There were requests for more geometry and a bit less topology.
I'll rearrange the order in which we'll do things: we'll do connections, curvature and Chern-Weil first, Poincaré-Hopf and Poincaré duality later.

But first we need a better definition of tensor products and exterior algebra than the one you've seen last semester. This is so we can say things like:

\[ H^* (M \times N) \cong H^* (M) \otimes H^* (N) \]

\[ H^* (\mathbb{R}^n) \cong \bigwedge^* (\mathbb{R}^n) \]

- For any vector spaces \( V, W \), \( V \otimes W \cong \text{Hom}(V, W) \)

- For any vector bundles \( E \to M, F \to M \), \( E \otimes F \cong \text{Hom}(E, F) \)

etc.

We need this language to define connections on vector bundles:

\[ \Gamma(E) : \text{space of global sections}. \]

**Def. (p 167 of text)** Let \( E \to M \) be a vector bundle. A connection \( \nabla \) on \( E \) is an \( \mathbb{R} \)-linear map

\[ \nabla : \Gamma(E) \to \mathcal{L}(M) \otimes \mathcal{A}^1(M) \Gamma(E) \]

so that

\[ \nabla (fs) = df \otimes s + f \nabla s \]

for all \( s \in \Gamma(E) \) and all \( f \in C^\infty(M) \).

Recall: if \( R \) is a commutative ring (think \( R = \mathbb{R} \) or \( R = C^\infty(M) \)) a module over \( R \) is an abelian group \( V \) together with a map

\[ R \times V \to V, \quad (r, v) \mapsto rv \] so that

\[ a(bv) = (ab)v \]
\[ 1_R \cdot v = v \]
\[ (a + b)v = av + bv \]
\[ a \cdot (v + w) = av + aw \]
\[ a \cdot 0 = 0 \]
\[ a \cdot (b v) = abs \cdot v \]
\[ a \cdot (b v) = ab \cdot v \]

Ex. If \( E \) is a vector bundle over a manifold \( M \), then \( \Gamma(E) = \{ s : M \to E | s \text{ is smooth} \} \).
in a module over the ring \( C^0(M) \):

\[ (fs)(m) = f(m)s(m) + fs(0), \text{ } s \in F(E), \text{ } m \in M. \]

**Ex.** Any abelian group is a module over the ring \( \mathbb{Z} \).

**Ex.** Any vector space over \( \mathbb{R} \) is a module over \( \mathbb{R} \).

\[ 4(v+w) = 4(v) + 4(w) \]

A map of \( R \)-modules is an \( R \)-linear map:

\[ f: V \to W, \quad f(av) = a(f(v)) \]

Recall given a ring \( R \) and a set \( S \) there exists a free module \( F(S) \) over \( R \) generated by \( S \). \( F(S) \) is defined (up to a unique iso) by a universal property: for any module \( V \) over \( R \) and any map of sets \( \varphi: S \to V \), there is a unique map of modules \( F(\varphi): F(S) \to V \) with \( F(\varphi)|_S = \varphi \).

It’s usually constructed as \( F(S) = \{ f: S \to R \mid f(s) = 0 \text{ for all but finitely many } s \} \).

So if \( |S| = n < \infty \) we may take \( F(S) = R \oplus \cdots \oplus R \).

The tensor product \( E_1 \otimes_R E_2 \otimes_R \cdots \otimes_R E_n \) of \( R \)-modules \( E_1, \ldots, E_n \) is an \( R \)-module together with a multi-linear map \( i: E_1 \times \cdots \times E_n \to E_1 \otimes_R \cdots \otimes_R E_n \),

\[ i(v_1, \ldots, v_n) = v_1 \otimes_R \cdots \otimes_R v_n \]

with the following universal property:

\[ \forall \text{ multi-linear map } f: E_1 \times \cdots \times E_n \to W \exists ! \text{ linear map } \overline{f}: E_1 \otimes_R \cdots \otimes_R E_n \to W \text{ so that } \]

\[ E_1 \otimes_R \cdots \otimes_R E_n \xrightarrow{\overline{f}} W \]

\[ E_1 \times \cdots \times E_n \xrightarrow{f} \]
Since $i$ is multilinear, we have \((\text{for } n=2)\)

\[
\forall \; v_1, v_2 \in V, \; \forall \; w_1, w_2 \in W, \; a, b \in \mathbb{R} \\
(a v_1 + b v_2) \otimes w_i = a (v_1 \otimes w_i) + b (v_2 \otimes w_i) \\
= v_1 \otimes (a w_i) + v_2 \otimes (b w_i) \\
v_1 \otimes (w_1 + w_2) = v_1 \otimes w_1 + v_2 \otimes w_2 \quad \text{etc.}
\]

**Proof**

**Existence**  Consider $F(E_1 \times \cdots \times E_n)$, the free module on the set $E_1 \times \cdots \times E_n$.

Let $N$ be the submodule generated by the elements of the form

\[
(x_i, \ldots, x_i, \ldots, x_n) - a (x_i, \ldots, x_n)
\]

\[
(x_i, \ldots, x_i, x_i', \ldots, x_n) - (x_i, \ldots, x_i, \ldots, x_n) - (x_i, \ldots, x_i', \ldots, x_n)
\]

For all $x_i \in E_i$, $x_i' \in E_i$, $a \in \mathbb{R}$.

Define $E_1 \otimes \cdots \otimes E_n = F(E_1 \times \cdots \times E_n) / N$.

