Last time: $\frac{\partial}{\partial t_i} |_{t=0} \cdot \frac{\partial}{\partial x_i} |_{x=0}$ is a basis of $T_{\theta} \mathbb{R}^m$

(here $t_1, \ldots, t_m : \mathbb{R}^m \to \mathbb{R}$ are the standard coordinates)

5.2 $\forall p \in M \forall \nu \in T_p M$, $\forall f \in \mathcal{C}^0(M)$, $\nu(f)$ depends only on flow where $\mathcal{U}$ is some (any) open nbhd of $p$.

For $f : M \to N$ defined $T_p f : T_p M \to T_{\phi(p)} N$ by $(T_p f)(\nu)(h) = \nu(\phi(h)) + \nu \in \mathcal{C}^0(N)$

Lemma 5.3 Let $M$ be a manifold, $p \in M$, $U$ open nbhd of $p$. Then $T_p i : T_p U \to T_p M$ is an isomorphism ($i : U \to M$).

**Sketch of proof**

Note that $\forall \nu \in T_p U \forall f \in \mathcal{C}^0(M)$ $$(T_p i)(\nu)(f) = \nu(f \circ i) = \nu(f|_U).$$

We construct an inverse $j : T_p M \to T_p U$ as follows.

Choose $\tau \in \mathcal{C}^0(M)$, so that supp $\tau \leq U$ and $\tau V = 1$ where $V$ is an open nbhd of $p$.

For $h \in \mathcal{C}^0(U)$ define $T_h \in \mathcal{C}^0(M)$ by $T_h(q) = \frac{\tau(q)}{h(q)} q \in U$ for $q \in \mathcal{C}^0(U)$ define $T_h \in \mathcal{C}^0(M)$ by $T_h(q) = \frac{\tau(q)}{h(q)} q \in U$

Now define $j : T_p M \to T_p U$ by $(j(h))(\nu) = \nu(\tau \cdot h)$.

Easy to check that $j(h)$ is linear. Moreover, $\forall h_1, h_2 \in \mathcal{C}^0(U)$ $j(h_1 h_2) = \nu(T(h_1 h_2)) = \nu(T \cdot h_1 h_2)$ since $\tau^2 h_1 h_2 = 0$ on $V$

- $\nu(\tau) \cdot (\tau h_1 (p) + (\tau h_1)(p) \nu(\tau h_2))$
- $(\nu(h_1 h_2) \cdot h_1(p) + h_1(p) \nu(h_2))$

$\Rightarrow j(h) \in T_p U.$

Now, $\forall \nu \in T_p M \forall f \in \mathcal{C}^0(M)$ $\nu(f)$ $$(T_p i \circ \tau)(\nu)(f) = j(h)(f \circ i) = \nu(\tau \cdot f|_U) = \nu(f)$$ since $\tau \cdot f|_U \equiv f$ on $V$

$\Rightarrow T_p i \circ \tau = \text{id}_{T_p M}$.

Similarly, for any $\nu \in T_p U \forall h \in \mathcal{C}^0(U)$ $$(j \circ T_p i)(\nu)(h) = (T_p i)(\nu)(\tau h) = \nu(\tau h|_U) = \nu(h)$$
Since \( Th = 0 \) on \( V \), \( \Rightarrow \) \( 0 \circ \tau_p^i = 0 \circ \tau_p^i = T_h^p \). 

Remark. In practice the map \( \tau^p : T_p U \rightarrow T_p M \) is "movable".

For example, let \( M = \mathbb{R}^n \) and \( U \) an open ball \( W \) of \( \mathbb{R}^n \).

We have \( \frac{\partial}{\partial x_i} |_{0} = \frac{\partial}{\partial y_i} |_{0} \in T_p W \)

and \( \left( \tau^p \right)^* \left( \frac{\partial}{\partial x_i} \right) = \frac{\partial}{\partial y_i} \left( f | W \right) = \frac{\partial}{\partial x_i} |_{0} f \quad \forall f \in C^\infty(\mathbb{R}^n) \)

\( \Rightarrow \) \( \frac{\partial}{\partial x_i} |_{0} \) and \( \frac{\partial}{\partial y_i} |_{0} \) is a basis of \( T_p W \).

Let \( p \in M \), \( \psi : U \rightarrow \mathbb{R}^n \) a coordinate chart with \( \psi(p) = 0 \).

We defined \( \frac{\partial}{\partial x_i} |_{p} \in T_p M \) by \( \left( \frac{\partial}{\partial x_i} \right) |_{0} \left( f \circ \psi^{-1} \right) = \frac{\partial}{\partial x_i} |_{0} \left( f \circ \psi^{-1} \right) \).

But \( \frac{\partial}{\partial x_i} \left( f \circ \psi^{-1} \right) = \left( T_p \left( (\psi \circ \psi^{-1}) \left( \frac{\partial}{\partial x_i} \right) \right) \right) \frac{\partial f}{\partial y_i} = \left( \tau^p \circ T \psi^{-1} \right) \left( \frac{\partial}{\partial y_i} \right) f \quad \forall f \in C^\infty(\mathbb{R}^n) \).

Now \( \psi : \psi(U) \rightarrow U \) is a diffeomorphism.

Hence \( \tau^{\psi^{-1}} : T_p \psi(U) \rightarrow T_p U \) is an isomorphism.

Also \( \tau^p : T_p U \rightarrow T_p M \) is an isomorphism.

\( \Rightarrow \) \( \tau^p \circ \tau^{\psi^{-1}} : T_p \psi(U) \rightarrow T_p M \) is an isomorphism.

