Last time, if $D \subset M$ is a regular domain, then $\int_{\partial D} \omega$ exists, and $\partial D$ is an orientable surface.

2. If $\omega = \omega_{\text{ext}(M)}$ is a volume form, $D \subset M$ regular domain and $\mathbf{v}$ is a vector field near $\partial D$ that points out of $D$, then

$$\int_{\partial D} \mathbf{v} \cdot (\partial_{\mathbf{n}} \omega) = \int_D \mathbf{v} \cdot \mathbf{v} \, dV$$

is a volume form on $\partial D$.

Started proving Stokes' theorem: $M$ orientable, $D \subset M$ regular domain, $\omega \in \Omega^{n-1}(M)$. Then

$$\oint_{\partial D} \omega = \int_D \text{d} \omega.$$

Special case: $M = \mathbb{R}^n$, $\omega = dx_1 \wedge \ldots \wedge dx_n$, $D = \{(x_1, \ldots, x_n) : x_1 < 0\}$ oriented by $dx_1 \wedge \ldots \wedge dx_n$, $\omega = \sum_{j=1}^n (-1)^{j-1} f_j \, dx_1 \wedge \ldots \wedge \hat{dx}_j \wedge \ldots \wedge dx_n$.

$R > 0$, supp $f_j \subset (-R, R)^n$. Then

$$\omega = \sum_{j=1}^n \frac{\partial f_j}{\partial x_j} \, dx_1 \wedge \ldots \wedge dx_n$$

and

$$\int_{\partial D} \omega = \sum_{j=1}^n \frac{\partial f_j}{\partial x_j} \int_{x_1 = 0}^{x_1 = R} \int_{x_2 \geq -R} \ldots \int_{x_n \geq -R} dV.$$
1) \( \bar{\bigcup_{j \in J}} U_j \supset \text{supp}\omega \)

2) \( U_j \)'s are connected

3) \( \psi_j : U_j \to \mathbb{R}^n \) are adapted to \( D \).

Note: \( \psi_j^*(d\text{rm}_U \wedge d\text{rm}_M) = f_j \mu \) for some \( f_j \in C^\infty(U) \)

Since \( U \) is connected, either \( f_j > 0 \) and then we set \( \text{sign} \psi_j = +1 \)
or \( f_j < 0 \) and then we set \( \text{sign} \psi_j = -1 \).

Let \( U_0 = M \setminus \text{supp}\omega \). Then \( \{U_0, U_1, \ldots, U_N\} \) covers \( M \).

\[ \Rightarrow \] a partition of \( 1 \) \( \left( \psi_j \right)_{j=0}^N \) with \( \text{supp} \psi_j = U_j \).

\[ \Rightarrow \left( \sum_{j=1}^N \psi_j \right) |_{\text{supp}\omega} = 1 \Rightarrow \omega = \sum_{j=1}^N \psi_j \omega \]

\[ \Rightarrow d\omega = \sum_{j=1}^N d(\psi_j \omega) \text{. Note: } \text{supp}(d\psi_j \omega) \subset U_j \text{ and} \]

\[ \int_{\text{supp}\omega} d(\psi_j \omega) = (\text{sign} \psi_j) \int_{U_j} \psi_j^*(d\omega) \text{.} \]

Now \( \int_D \omega = \sum_{j=1}^N \int_{D \cap U_j} d(\psi_j \omega) = \sum_{j=1}^N (\text{sign} \psi_j) \int_{U_j} \psi_j^*(d\omega) \text{.} \)

Recall \( \psi_j(D \cap U_j) = \chi \leq 0 \text{ and } \psi_j(U_j) = v_j \).

\[ \int_{\text{supp}\omega} \psi_j^*(d\omega) = \int_{\chi \leq 0 \cap \psi_j(U_j)} d(\psi_j^\ast \omega) \]

\[ \Rightarrow \int_D \omega = \sum_{j=1}^N \int_{U_j \cap \partial D} (\psi_j \omega) \text{d}D = \int_{\partial D} (\sum_{j=1}^N \psi_j \omega) \text{d}D = \int_{\partial D} \omega \text{d}D. \]
Corollary: Suppose $M$ is a compact oriented manifold, $\omega \in \Omega^{\dim M - 2}(M) = \Omega^{\dim M - 1}(M)$. Then
\[ \int_M \omega = 0. \]

