Last time: Example of a non-Hausdorff manifold:

- $\mathbb{RP}^n$ and $\pi: \mathbb{R}^{n+1}_+ - \rightarrow \mathbb{RP}^n \in C^\infty$ (smooth)
- notion of a diffeomorphism.

Digression (Connected and path connected)

"Recall" A topological space $X$ is path-connected if $\forall x_0, x_1 \in X$

exists continuous map $\gamma: [0, 1] \rightarrow X$ (a path) s.t.

$\gamma(0) = x_0$ and $\gamma(1) = x_1$

A topological space $X$ is connected if $\forall U, V \subseteq X$ open

with $U \cup V = X$, $U \cap V = \emptyset$

either $U = \emptyset$ or $V = \emptyset$.

"$X$ can't be written as a disjoint union of two non-empty open sets."

Fact $[0, 1]$ is connected.

Consequence path connected spaces are connected

Fact For manifolds "path connected" = "connected"

Lemma 3.1 Let $\phi_a: U_a \rightarrow \mathbb{R}^m$, $a \in A$ be an atlas on a connected

manifold $M$. Then $\forall a, b \in A$ $\phi_a = \phi_b$.

We call $m = m_M$ (as $A$) the dimension of the manifold $M$.

Why is Lemma 3.1 true? Suppose $\phi_a: U_a \rightarrow \mathbb{R}^m$, $\phi_b: U_b \rightarrow \mathbb{R}^m$

are two coordinate charts on $M$ and $U_a \cap U_b \neq \emptyset$.

We then have a $C^\infty$ map

$$ f = \phi_a \circ \phi_b {^{-1}} \mid_{\phi_b(U_a \cap U_b)} : \phi_b(U_a \cap U_b) \rightarrow \phi_a(U_a \cap U_b) $$

It has a $C^\infty$ inverse

$$ g = \phi_b \circ \phi_a {^{-1}} \mid_{\phi_a(U_a \cap U_b)} : \phi_a(U_a \cap U_b) \rightarrow \phi_b(U_a \cap U_b). $$
\[
\begin{align*}
D(g \circ f) &= \frac{\partial g}{\partial y} \cdot \frac{\partial f}{\partial x} \quad \text{in } \mathbb{R}^n \\
\text{Chain rule } \Rightarrow \\
\text{Id} &= D(g \circ f)(x) = Dg(y) \circ Df(x) : \mathbb{R}^{m_2} \to \mathbb{R}^{m_2} \\
\text{Id} &= D(f \circ g)(y) = Df(x) \circ Dg(y) : \mathbb{R}^{m_3} \to \mathbb{R}^{m_3} \\
\Rightarrow Df(x) : \mathbb{R}^{m_2} \to \mathbb{R}^{m_3} \text{ is an invertible linear map} \\
\Rightarrow m_2 = m_3.
\end{align*}
\]

In general, given \( x_0, x_1 \in M \) pick a path \( \gamma : (0, 1) \to M \) and cover \( M \) with finitely many coord charts \( \phi_0 : U_0 \to \mathbb{R}^{m_0}, \phi_1 : U_1 \to \mathbb{R}^{m_1}, \ldots, \phi_k : U_k \to \mathbb{R}^{m_k} \)

\[ \Rightarrow m_{0_0} = m_{0_1} = \ldots = m_{0_k}. \]

**Products of manifolds**

Recall the product topology.

Let \( (X, T_X), (Y, T_Y) \) be two topological spaces. Their product is the set \( X \times Y \) with the product topology \( T_{X \times Y} \).