\( \left( \frac{\partial}{\partial y_i} \right) |_{0} \) is a basis of \( T_p \psi(U) \). \( \Rightarrow \) \( \left( \frac{\partial}{\partial x_i} \right) |_{0} \) is a basis of \( T_p M \). 

\[ \square \]

Lemma 6.1: let \( V \) be a finite dimensional vector space over \( \mathbb{R} \).

Then \( T_p V \) is canonically isomorphic to \( V \).

Proof. We know that \( V \) is a manifold of dimension \( \dim V \).

\( \Rightarrow T_p V \) is a vector space of dimension \( \dim V \).

\( \Rightarrow T_p V \) is isomorphic to \( V \) (but this may involve choices).

Now given \( v \in V \) we define \( D_v \) \( \in T_p V \) by \( D_v |_p \left( f \right) = \frac{\partial}{\partial t} |_{0} f \left( p + tv \right) \). Not hard to check.
\[ DV|_p \in T_p V, \text{ is } DV|_p \text{ is a derivation of } C^\infty(M) \]

Moreover the map \( \Phi: V \to T_p V \)
\[ \Phi(v) = DV|_p \text{ is linear. } \]
\[ \ker \Phi = \{ v \in V \mid DV|_p f = 0 \quad \forall f \in C^\infty(V) \} \]

For any \( v \in V, \ u \neq 0 \), \( \exists l \in V^* \) s.t. \( l(v) \neq 0 \).
\[ l \in C^\infty(V) \text{ and } DV|_p l = \frac{d}{dt}|_0 l(p+tv) = \frac{d}{dt}|_0 (l(t)v + tl(v)) = l(v) \neq 0. \]
\[ \Rightarrow \ker \Phi = \{0\}. \text{ Dimension count } \Rightarrow \]
\[ \Phi \text{ is an isomorphism of vector spaces } \Phi \]

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We now generalize.

**Definition** Let \( M \) be a manifold, \( p \in M \). A curve through \( p \)

\[ \gamma: (a, b) \to M \text{ for some } a < c \leq b \]

so that \( \gamma(0) = p \).

**Construction** Given a curve \( \gamma: (a, b) \to M \) through \( p \)

We define \( \dot{\gamma}(0) \in T_p M \) by

\[ \dot{\gamma}(0)f = \frac{d}{dt}|_0 (f \circ \gamma)(t) \quad \forall f \in C^\infty(M) \]

**Note** The construction makes sense: \( \forall \gamma, \mu \in C^\infty(M) \)

\[ \gamma(0)(\gamma + \mu) = \frac{d}{dt}|_0 (\gamma + \mu) \circ \gamma \]

and

\[ \dot{\gamma}(0)(fg) = \frac{d}{dt}|_0 (f \cdot \gamma(0)) \circ \gamma \]

\[ = \frac{d}{dt}|_0 (f \cdot \gamma(0)) \cdot g(\gamma(0)) + \frac{d}{dt}|_0 g(\gamma(0)) \cdot f(\gamma(0)) \]

Claim The map \( (\gamma(0)): \text{ curves through } p \to T_p M \)

\[ x \mapsto \dot{\gamma}(0) \]

is onto.
Reason let \( \varphi = (x_1, \ldots, x_m) : U \to \mathbb{R}^m \) be a coordinate chart with \( p \in U \), \( \varphi(p) = 0 \).

Given \( v \in T_pM \) \( \exists a_i, a_m \in \mathbb{R} \) s.t. \( v = \sum a_i \frac{\partial}{\partial x_i} \big|_p \).

Then \( v \in C^\infty(M) \)

\[ v(f) = \left( \sum a_i \frac{\partial}{\partial x_i} \big|_p \right) (f) = \sum a_i \frac{\partial}{\partial x_i} \big|_0 (f \circ \varphi^{-1}) \]

\[ = \frac{d}{dt} \big|_0 (f \circ \varphi^{-1} \big|_{ \varphi^{-1}(0) = t}) \]

Now set \( \xi(t) = (\varphi^{-1}(0, \ldots, t)) \).

Then \( \xi(0) = \varphi^{-1}(0, \ldots, 0) = p \)

and \( \dot{\xi}(0) f = \frac{d}{dt} \big|_0 f(\varphi^{-1}(0, \ldots, t)) = v(f) \).

The map \( \xi(0) \) is not linear and it's not 1-1.

Define a relation \( \sim \) on curves through \( p \) by

\[ \xi \sim \sigma \iff \xi(0) = \sigma(0) \] “first order tangency”

We then have a bijection

\[ \left( \text{curves through } p \right) / \sim \to T_pM. \]

Recall given \( F : M \to N \) we defined \( T_pF : T_pM \to T_{F(p)}N \)

by \( (T_pF)(v) f = v(F \circ F'). \forall f \in C^\infty(N) \)

Now we know \( v = \dot{\xi}(0) \) for some \( \xi : (0, b) \to M \).

What is \( (T_pF)(\dot{\xi}(0)) \)?

A. \( \forall f \in C^\infty(N) \)

\[ \left( (T_pF)(\dot{\xi}(0)) \right) f = \dot{\xi}(0) (F \circ F) = \frac{d}{dt} \big|_0 (F \circ F \circ \varphi^{-1})(t) \]

\[ = \frac{d}{dt} \big|_0 f(\varphi^{-1}(0, \ldots, t)) = (F \circ F)'\circ(0) f \]

\[ \Rightarrow (T_pF)(\dot{\xi}(0)) = (F \circ F)'(0) \]

This is useful in computations.