Proof: Let $D = M$. Then $\partial D = \emptyset$. Stokes' theorem \Rightarrow
\[ \int_M \omega = -\int_{\partial M} \omega = 0. \]

Divergence: Let $M$ be an orientable manifold, $\mu \in \Omega^{\dim M}(M)$ a volume form, and $X$ a vector field on $M$.
\[ \Rightarrow \forall q \in M, \quad \mu^{\dim M - 1}(T_q M) = 1, \]
\[ \Rightarrow (L_X \mu)_q = f(x) \mu_q \quad \text{for some} \quad f \in C^\infty(M). \]

Definition: The divergence of $X$ with respect to $\mu$ is the function $\text{div}_\mu(X) \in C^\infty(M)$ so that
\[ L_X \mu = \text{div}_\mu(X) \mu. \]

Divergence Theorem: Let $M$ be an orientable manifold, $\mu \in \Omega^{\dim M}(M)$ a volume form, $D \subseteq M$ a compact regular domain. Then for any vector field $X$
\[ \int_D \text{div}_\mu(X) \mu = \int_{\partial D} \mu \quad (= \int_D (L_X \mu) - \int_D). \]

Proof: By Cartan's formula,
\[ \text{div}_\mu(X) \mu = L_X \mu = d \mu(X) + i(X) d \mu = d \mu(X). \]

Stokes' theorem:
\[ \int_D \text{div}_\mu(X) \mu = \int_D d \mu(X) = \int_{\partial D} \mu. \]

De Rham cohomology
Let $\mathcal{M}$ be a manifold. A $k$-form $\omega \in \Omega^k(\mathcal{M})$ is closed if $d\omega = 0$. A $k$-form $\beta$ is exact if $\beta = d\gamma$ for some $(k-1)$-form $\gamma$.

Note: exact $k$-forms $\subseteq$ closed $k$-forms. This is because $d(d\gamma) = 0 \quad \forall \gamma \in \Omega^{k-1}(\mathcal{M})$

**Definition:** The $k^{th}$ de Rham cohomology of a manifold $\mathcal{M}$ in the vector space

$$H^k(\mathcal{M}) = \frac{\text{(closed $k$-forms on $\mathcal{M}$)}}{\text{(exact $k$-forms on $\mathcal{M}$)}}$$

$$= \frac{\ker(d : \Omega^k(\mathcal{M}) \to \Omega^{k+1}(\mathcal{M}))}{\text{im}(d : \Omega^{k-1}(\mathcal{M}) \to \Omega^k(\mathcal{M}))}$$

We'll prove: if $\mathcal{M}$ is compact, $\dim H^k(\mathcal{M}) < \infty$.

Note: By definition $\Omega^0(\mathcal{M}) = \mathbb{R}$. Hence

$$H^0(\mathcal{M}) = \{ f \in \Omega^0(\mathcal{M}) \mid df = 0 \} / \mathbb{R} = \{ f \in \Omega^0(\mathcal{M}) \mid df = 0 \}$$

Note: if $f \in C^\infty(\mathbb{R}^m)$ and $df = 0$, then $\frac{df}{dx^i} = 0 \quad \forall i$

$\implies f$ is constant.

Consequently

$$H^0(\mathcal{M}) = \text{the space of locally constant functions on }\mathcal{M}.$$ 

$$\text{Ex: } M = (0,1) \cup (2,3) \quad H^0(\mathcal{M}) = \{ f \in C^\infty((0,1) \cup (2,3)) \mid f \equiv 0 \}$$

$$= \{ f(x) = c_1 x \in (0,1) \} \cup \{ f(x) = c_2 x \in (2,3) \} \text{ s.t. } f(1) = f(3)$$

$\implies H^0(\mathcal{M}) = 1^\mathbb{R}_0(\mathcal{M}) \quad \mathcal{M}_0(\mathcal{M}) = \text{set of connected components of }\mathcal{M}$

Next time: Tools for computing $H^k(\mathcal{M}) = \bigoplus_{i=0}^\infty H^k(\mathcal{M})$. 