\( T_{X \times Y} \) is the smallest topology so that the two projections \( p_1 : X \times Y \to X \), \( p_2 : X \times Y \to Y \)

\[ (x, y) \mapsto (x, y) \]

are continuous. The condition translates into:

\[ W \subseteq X \times Y \text{ is open } \Leftrightarrow \forall (x, y) \in W \\
\exists U \subseteq X, V \subseteq Y \text{ open} \\
\quad (x, y) \in U \times V \subseteq W \]
Now suppose \((M, \phi_m: U_m \to \mathbb{R}^m), (N, \psi_n: V_n \to \mathbb{R}^n)\) are two manifolds. We make their product \(M \times N\) into a manifold as follows:

1. \(U_m \times V_n \{(x, y) \in U \times V \mid \phi_m(x) \cap \psi_n(y) \neq \emptyset\}\) covers \(M \times N\).
2. \(\phi_m \times \psi_n: U_m \times V_n \to \mathbb{R}^m \times \mathbb{R}^n \{(x, y) \in U \times V \mid \phi_m(x) \cap \psi_n(y) \neq \emptyset\}\) is an atlas on \(M \times N\).

The two projections \(p_1: M \times N \to M, p_2: M \times N \to N\) are smooth.

(universal property) For any manifold \(Q\), a map \(f: Q \to M \times N, f_Q = (f_1, f_2)(Q)\) is \(C^0\) if \(f_1: Q \to M, f_2: Q \to N\) are \(C^0\).

Proof: Exercise.

Facts: For any two smooth maps \(f: Q \to M, g: M \to N\) on manifolds, \(g \circ f: Q \to N\) is smooth.

A manifold \(M, id_M: M \to M\) is smooth. Consequently, manifolds and smooth maps form a category.

Proof: Exercise.

Notation: Let \(M\) be a manifold.

\(C^0(M) = \{ f: M \to \mathbb{R} \mid f \text{ is smooth} \} \).

Easy to see:

1. \(C^0(M)\) is a vector space \(\mathbb{R}\) with the two operations defined "point-wise":

\[
\forall x \in \mathbb{R} \quad \forall f, g \in C^0(M) \quad (\lambda f)(x) := \lambda f(x)
\]

\[
\forall f, g \in C^0(M) \quad (f + g)(x) := f(x) + g(x)
\]

2. The map \(C^0(M) \times C^0(M) \to C^0(M), (f, g) \mapsto f \cdot g\)
\[(f,g)(x) = f(x) \cdot g(x) \quad \forall x \in \mathbb{R}\]

**Linear**

\[\Rightarrow C^0(M) \text{ is a commutative algebra/IR.}\]

We'll need this fact when we define tangent vectors and vector fields.

Next topic: paracompactness and partitions of unity.

**Definition** Let \( X \) be a topological space. An open cover of \( X \) is a collection of open sets \( \{U_d\}_{d \in D} \) s.t. \( \bigcup U_d = X \).

An open cover is locally finite if for each \( x \in X \) there exists an open neighborhood \( W \) of \( x \) in \( X \) (or \( W \subseteq X \) is open and \( x \in W \)) s.t.

\[U_d \cap W = \emptyset \quad \text{for all but finitely many } d,\]

and

\[\mathbb{R} \setminus \left( \left[ n, n + 2 \right] \cup \left[ -n, n \right] \right) \text{ is a locally finite cover of } \mathbb{R}.\]

A locally finite cover of \( \mathbb{R} \) is not a locally finite cover of \( \mathbb{R} \).

**Definition** An open cover \( \{V_d\}_{d \in D} \) is a refinement of an open cover \( \{U_d\}_{d \in D} \) of a top space \( X \) if for all \( d \in D \), s.t. \( V_d \subseteq U_d \).

**Definition** A Hausdorff topological space \( X \) is paracompact if every open cover has a locally finite refinement.

Compare with: A Hausdorff topological space \( X \) is compact if every open cover \( \{U_d\}_{d \in D} \) has a finite subcover; i.e.,

\[\exists \alpha \in \mathbb{N}, \exists i_d : 1 \leq i_d \leq \alpha \text{ s.t. } \bigcup_{d \in D} U_{i_d} \text{ is a cover of } X.\]