Last time: Prove that $\tilde{H}_{\cdot}^k(\mathbb{R}^n) = \{0\}$ for $k \neq n$.

Consider $\tilde{S}^n: H^k_c(\mathbb{R}^n) \to \mathbb{R}$, $[\sigma] \mapsto \int \tau\cdot \sigma$

a well-defined isomorphism.

**Remark 41.0**

$B_{\mathbb{R}}^n(0) \subset \mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^n$.

$\Rightarrow \tilde{S}^n: H^k_c(B_{\mathbb{R}}^n(0)) \to \mathbb{R}$

$[\sigma] \mapsto \int\sigma$ is a well-defined iso.

In particular, for $\omega \in \Omega^k_c(B_{\mathbb{R}}^n(0))$, $0 = \int\omega \Rightarrow \omega = dy$ for some $y \in \Omega^{n-1}_c(B_{\mathbb{R}}^n(0))$.

$\Rightarrow \int\omega = 0 \in H^k_c(B_{\mathbb{R}}^n(0))$.

**Goal for today**: Let $M$ be a connected orientable manifold. Then $H^k_c(M) = \{0\}$ where $m = \dim M$.

We first prove

**Lemma 41.1** Suppose $M$ is a connected orientable manifold, $U, V \subset M$ two open sets, which are diffeomorphic to open balls. Then $\forall \omega \in \Omega^k_c(U)$

$\exists \omega' \in \Omega^k_c(V)$

$s.t. \int\omega = \int\omega'$ in $H^k_c(M)$ ($m = \dim M$).

**Proof** (i) Choose an orientation of $M$. Suppose $U \cap V \neq \emptyset$.

Let $c = \int\omega$.

Then $\exists W \subset U \cap V$ so that $W$ is diffeomorphic to $B_{\mathbb{R}}^n(0) \subset \mathbb{R}^n$ (for some $n$).

Let $\gamma: W \to B_{\mathbb{R}}^n(0)$ be the diffeomorphism. Choose $f \in C^\infty_c(B_{\mathbb{R}}^n(0))$ so that $f = c$.

Let $\gamma' = \gamma^*(f dr_1 - \cdots \cdots dr_m)$

$f_{\mathbb{R}}(0)$

Let $\omega' = \gamma^*(f \omega - f \omega')$.

Then $\int\omega' = \int\omega' = \int f = c$.

$\Rightarrow \int\omega - \int\omega' = \int(f\omega - f\omega') = 0$.

By 41.0 again,

$\exists \xi \in \Omega^{k-1}_c(V) \subset \Omega^{k-1}_c(M) \Rightarrow \omega - \omega' = d\xi$. 

41.1
Also suppose $w \in W \subset U$. So $w \in \Omega^m_c(U)$.

(3) Now suppose $UNV = \emptyset$. Pick $p \in V$, $q \in U$. Since $M$ is connected, it's path connected. There exists a continuous path $\gamma : [0,1] \to M$ with $\gamma(0) = p$, $\gamma(1) = q$. Since $\gamma([0,1])$ is compact, there exists $W_1, \ldots, W_h \in M$ open such that $W_i \cap W_j = \emptyset$ for $i \neq j$. $W_1 = V$, $W_h = U$ and $W_i \cap W_j = \emptyset$.

By (1) above, $\exists w^{(1)} \in \Omega^m_c(W_1)$ such that $[w] = [w^{(1)}] \in H^m_c(M)$.

Given $w^{(1)} \in \Omega^m_c(W_1)$ and $w^{(2)} \in \Omega^m_c(W_2)$, let $\{w^{(0)}, w^{(2)}\} = [w^{(3)}] \in H^m_c(M)$.

$\vdots$

Given $w^{(k)} \in \Omega^m_c(W_k)$, let $\{w^{(k-1)}, w^{(k)}\} = [w^{(k+1)}] \in H^m_c(M)$.

Thus let $M$ be a connected orientable manifold. Then $H^m_c(M) \cong \mathbb{R}$ where $m = \dim M$.

Proof. Fix $U \subset M$ open. Let $U$ be diffeomorphic to a ball.

