Recall a functor $F: C \to D$ assigns to each object $c$ of $C$ an object $F(c)$ of $D$, to each morphism $c \to c'$ of $C$ a morphism $F(c) \xrightarrow{F(f)} F(c')$ of $D$ so that

1. $\forall c \in C \enspace F(id_c) = id_{F(c)}$
2. $\forall c \xrightarrow{\delta c'} c'' \in C, \enspace F(\delta) \circ F(\delta) = F(\delta \circ \delta)$

II. $\text{Vect}_{\text{top}} = \text{category of finite dim vector spaces and linear iso morphisms}$ $F: (\text{Vect}_{\text{top}})^n \to \text{Vect}_{\text{top}} \in \text{C}^\infty$ (it smooth) if

$\forall (V_i, V_n) \in (\text{Vect}_{\text{top}})^n$

$F: \text{Hom}( (V_i, V_n), (V_i, V_n)) \to \text{Hom} (F(V_i, V_n), F(V_i, V_n))$

$GL(V_i) \times \cdots \times GL(V_n)$

and $F \in \text{C}^\infty$.

**Theorem 30.** Let $M$ be a manifold, $E_1 \to M$, \ldots, $E_n \to M$ vector bundles, $\{U_{i_1}, \ldots, U_{i_n}\}$ an open cover of $M$ s.t. $E_{i_j}|_{U_{i_j}}$ is trivial for $i_j = 1, \ldots, n$ (this can be arranged).

$\{\varphi_{i_1}: U_{i_1} \to GL(\mathbb{R}^{k_1}) \mid i_1 = 1, \ldots, n\}$

The corresponding Čech cocycles and $F: (\text{Vect}_{\text{top}})^n \to \text{Vect}_{\text{top}}$ a $\text{C}^\infty$ functor. Then $\tilde{F}(U_{i_1}) : U_{i_1} \to GL(F(\mathbb{R}^{k_1}, \mathbb{R}^{k_n}))$

defined by

$\tilde{F}(U_{i_1}) = F(\varphi_{i_1}, \ldots, \varphi_{i_n})$ \text{ for } $U_{i_1}$ \text{ a } Čech cocycle, hence defines a vector bundle $F(E_1, \ldots, E_n) \to M$.

Moreover, $\forall q \in M$, $(F(E_1, \ldots, E_n))_q = F(E_{i_1}, \ldots, E_{i_n})_q$

**Proof.** To keep the notation manageable assume $n = 2$.

Then $U_{i_1}, U_{i_2} \in U_{i_1}$

$\tilde{F}(U_{i_1}) = F(\varphi_{i_1}, \varphi_{i_2}) \overset{F}{=} F(id_{\mathbb{R}^{k_1}}, id_{\mathbb{R}^{k_2}}) = id_{F(\mathbb{R}^{k_1}, \mathbb{R}^{k_2})}$
\[ T_{ap}(T) \circ T_{bp}(T) = F(\varphi_{ap}^{(1)}, \varphi_{ap}^{(2)}) \circ F(\varphi_{bp}^{(1)}, \varphi_{bp}^{(2)}) \]
\[ = F(\varphi_{ap}^{(1)} \circ \varphi_{bp}^{(1)}, \varphi_{ap}^{(2)} \circ \varphi_{bp}^{(2)}) \]
\[ = F(\text{id}_{R^k_1}, \text{id}_{R^k_1}) = \text{id}_F(R^k_1, R^k_2) \]

Similarly, for any \( \alpha, \beta, \gamma, \forall q \in \mathbb{R}_p \pi 
\[ T_{ap}(T) \circ T_{\beta q}(T) \circ T_{\gamma q}(T) = \text{id}_F(R^k_1, R^k_2). \]

**Next goal: Stokes' theorem**

Recall, the Fundamental theorem of calculus says:
\[ \forall f \in C^1([a,b]) \]
\[ \int_a^b f'(x) \, dx = f(b) - f(a). \]

We can rewrite this as
\[ \int_{[a,b]} \frac{df}{dx} \, dx = \int_{\partial([a,b])} f \, ds \]

This holds for an oriented 1-dimensional manifold in 1-dimensional oriented manifold with boundary.

