Last time we defined partitions of a subdomain to covers, and sketched a proof that partitions of 1 exist.

- We defined a vector \( v \)-tangent to a manifold \( M \) at a point \( p \) to be a derivation \( v : C^\infty(M) \to \mathbb{R} \), a linear map such that \( v f, g \in C^\infty(M) \)

\[
v(fg) = v(f)g(p) + f(p)v(g), \text{ i.e. } v \text{ is a derivation}
\]

We defined \( T_p M = \{ v : C^\infty(M) \to \mathbb{R} \mid v \text{ is a derivation} \} \) as the space of vectors tangent to \( M \) at \( p \).

\( T_p M \) is a vector space.

We proved:
- If \( f : C^\infty(M) \to \mathbb{R} \) is constant then \( v f = 0 \) \( \forall v \in T_p M \).

Hadamard's lemma:

\[
\frac{df}{dx_i} \quad \text{so that}
\]

\( f(x) = f(0) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(0) x_i(x) \quad \text{and} \quad \frac{\partial f}{\partial x_i}(0) = \frac{\partial f}{\partial x_i}(0) \).

Note:

\[
\frac{2}{\partial x_i} \text{ is } T_0 \mathbb{R}^n \quad \text{since } v_j = \frac{\partial}{\partial x_j} \in C^\infty(\mathbb{R}^n)
\]

\[
\frac{\partial f}{\partial x_j}(0) = \frac{\partial f}{\partial x_j}(0) + f(0) \frac{\partial h}{\partial x_j}(0)
\]

Lemma 5.1: \( \{ \frac{2}{\partial x_i} \mid i = 1, \ldots, n \} \) is a basis of \( T_0 \mathbb{R}^n \).

Proof: Suppose

\[
\sum c_i \frac{\partial f}{\partial x_i} |_0 = 0 \quad \text{for some } c_i \in \mathbb{R}.
\]

Then \( v_j \)

\[
0 = O(x_j) = \sum c_i \frac{\partial}{\partial x_i} |_0 \rho_j(x_j) = \sum c_i \delta \rho_j(x_j) = \rho_j(x_j)
\]

\[
\Rightarrow \quad \rho_j(x_j) = 0 \quad \text{hence independent.}
\]

Moreover, \( \forall f \in C^\infty(M) \) \( \forall v \in T_0 \mathbb{R}^n \)

\[
v(f) = v(f(0)) + \sum x_j \frac{\partial f}{\partial x_j}(0) = v(f(0)) + \sum u(x_i) \frac{\partial x_i}{\partial x_j} |_0
\]

\[
+ \sum_{i=1}^n x_j(0) \frac{\partial v}{\partial x_j}(0) = 0 + \left( \sum u(x_i) \frac{\partial x_i}{\partial x_j} |_0 \right)(f).
\]
Aside. Recall that if \( \{e_1, \ldots, e_n\} \) is a basis of a vector space \( V \)
then \( \exists \) linear functionals \( e_1^*, \ldots, e_n^* : V \rightarrow \mathbb{R} \)
so that \( e_i^*(e_j) = \delta_{ij} \).
Moreover, \( \forall v \in V \)
\[ u = \sum_{i=1}^{n} e_i^*(v) e_i. \]

The functions \( x_1, \ldots, x_n : \mathbb{R}^n \rightarrow \mathbb{R} \)
give rise to a basis of \( (\mathbb{R}^n)^* \) dual to \( \{ \frac{\partial}{\partial x_i} \}_{i=1}^{n} \).
This dual basis is denoted by \( \{ dx_i, \ldots, dx_n \} \) and defined by
\[ dx_i(v) := x_i(v). \]
So (**) can be rewritten as
\[ U = \sum_{i=1}^{n} dx_i(v) \frac{\partial}{\partial x_i}. \]

Remark. It also follows that any \( v \in \mathbb{T}_{x} \mathbb{R}^n \) is of the form
\[ v(t) = (\sum v_i \frac{\partial}{\partial x_i}) f \]
where \( v, f : \mathbb{R} \rightarrow \mathbb{R} \)
and so \( v \) is the directional derivative in the direction \( (v_1, \ldots, v_n) \in \mathbb{R}^n \).

Back to manifolds.

Aside. You checked that if \( M \) is a manifold, \( U \subset M \) an open subset
then \( U \) is a manifold and the inclusion \( i : U \rightarrow M \)
is a \( C^\infty \) map
\[ i(q) = q \quad \forall q \in U \]
\[ \Rightarrow \forall f \in C^\infty(M), \quad f i \in C^\infty(U). \]
One usually writes \( f |_{U} \) for \( f i \).
Note that the restriction map: \( C^\infty(M) \rightarrow C^\infty(U) \), \( f \mapsto f|_U \) need not be onto. For example (let \( M = \mathbb{R} \), \( U = (-\pi/2, \pi/2) \))

\[ f(x) = \tan(x) \in C^\infty((-\pi/2, \pi/2)) \] but there is no \( f \in C^\infty(\mathbb{R}) \) st.

\[ f(-\pi/2, \pi/2) = \tan(x) \]

Change of notation: denote the standard coordinate functions on \( \mathbb{R}^m \) by \( R_i = r_i \). Thus for \( \alpha = (a_1, \ldots, a_m) \in \mathbb{R}^m \)

\[ R_i(\alpha) = a_i, \quad 1 \leq i \leq m. \]

Let \( M \) be a manifold, \( p \in M \), \( \psi : U \rightarrow \mathbb{R}^m \) a coordinate chart with \( p \in U \). Note, by replacing \( \psi \) by \( \psi - \psi(p) \) if necessary, we may assume that \( \psi(p) = 0 \).

