Last time: Defined embedded submanifold \( Z \) of a manifold \( M \) to be a subset \( Z \subseteq M \) so that \( \forall p \in Z \) \( \exists \) a chart \( \psi : U \to \mathbb{R}^m \) on \( M \) which is adapted to \( Z \):

\[
\psi(U \cap Z) = \psi(U) \cap (\mathbb{R}^2 \times \{0\}) \subseteq \mathbb{R}^2 \times \mathbb{R}^{m-2}
\]

\( m-2 \) : codimension of \( Z \).

Exercise: An embedded submanifold \( Z \) of a manifold \( M \) in a manifold, \( \psi \) inclusion map \( i: Z \to M \) is \( C^\infty \).

Note: \( W \) is an open subset of a manifold \( M \), then \( W \subseteq M \) is an embedded submanifold of codimension \( 0 \).

We want to prove!

Regular value theorem: Let \( F: M \to N \) be a \( C^\infty \) map, \( c \in F(M) \subseteq N \)

regular value of \( F \) (i.e. \( \forall p \in F^{-1}(c) \) \( T_p F: T_p M \to T_c N \) is onto)

Then \( F^{-1}(c) \) is an embedded submanifold of \( M \). Moreover, \( \forall p \in F^{-1}(c) \)

\[
T_p \left( F^{-1}(c) \right) = \ker \left( T_p F : T_p M \to T_c N \right)
\]

and \( \dim F^{-1}(c) = \dim M - \dim N \).

Remark: If \( p \in F^{-1}(c) \), \( \gamma: (a, b) \to F^{-1}(c) \) is a path through \( p \)

Then \( F \circ \gamma(t) = c \ \forall t \), \( \Rightarrow \)

\[
0 = T_0 \left( F \circ \gamma \right) \left( \frac{d}{dt} \right) = T_p F \circ T_0 \gamma \left( \frac{d}{dt} \right) = T_p F \left( \dot{\gamma}(0) \right).
\]

Since \( \gamma \) is arbitrary, \( T_p \left( F^{-1}(c) \right) \subseteq \ker \left( T_p F \right) = \dim T_p M - \dim T_c N \)

So once we know \( \dim F^{-1}(c) = \dim M - \dim N \), it follows that

\[
T_p \left( F^{-1}(c) \right) = \ker \left( T_p F \right).
\]

Key analysis fact:

Inverse function theorem: Suppose \( U \subseteq \mathbb{R}^m \) is open, \( F: U \to \mathbb{R}^n \)

in \( C^\infty \), \( p \in U \), \( T_p F: T_p U \subseteq \mathbb{R}^n \to T_F(p) \mathbb{R}^n = \mathbb{R}^n \)

is a isomorphism. Then \( \exists \) open \( V \) of \( p \) in \( U \), open \( W \)

\( W \subseteq \mathbb{R}^n \) of \( F(p) \) so that \( \exists \)

\[
f | V: V \to W \text{ is a diffeomorphism.}
\]
Notation. Given a $C^\infty$ map $F: \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^n$ and $(a,b) \in \mathbb{R}^k \times \mathbb{R}^l$ we define

$$\frac{\partial F}{\partial x} (a,b) := T_{(a,b)} F |_{\mathbb{R}^k \times \mathbb{R}^l}.$$ 

$$\frac{\partial F}{\partial y} (a,b) := T_{(a,b)} F |_{\mathbb{R}^k \times \mathbb{R}^l}.$$ 

So $T_{(a,b)} F = \left( \frac{\partial F}{\partial x} (a,b) \mid \frac{\partial F}{\partial y} (a,b) \right)$ as block matrices/linear maps.

Implicit Function Theorem. Suppose $F: \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$ is $C^\infty$ and $F(a,b) : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism, $c = F(a,b)$.

Then for each $(a,b)$ in $\mathbb{R}^k \times \mathbb{R}^n$, a neighborhood $U$ of $a$ in $\mathbb{R}^k$ and a $C^\infty$ map $g: U \to \mathbb{R}^n$ so that $F(x,y) \in W$.

$$F((x,y)) = c \iff y = g(x), \quad i.e.$$

$$F^{-1}(c) \cap W = \text{graph}(g) = \{ (x, y) : g(U) \cap \mathbb{R}^n \}.$$ 

Proof. Consider $H: \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^k \times \mathbb{R}^n$, $H(x,y) = (x, F(x,y))$.

Then $H(a,b) = (a, F(a,b)) = (a, c)$.

$$T_{(a,b)} H = \begin{pmatrix} I & 0 \\ \frac{\partial F}{\partial x} (a,b) & \frac{\partial F}{\partial y} (a,b) \end{pmatrix}.$$ 

Since $\frac{\partial F}{\partial y} (a,b)$ is invertible, so is $T_{(a,b)} H$.

Inverse Function Theorem. For each $(a,b)$ in $\mathbb{R}^k \times \mathbb{R}^n$, a neighborhood $V$ of $c$ in $\mathbb{R}^n$ so that $H|_V: V \to U \times V$ is a diffeo. [Why?]

Let $G(u,v) = H^{-1}(u,v)$, $v = G(u,v) \in U \times V$.

Then $G(c,u) = (G_1(c,u), G_2(c,u))$ where $G_1: U \times V \to \mathbb{R}^k$, $G_2: U \times V \to \mathbb{R}^n$.