**Green's theorem**
\[ D \subseteq \mathbb{R}^2 \text{ domain with (smooth) boundary } \partial D, \]
\[ \int_D P(x,y) \, dx + Q(x,y) \, dy = \int_{\partial D} (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) \, ds \]

Set \( \alpha = P \, dx + Q \, dy \), \( da = dx + dy \)

Green's theorem then becomes
\[ \int_D \alpha = \int_{\partial D} da \]

*(Generalized) Stokes theorem* says:

Let \( M \) be an oriented manifold, \( D \subseteq M \) a domain with \( C^\infty \) boundary \( \partial D \). Then, for \( \omega \in \Omega^p(M) \)
\[ \int_D \omega = \int_{\partial D} \omega \] provided \( \partial D \) is oriented correctly.
We need to make sense of $d_0$. To this end we construct a sequence of linear maps $d^i_m: \Omega^i(M) \to \Omega^{i+1}(M)$ called the exterior derivative.

**Notation** $\Omega^i(M) = \bigoplus_{i=0}^{\infty} \Omega^i(M)$, $d^i_m: \Omega^i(M) \to \Omega^{i+1}(M)$

where $d^i_m = \bigoplus_{i=0}^{\infty} d^i_m$.

**Theorem** For any manifold $M$ there is a unique $R$-linear map $d^i_m: \Omega^i(M) \to \Omega^{i+1}(M)$ called the exterior derivative so that

1. $\forall f \in \Omega^0(M) = C^0(M)$, $d^i_m f = df$
2. $\forall U \in M$ open $\forall \omega \in \Omega^i(M)$, $(d^i_m \omega)|_U = d^i m (\omega|_U)$
3. $\forall \omega \in \Omega^k(M), \forall \eta \in \Omega^l(M)$
   
   $d^i_m (\eta \wedge \omega) = (d^i_m \eta) \wedge \omega + (-1)^k \eta \wedge (d^i m \omega)$

4. $d^i_m (d^i m \omega) = 0 \quad \forall \omega \in \Omega^i(M)$.

**Remark** Once we prove existence and uniqueness, we'll write $d$ for $d^i_m$.

**Proof (uniqueness)** Suppose $\forall M, \exists d^i m: \Omega^i(M) \to \Omega^{i+1}(M)$ satisfying (1)-(4) above.

Fix a manifold $M$. Pick a coordinate chart $(x_1^1, \ldots, x_n^1): U \to \mathbb{R}^n$.

Then $\forall \omega, \forall \omega \in \Omega^k(M)$

$\omega = \sum_{|I|=k} a^I dx^I = \sum a^I_{i_1, \ldots, i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$,

where $a^I_{i_1, \ldots, i_k} \in C^0(U)$.

**Claim** $d^i_m (dx^I) = 0$.

**Proof** Induction on $k = |I|$. If $k = 1$, \( \forall i \)

$d^i_m (dx_i) = d^i m (d^1 m x_i) = 0$, by (4).

Suppose claim holds for all $I$ with $|I| < n$. Then

$d^i_m (dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = d^i m (dx_{i_1} \wedge (dx_{i_{k+1}} \wedge \cdots \wedge dx_{i_{k+n}}))$

$+ (-1)^k dx_{i_1} \wedge d^i m (dx_{i_{k+1}} \wedge \cdots \wedge dx_{i_{k+n}}) = 0$. 

$\in C^0(U)$.
\[ (a, b) \times (c, d) = (ac - bd, ad + bc), \] where \( k = 1 \).

Then, by \( \lambda (a, b) = (\lambda a, \lambda b) \) and \( (a + b) \times (c + d) = (ac + ad + bc + bd, bc + bd + ad + ab) \) \( \times \) (1) holds.

(1) holds by checking (a) in a computation. Since \( da \) is \( \mathbb{R} \)-linear and \( a \) is

\[ d(x_1) = d(\exp_1) = d(\exp_1(1, 0)) = \sqrt{2} d(\exp_1(1, 0)) = 1 + \sqrt{2}, \]

Then \( b = \sqrt{2} \) \( \mathbb{C} \)-linear and \( a \) is

\[ d(\alpha) = (\alpha, d\alpha), \] where \( \alpha \) is

Since we use a\( a \) arbitrary, \( d\alpha = d\alpha \), and \( 25 \) of (1) holds.

Thus prove uniqueness of \( d \).

Now \( d(\alpha) = (\alpha, d\alpha) \) is another map with properties (1)-(4).

Therefore, \( d(x_1) = d(\exp_1(1, 0)) = \frac{1}{2} d(\exp_1(1, 0)) = \frac{1}{2} (1 + \sqrt{2}). \)