For any \( f \in C^\infty(M) \), \( f \circ \psi^{-1} \in C^\infty(\psi(U)) \) (by def. of \( C^\infty(M) \)).

We define

\[ \frac{\partial}{\partial x_i} f : C^\infty(M) \rightarrow \mathbb{R} \]

by

\[ \frac{\partial}{\partial x_i} f = \frac{\partial}{\partial R_i} (f \circ \psi^{-1}) = \frac{\partial}{\partial R_i} (f \circ \psi^{-1}). \]

Our goal is to show that \( \left\{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m} \right\} \) is a basis of \( T_p M \).

Let \( x_i := R_i \circ \psi \in C^\infty(U) \). Then \( \forall f \in C^\infty(M) \)

\[ f|_U = f \circ \psi^{-1} \circ \psi. \]

Thus \( \forall q \in U \)

\[ f(q) = (f \circ \psi^{-1})(\psi(q)) = (f \circ \psi^{-1})(x_1(q), \ldots, x_m(q)) \]

or

\[ f|_U = (f \circ \psi^{-1})(x_1, \ldots, x_m) \]

Lemma 5.2 Let \( M \) be a manifold, \( p \in M \), \( v \in T_p M \).

For any \( f \in C^\infty(M) \), \( v(f) \) depends only on the values
of $f$ in an arbitrarily small neighborhood $W$ of $p$:

\[ V \text{ open nbd of } p, \forall f_1, f_2 \in C^\infty(M), f_1|_W = f_2|_W \implies v(f_1) = v(f_2). \]

**Proof.** There is $p \in C^\infty(M)$ so that $\text{supp } p \subseteq W$ and $p \equiv 1$ near $p$.

Let $h = p \cdot (f_1 - f_2)$.

For $x \in W$, $h(x) = p(x) \cdot (f_1(x) - f_2(x)) = 0$ since $h|_W = f_1|_W - f_2|_W$.

For $x \notin W$, $p(x) = 0$ since $\text{supp } p \subseteq W$, so $h(x) = 0$ as well.

Thus $p \cdot (f_1 - f_2) \equiv 0$.

\[ 0 = v(p) = v(p \cdot (f_1 - f_2)) = v(p) \cdot (f_1 - f_2) + p(v(f_1) - v(f_2)) \]

\[ = v(p) \cdot 0 + 1 \cdot (v(f_1) - v(f_2)). \]

\[ \implies v(f_1) = v(f_2). \quad \square \]

Let $f : M \to N$ be a $C^\infty$ map between two manifolds. We define

\[ df_p = T_p f : T_p M \to T_{f(p)} N \]

by setting

\[ ((T_p f)(v))(h) = v(h \circ f) \quad \forall h \in C^\infty(N). \]

**Exercise.** i) $\forall u \in T_p M$, $(T_p f)(u)$ is a tangent vector to $N$ at $f(p)$, that is, $(T_p f)(u) : C^\infty(N) \to \mathbb{R}$ is a derivation.

So $T_p f : T_p M \to T_{f(p)} N$ is well-defined.

(ii) $T_p f$ is linear

(iii) $T_p (\text{id}_M) = \text{id}_{T_p M}$

(iv) $\forall M \to N, N \xrightarrow{g} Q$ two $C^\infty$ maps

\[ T_p (g \circ f) = T_{f(p)} g \circ T_p f. \]

Note: (iii)+(iv) $\implies$ for any diffeomorphism $f : M \to N$, $\forall p \in M$ $T_p f : T_p M \to T_{f(p)} N$ is an isomorphism of vector spaces.

Reason $T_p g : N \to M, C^\infty$ s.t. $g \circ f = \text{id}_M$, $f \circ g = \text{id}_N$

$\implies \text{id}_{T_p M} = T_p (g \circ f) \circ T_p f, \quad \text{id}_{T_{f(p)} N} = T_{f(p)} g \circ T_p f$. 
Lemma 5.3: Let $M$ be a manifold, $p \in M$, $U$ an open nbhd of $p$, and $i : U \to M$ the inclusion map. Then
\[ T_p i : T_p U \to T_p M \]
is an isomorphism of vector spaces.

Proof: Note that for any $w \in T_p U$, $f \in C^\infty(M)$
\[ ((T_p i)(w))(f) = w(f \circ i) = w(f|_U) \]
Now fix a bump function $\tau \in C^\infty(M)$ with $\tau \equiv 1$ near $p$
say on an open nbhd $V$ of $p$ and supp $\tau \subseteq U$ (so $V \subseteq U$).