Now

$$(u,v) = H(G_1(u,v), G_2(u,v)) = (G_1(u,v), F(G_1(u,v), G_2(u,v))).$$

$$\Rightarrow G_1(u,v) = u, \quad \Rightarrow v = F(u, G_2(u,v)). \quad \forall (u,v) \in U \times V.$$

$$\Rightarrow c = F(x, G_2(x, c)) \quad \forall x \in U.$$

Let $g(x) = G_2(x,c)$.

Then graph$(g) = \{(x, G_2(x,c)) \mid x \in U \subset F^{-1}(c)$.}
Conversely, \( \psi(x,y) \circ W \circ F^{-1}(c) \), \( F(x,y) = c \) and 
\[
\begin{align*}
(x,y) &= \psi_1(H(x,y)) = \varphi_{(x,F(x,y))} = \varphi_{(x,c)} = (\varphi_1(x,c), \varphi_2(x,c)) \\
&= (x, \varphi(x)).
\end{align*}
\]

\[\Rightarrow \ W \circ F^{-1}(c) \subseteq \text{graph}(\varphi). \]

**Proof of the regular value theorem.** Suppose \( c \in N \) is a regular value of \( F: M \to N \) and \( F^{-1}(c) \neq \emptyset \).

Fix \( p \in F^{-1}(c) \) and pick two coordinate charts 
\[ \varphi: U \to \mathbb{R}^m \text{ on } M, \quad \psi: V \to \mathbb{R}^n \text{ on } N \]
with \( p \in U \) and \( F(p) \in V \).

Since \( F \) is smooth,
\[
\psi \circ F \circ \varphi^{-1} \mid \psi(\varphi(F(V) \cap U)) \to \mathbb{R}^n \text{ \& } C. \]

Claim: \( \psi(c) \) is a regular value of \( \bar{F} = \psi \circ F \circ \varphi^{-1} \).

Reason: \( \forall r \in \varphi^{-1}(\psi(c)) \Rightarrow \psi(c) = \psi(F(\varphi^{-1}(r))) \)
\[ \Rightarrow c = F(\varphi^{-1}(r)) \]
\[ \Rightarrow (\varphi^{-1}(r)) \in F^{-1}(c). \]

Hence, \( \bar{T}_c \bar{F} = T_{\varphi^{-1}(c)} \psi \circ T_{\varphi^{-1}(c)} F \circ T_c \varphi \).

Derivatives of coordinate charts are isomorphisms:
\( T_{\varphi^{-1}(c)} F \) is onto since \( \varphi^{-1}(c) \in F^{-1}(c) \) and \( c \)
is a regular value, \( \bar{T}_c \bar{F} : \bar{T}_c \mathbb{R}^m \to \bar{T}_{\psi(c)} \mathbb{R}^n \)
is onto.

We now argue: \( (\bar{F})^{-1}(\psi(c)) \) is an embedded submanifold of \( \varphi(F(V) \cap U) \) of dimension \( m-n \).

Note since \( \varphi \mid \varphi(F(V) \cap U) : \varphi(F(V) \cap U) \to \varphi(F(V) \cap U) \)
is a diffeomorphism, it follows that 
\[ F^{-1}(c) \cap (F^{-1}(V) \cap U) \]is an embedded submanifold.
We therefore may assume: $M = \mathbb{R}^m$, $N = \mathbb{R}^n$, $c \in \mathbb{R}^n$ is a regular value of $F: \mathbb{R}^m \to \mathbb{R}^n$.

Fix $p \in F^{-1}(c)$. Let $X = \ker (T_p F: \mathbb{R}^m \to \mathbb{R}^n) \subset \mathbb{R}^m$.
Pick a complement subspace $Y \subset \mathbb{R}^m$ so that
\[ \mathbb{R}^m \cong X \times Y. \]
Then $p \mapsto (a, b) \in X \times Y$.

Since $T_p F: \mathbb{R}^m \to \mathbb{R}^n$ is onto and $X = \ker (T_p F)$,
\[ T_p F|_Y : Y \to \mathbb{R}^n \] is an isomorphism.

Implicit function theorem $\Rightarrow$ Near $p = (a, b)$, $F^{-1}(c)$ is a graph of function $g: U \to V$, where $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$.

$\Rightarrow$ Near $p = (a, b) \in X \times Y \subset \mathbb{R}^m$

$F^{-1}(c)$ is an embedded submanifold.

Ex. Let $\text{Sym}^2(\mathbb{R}^n) = \text{vector space of symmetric } n \times n \text{ matrices}$
\[ = \{ A \in M_n(\mathbb{R}) \mid A^T = A \}. \]

Consider $F: \text{GL}(n, \mathbb{R}) \to \text{Sym}^2(\mathbb{R}^n) \cong \mathbb{R}^{\frac{n^2+n}{2}}$
\[ F(A) = A^T A. \]

We'll prove next time: $I \subset \text{Sym}^2(\mathbb{R}^n)$ is a regular value of $F$.

Regular value theorem $\Rightarrow$
\[ F^{-1}(I) = \{ B \in \text{GL}(n, \mathbb{R}) \mid B^T B = I \} =: O(n) \]
is an embedded submanifold of $\text{GL}(n, \mathbb{R})$.
\[ \dim O(n) = n^2 - \frac{n^2 + n}{2} = \frac{n^2 - n}{2} \]
\[ T_I O(n) = \ker T_I F = \{ A \in M_n(\mathbb{R}) \mid A^T A = O \}. \]

We'll compute this