\( [X, Y] f = X(Y(f)) - Y(X(f)) \quad \forall f \in C^\infty(M) \).

**Ex** \( V = \mathbb{R}^3 \)

\( [v, w] = v \times w \), cross product.

**Ex** \( V \) any vector space \( [v, w] = 0 \) \( \forall v, w \in V \) \( V \) is an abelian Lie algebra.
Recall if $f : M \to N$ is a map between manifolds and $X \in \Gamma(TM)$ is a vector field, we cannot, in general, push it forward by $f$ to a vector field on $N$.

$\text{Ex. } M = \mathbb{R}^2$, $f(x,y) = x$

$X(x,y) = x \frac{\partial}{\partial x}$

while $df(x,y)(X(x,y)) = y \frac{\partial}{\partial x}$

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**Definition**

If $f : M \to N$, $X$ as above, $Y \in \Gamma(TN)$

$X$ and $Y$ are $f$-related if

$$(df)_x (X(x)) = Y(f(x)) \quad \forall x \in M.$$  

$\text{Ex. } f : \mathbb{R}^2 \to \mathbb{R}^2$ as above, $X(x,y) = \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$

$Y(x) = \frac{\partial}{\partial x}$

$X$ and $Y$ are $f$-related.

---

**Theorem**

If $f : M \to N$ is a map of manifolds

$X_1, X_2 \in \Gamma(TM)$, $Y_1, Y_2 \in \Gamma(TN)$

$X_1, Y_1$ ($X_2, Y_2$) are $f$-related

$\implies [X_1, X_2]$ and $[Y_1, Y_2]$ are $f$-related.

---

**Proposition**

If $f : M \to N$ embedding, $X_i$ $f$-related to $Y_i$

$\iff Y_i|_M$ is tangent to $M$.

Then so is $[Y_1, Y_2]$. 

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**Corollary**

If $Y_1, Y_2$ are tangent to $M$

then so is $[Y_1, Y_2]$. 

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