Define $j : T_p M \to T_p U$ by
\[ (j(w))(h) = \tau(h \circ \iota) \]
Here $\iota \in C^\infty(M)$ is defined by
\[ \iota(q) := \begin{cases} 1 & q \in U \\ 0 & q \not\in U \end{cases} \]

Then $\forall \lambda_1, \lambda_2 \in \mathbb{R}$, $h_1, h_2 \in C^\infty(U)$
\[ (j(w))(\lambda_1 h_1 + \lambda_2 h_2) = \tau(\lambda_1 h_1 + \lambda_2 h_2) = \lambda_1 \tau(h_1) + \lambda_2 \tau(h_2) \]
\[ = \lambda_1 j(w)(h_1) + \lambda_2 j(w)(h_2) \]
\[ \Rightarrow j(w) : C^\infty(U) \to \mathbb{R} \text{ is linear.} \]

Also $j(w)(h_1 h_2) = \tau(\iota(h_1 h_2))$
\[ = \tau(h_1 h_2) \quad \text{since } \tau^2 h_1 h_2 = \tau(h_1 h_2) \text{ on } V. \]
\[ = \tau(h_1)(h_2(p)) + [h_1(p) \cdot \tau(h_2)] \]
\[ \Rightarrow j(w) : C^\infty(U) \to \mathbb{R} \text{ is a derivation. So } j(w) \iota T_p U. \]

Easy to check: $\forall j : T_p M \to T_p U$ is linear.

Now $\forall w \in T_p M$, $\forall f \in C^\infty(M)$
\[ ((T_p i \circ j)(w)) f = j(w)(f|_U) = \tau(f|_U) = \tau(f) \]
\[ \text{since } \tau \cdot f|_U = f \text{ on } V. \]

Similarly, $\forall w \in T_p U$ and $h \in C^\infty(U)$
\[ ((\iota \circ T_p i)(w)) h = (T_p i(w))(\tau h) = w(\tau h|_U) = w(h) \]
\[ \text{since } \tau h = h \text{ on } V. \]
\[ \square \]
Remark. In practice the map $T_p i : T_p U \to T_p M$ is "invisible".

"Ex." $M = \mathbb{R}^m$, $U$ nbhd of $0$. $\forall f \in \mathcal{C}^\infty(\mathbb{R}^m)$,

\[
(T_p i)(\frac{\partial}{\partial x_i}) \cdot f = \frac{\partial}{\partial y_i} (f \circ \varphi) = \frac{\partial}{\partial x_i} (f).
\]

So $(T_p i)(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i}$.

"Ex." $\varphi : U \to \mathbb{R}^m$ is a coordinate chart on a manifold $M$, $p \in U$.

We define $\frac{\partial}{\partial x_i} \big|_p \in T_p U$ by

\[
\frac{\partial}{\partial x_i} \big|_p \cdot h = \frac{\partial}{\partial y_i} (h \circ \varphi^{-1}).
\]

Then $(T_p i)(\frac{\partial}{\partial x_i} \big|_p) f = \frac{\partial}{\partial x_i} \big|_p (f \circ \varphi^{-1}) = \frac{\partial}{\partial y_i} (f \circ \varphi^{-1})$

\[
= \frac{\partial}{\partial y_i} (f \circ \varphi^{-1}) = \frac{\partial}{\partial x_i} \big|_p (f)
\]

$\Rightarrow \{ \frac{\partial}{\partial x_i} \big|_p \} \subseteq$ a basis of $T_p U \subseteq \{ \frac{\partial}{\partial y_i} \big|_p \} \subseteq$ a basis of $T_p M$.

Note \( (T_p \varphi) (\frac{\partial}{\partial x_i} \big|_p) = \frac{\partial}{\partial y_i} \big|_p \), \( T_p \varphi : T_p U \to T_0 (\varphi(U)) \)

Reason: $\forall h \in \mathcal{C}^\infty(\varphi(U))$

\[
((T_p \varphi) (\frac{\partial}{\partial x_i} \big|_p)) h = \frac{\partial}{\partial x_i} \big|_p (h \circ \varphi) \quad \text{(def of } T_p \varphi) \]

\[
= \frac{\partial}{\partial y_i} ((h \circ \varphi) \circ \varphi^{-1}) \quad \text{(def of } \frac{\partial}{\partial y_i} \big|_p \}
\]

\[
= \frac{\partial}{\partial y_i} \big|_p \cdot h
\]

Since $\varphi : U \to \varphi(U)$ is a diffeo, $T_p \varphi : T_p U \to T_0 (\varphi(U))$

is an iso. $\Rightarrow \frac{\partial}{\partial y_i} \big|_p \subseteq$ a basis of $T_0 (\varphi(U))$

\[
\Rightarrow \frac{\partial}{\partial x_i} \big|_p \subseteq \text{a basis of } T_p U \subseteq T_p M.
\]