We have a canonical inclusion $\Omega^m_c(U) \to \Omega^m_c(M)$, which is a map of cochain complexes, hence induces

$\iota_* : H^m_c(U) \to H^m_c(M)$

We argue that $\iota_* : H^m_c(U) \to H^m_c(M)$ is an isomorphism.

(Remember: $H^m_c(U) \cong \mathbb{R}$ since $U$ is diffeomorphic to a ball).

If $w = d\eta \in \Omega^m_c(U)$ and $0 = \iota_*(lw)$, then $w = d\eta$ for some $\eta \in \Omega^{m-1}_c(M)$.

Fix an orientation of $M$. Then

$\int_M w = \int_M (d\eta) = \int_M \eta = 0$ since $\partial M = \emptyset$.

On the other hand, $\int_M w = \int_M w$ and $\int_M w = 0$ implies that $\iota_* : H^m_c(U) \to H^m_c(M)$ is injective.

We now argue that $\iota_* : H^m_c(U) \to H^m_c(M)$ is onto.
Since \( \text{supp}\, \omega \) is compact \( \exists V_i , \ldots , V_k \subset M \) open so that \( V_i \) is open ball \( U_i \) and \( \text{supp}\, \omega \subset U_i \) for \( i = 1, \ldots , k \).

Consider the cover \( (M, \text{supp}\, \omega) \), \( V_i \), \( i \leq k \) of \( M \).

Choose a partition of unity \((\varphi_0 , \ldots , \varphi_k)\) subordinate to the cover. Then \( \text{supp}\, \varphi_i \subset V_i \) for \( i = 1, \ldots , k \)

\[ \sum_{i=0}^{k} \varphi_i \omega = 1 \text{ in } H^m_c(M) \]

By 41.1, \( \exists \Sigma \in \bigwedge^m_c(U) \) s.t. \( \{ \Sigma \} \subset [\varphi_i \omega] \text{ in } H^m_c(M) \)

\[ \{ \Sigma \} = \sum_{i=0}^{k} \{ \varphi_i \omega \} = \sum_{i=0}^{k} \varphi_i \omega = \omega \text{ in } H^m_c(M) \]

\( \{ \omega \} = i^* \{ \Sigma \} \)

**Definition** A continuous map \( f : X \rightarrow Y \) between two topological spaces is proper if \( V \subset Y \) compact, \( f^{-1}(V) \) is compact.

**Ex** Any homeomorphism \( f : X \rightarrow Y \) is proper. This is because \( f^{-1}(V) = g^{-1}(V) \) where \( g = f^{-1} \)

and images of compact sets under continuous maps are compact.

**Ex** Any continuous map between compact Hausdorff spaces is proper: if \( X, Y \) compact, \( f : X \rightarrow Y \) continuous and \( K \subset Y \) compact, then \( K \) is closed \( \rightarrow f^{-1}(K) \) is closed \( \rightarrow f^{-1}(K) \) is compact.

**Lemma 41.2**
A proper map \( f : M \rightarrow N \) between two manifolds gives rise to \( f^* : H^*_c(N) \rightarrow H^*_c(M) \)

**Proof** Enough to check: if \( \omega \in \bigwedge^m_c(N) \) and \( \text{supp}\, \omega \) is compact then \( \text{supp}\, (f^*\omega) \) is compact as well.

Now \( \text{supp}\, (f^*\omega) = \{ x \in M \mid (f^*\omega)_x \neq 0 \} = \{ x \in M \mid \omega_{f(x)} \neq 0 \} \)
Now, \( \forall u \in U \), \( f^{-1}(u) \subseteq f^{-1}(\overline{u}) \) since \( f \) is continuous. 

\[ \Rightarrow \text{supp}(f^*w) = f^{-1}(\{ y \in U \mid wy \neq 0 \}) \subseteq f^{-1}(\{ y \in U \mid wy \neq 0 \}) = f^{-1}(\text{supp}w) \]

Since \( \text{supp}w \) is compact & \( f \) is proper, \( f^{-1}(\text{supp}w) \) is compact.

\( \text{supp}(f^*w) \) is closed and is contained in \( f^{-1}(\text{supp}w) \), which is compact.

\[ \Rightarrow \text{supp}(f^*w) \text{ is compact} \]

We'll use this fact to define degrees of proper maps.