We have a canonical map

\[
i : E_1 \times \cdots \times E_n \rightarrow F(E_1 \times \cdots \times E_n) / N
\]

\[
i (x_1, \ldots, x_n) = (x_1, \ldots, x_n) + N
\]

It's not hard to check that $i$ is multilinear.

**Universal property?**

Any multilinear map $f : E_1 \times \cdots \times E_n \rightarrow W$ defines a linear map

\[
F(f) : F(E_1 \times \cdots \times E_n) \rightarrow W.
\]

If multilinear $\Leftrightarrow F(f) | N = 0$.

Hence $F(f)$ descends to a linear map

\[
\overline{f} : F(E_1 \times \cdots \times E_n) / N \cong E_1 \otimes \cdots \otimes E_n \rightarrow W.
\]

By construction $\overline{f} i = f$.

**Note:** $\forall \; \otimes v_i \in E_i$; $\forall \; v_i \in E_i$; $\forall \; i$; $\forall \; i$; $\forall \; i$; $\forall \; i$; $\forall \; i$; $\forall \; i$.

**Uniqueness**  Suppose $\overline{i} : E_1 \otimes \cdots \otimes E_n \rightarrow (E_1 \otimes \cdots \otimes E_n)$, $j=1, 2$. 

are two multilinear maps with the universal property.

We have

\[ (E_1 \otimes \ldots \otimes E_n)^{(1)} \quad (E_1 \otimes \ldots \otimes E_n)^{(2)} \]

\[ i_2 \quad i_1 \]

\[ E_1 \times \ldots \times E_n \]

By the universal property of \( i_2 \), there exists a linear map \( \overline{i_1} : (E_1 \otimes \ldots \otimes E_n)^{(1)} \rightarrow (E_1 \otimes \ldots \otimes E_n)^{(2)} \) such that \( \overline{i_1} \circ i_2 = \text{id} \) commutes.

So that

\[ (E_1 \otimes \ldots \otimes E_n)^{(1)} \xrightarrow{\overline{i_1}} (E_1 \otimes \ldots \otimes E_n)^{(2)} \]

\[ i_2 \quad i_1 \]

Similarly, there exists a linear map \( \overline{i_2} : (E_1 \otimes \ldots \otimes E_n)^{(2)} \rightarrow (E_1 \otimes \ldots \otimes E_n)^{(1)} \) such that \( \overline{i_2} \circ i_1 = \text{id} \) commutes.

Hence

\[ (E_1 \otimes \ldots \otimes E_n)^{(1)} \xrightarrow{i_2 \circ \overline{i_1}} (E_1 \otimes \ldots \otimes E_n)^{(2)} \]

\[ i_1 \quad \overline{i_1} \]

\[ (E_1 \times \ldots \times E_n) \]

By uniqueness, \( \overline{i_2} \circ \overline{i_1} = \text{id} \).

Similarly, \( \overline{i_1} \circ \overline{i_2} = \text{id} \).

\( \Rightarrow \overline{i_1}, \overline{i_2} \) are the desired unique iso's. \( \Box \)

Remark: For any \( R \)-module \( V \), \( R \times V \rightarrow V \), \( (a, v) \mapsto av \) is \( R \)-bilinear. \( \Rightarrow \exists ! \) linear map

\[ R \otimes R \rightarrow V \quad \text{with} \quad a \otimes v \mapsto av \]

We also have a linear map \( \psi : V \rightarrow R \otimes R \), \( \psi(v) = 1 \otimes v \)

\[ (\psi \circ \psi)(a \otimes v) = 1 \otimes av = a \otimes v \Rightarrow \psi \circ \psi = \text{id}_{R \otimes R} \]

\[ \psi(\psi(v)) = \psi(1 \otimes v) = 1 \cdot v = v \Rightarrow \psi \circ \psi = \text{id} \]

\( \therefore R \otimes R \cong V \).
Lemma Let $A_i : E_i \to F_i$ $(i = 1, \ldots, n)$ be maps of $R$-modules.

Then $\exists!$ $R$-linear map $\bigoplus_i A_i \otimes \bigotimes_i A_n : \bigotimes_i E_i \otimes \bigotimes_i F_i \to \bigotimes_i E_i \otimes \bigotimes_i F_i$

with $(A_1 \otimes \cdots \otimes A_n)(v_1 \otimes \cdots \otimes v_n) = (A_1 v_1) \otimes \cdots \otimes (A_n v_n)$

Proof. Since the set $\{v_i \otimes \cdots \otimes v_n | v_i \in E_i\}$ generates $E_1 \otimes \cdots \otimes E_n,$

$A_1 \otimes \cdots \otimes A_n$ is unique.

(Existence) Consider $f : \bigotimes_i E_i \to \bigotimes_i F_i,$

$f(v_1, \ldots, v_n) = A_1 v_1 \otimes \cdots \otimes A_n v_n$

$f$ is multilinear [why?].

$\Rightarrow \exists! \ f = A_1 \otimes \cdots \otimes A_n$ with

$(A_1 \otimes \cdots \otimes A_n)(v_1 \otimes \cdots \otimes v_n) = f(v_1, \ldots, v_n) = (A_1 v_1) \otimes \cdots \otimes (A_n v_n) . \quad \Box$

Next time: $(\sum_{\alpha \in A} E_{\alpha}) \otimes F = \sum_{\alpha \in A} (E_\alpha \otimes F)$