1. Prove that a sphere $S^n$ has a nowhere 0 vector field \( \iff n \) is odd.

**Hint.** (a) A nowhere zero vector field \( X \) defines a smooth map \( \psi: S^n \to S^n \). Use \( \psi \) to define a homotopy between \( \text{id} : S^n \to S^n \) and \( A : S^n \to S^n \), \( A(x) = -x \). If you get stuck try \( n = 1 \), \( \psi(x, y) = (-y, x) \).

(b) If \( n = 2k - 1 \) then \( S^n \subset C^k \). Now consider \( S^n \to C^k \), \( p \mapsto \mathbf{r}(p) \).

2. (a) Let \( \psi : M \to M \) be an involution, i.e., \( \psi \circ \psi = \text{id } M \).

Let \( \Omega^*_+ (M) = \{ \omega \in \Omega^*(M) \mid \psi^* \omega = \omega \} \)

\( \Omega^*_- (M) = \{ \omega \in \Omega^*(M) \mid \psi^* \omega = -\omega \} \).

Show that \( (\Omega^*_+ (M), d) \) are subcomplexes of \( (\Omega^*(M), d) \) and that \( H^*(M) = H^*(\Omega^*_+ (M), d) \oplus H^*(\Omega^*_- (M), d) \).

(b) Now let \( M = S^n \) and \( \psi(x) = -x \).

Show that \( H^*(\mathbb{R}P^n) = H^+ (S^n) \), where \( \mathbb{R}P^n = S^n / \mathbb{R} \times \) in the real projective space.

(c) Prove that \( H^n (S^n) = \begin{cases} H^n _+ (S^n) & n \text{ odd} \\ H^n _- (S^n) & n \text{ even} \end{cases} \)

(d) What is \( H^*(\mathbb{R}P^n) \)?

3. Let \( M \) be a manifold, \( \psi : U \subset V \subset \mathbb{R}^n \) a coordinate chart and \( f : M \to \mathbb{R} \) a \( C^\infty \) function. Let \( h(x_1, \ldots, x_n) = (J f (\psi^{-1}))(x_1, -x_n) \) be a local expression for \( f \). Prove: \( df \mid U \) is transverse to the zero section of \( T^* M \mid U \to U \) \( \iff \) the matrix

\( \left( \frac{\partial h}{\partial x_i} (\psi(p)) \right) \) is nondegenerate for any \( p \in U \) with \( df(p) = 0 \).