Last time Defined vector bundles, maps of vector bundles and spaces of sections \( \Gamma(E) \) of a vector bundle \( E \rightarrow M \).

Sketched a proof that if \( f \in C^0(M) \), \( f \in \Gamma(E) \), \( f \in \Gamma(E) \).

Time before last: Constructed \( s_m: \Omega^d_{\ast}(M) \rightarrow \mathbb{R} \)

provided \( M \) is oriented.

**Fact:** if \( N \subseteq M \) a submanifold with \( \dim M - \dim N > 0 \)
then \( \omega = \Omega^d_{\ast}(M) \), \( S^m \omega = \int_{M \cap N} \omega \)

For a different perspective see Lee, Prop 16.8

**Note:** if \( M \) is a (compact) embedded submanifold of a manifold \( N \)
and \( \omega \in \Omega^d_{\ast}(N) \), one writes \( i^\ast \omega \) for \( \int_M i^\ast \omega \)
where \( i: M \rightarrow N \) is the embedding/inclusion.

\[
\text{Ex: } \int_{S^1} \left( -\frac{y}{x^2 + y^2} \, dx + \frac{x}{x^2 + y^2} \, dy \right) = ? \quad \text{Here } S^1 = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \}
\]

Consider \( u: (0, 2\pi) \rightarrow S^1 \), \( u(\theta) = (\cos \theta, \sin \theta) \)

\[
\begin{align*}
\varphi((0, 2\pi)) &= S^1 \setminus \{(1,0)\} \quad \text{and} \quad S^1 = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \} \\
\int_{S^1} -\frac{y}{x^2 + y^2} \, dx + \frac{x}{x^2 + y^2} \, dy &= \int_0^{2\pi} -\frac{\sin \theta}{\cos^2 \theta + \sin^2 \theta} \, d(\cos \theta) + \frac{\cos \theta}{\cos^2 \theta + \sin^2 \theta} \, d\sin \theta \\
&= \int_0^{2\pi} \sin^2 \theta \, d\theta + \cos \theta \, d\theta = \int_0^{2\pi} \cos \theta \, d\theta = 2\pi
\end{align*}
\]

**Note** \( \theta = \tan^{-1}\left( \frac{y}{x} \right) \) which is defined on \( \mathbb{R}^2 \setminus \{(0,0)\} \).

But \( d\theta = d\tan^{-1}\left( \frac{y}{x} \right) = -\frac{y}{x^2 + y^2} \, dx + \frac{x}{x^2 + y^2} \, dy \) is defined on \( \mathbb{R}^2 \setminus \{(0,0)\} \).

**Definition.** Let \( E \rightarrow M \) be a vector bundle. A local section of \( p \)
in an open subset \( U \subseteq M \) and a \( C^0 \) map \( s: U \rightarrow E \) so that \( p \circ s = id_U \).
Remarks
1. Suppose \( V \to E \to M \) is a vector bundle, \( \{ U_x \}_{x \in A} \) an open cover of \( M \) and \( \{ s_x : U_x \to E \}_{x \in A} \) are local sections.

Given a partition of unity \( \{ \rho_x \}_{x \in B} \) s.t. \( \text{supp} \rho_x \subseteq U_x(x) \)
\[ S := \sum \rho_x s_x \text{ in a global section of } E \to M \]

2. Since \( x \in M \), \( \forall \text{ local } \} \{ \rho_x \} \) \( \text{ of } \ U \) \( \in \text{ E } \{ \ U \times V \} \)
and \( f : (U \times V) \to C^0(U, V) \) there are lots of local sections. \( \Rightarrow f(E) \) is nonempty.

Since \( f(E) \) is a module over \( C^0(M) \), it's an m.f. dim vector space over \( \mathbb{R} \).

Definition Let \( E \to M \) be a vector bundle. For any \( q, q_2 \in M \)
the vector spaces \( E_q, E_{q_2} \) have the same dimension.

We define the rank of \( E \) to be that dimension
\[ \text{rank } E := \dim E_q \text{ for some (hence any) } q \in M. \]

\( \text{rank } E \) is also the dimension of the typical fiber of \( E \).

Note
If \( E \to M \) is a vector bundle of rank \( k \), there is
no loss of generality to assume that the typical fiber
in \( \mathbb{R}^k \). If the typical fiber is \( V \) and \( \dim V = k \), fix
an \( \psi : V \to \mathbb{R}^k \). Then \( E_q \to U \times V = U \times \mathbb{R}^k \) etc.

Lemma 28.1 Let \( M \times \mathbb{R}^k \to M \) be a product bundle (of rank \( k \))
and \( f : M \times \mathbb{R}^k \to M \times \mathbb{R}^k \) an isomorphism of vector bundles.
Then \( f \) is of the form
\[ f(q, v) = (q, g(q)v) \]
where \( g : M \to GL(k, \mathbb{R}) \) is a \( C^0 \) map.

Proof Since \( f \) is a map of vector bundles, \( f \) is of the form
\[ f(q, v) = (q, \psi(q, v)) \]
for some \( C^0 \) map \( \psi : M \times \mathbb{R}^k \to \mathbb{R}^k \).
In particular, for each fixed \( v \in \mathbb{R}^k \), \( q \mapsto \psi(q, v) \) is \( C^0 \).

Let \( (e_1, \ldots, e_k) \) be the standard basis of \( \mathbb{R}^k \). Then the functions \( a_j : M \to \mathbb{R}^k \), \( a_j(q) = \psi(q, e_j) \) are \( C^0 \).

Now \( a_j(q) = \left( \frac{a_{ij}(q)}{a_{ki}(q)} \right) \) where \( a_{ij} : M \to \mathbb{R} \) are \( C^0 \).

\[ \Rightarrow q : M \times \mathbb{R}^k \to \mathbb{R}^k \text{ of the form } q(q, v) = (a_{ij}(q)) \left( \begin{array}{c} v_i \\ v_k \end{array} \right) \]

Moreover, for each \( q \in M \), \( v \mapsto \psi(q, v) \) is an isomorphism of \( \mathbb{R}^k \)

\[ = (a_{ij}(q)) \in \text{GL}(k, \mathbb{R}) \]

\[ \square \]

**Lemma 28.2** Let \( E \xrightarrow{\phi} M \) be a vector bundle of rank \( k \), \( U_x \in X \) an open cover of \( M \) so that \( E|_{U_x} \) is trivial. Let \( U_{x,y} = U_x \cap U_y \), \( U_{x,y,z} = U_x \cap U_y \cap U_z \)

for all \( x, y, z \in X \). There exists a family of \( C^0 \) maps

\[ \psi_{x,y} : U_{x,y} \to \text{GL}(k, \mathbb{R}) \]

so that

1) \( \psi_{x,x}(q) = \text{id} \) \( \forall q \in U_x \)
2) \( \psi_{x,y}(q) \cdot \psi_{y,z}(q) = \psi_{x,z}(q) \) \( \forall q \in U_{x,y} \cap U_{y,z} \)
3) \( \psi_{x,y}(q) \cdot \psi_{y,x}(q) \cdot \psi_{x,z}(q) = \text{id} \) \( \forall q \in U_{x,y} \cap U_{y,x} \cap U_{z,x} \)

Here \( \circ \) is the multiplication in \( \text{GL}(k, \mathbb{R}) \), i.e., composition of maps.

**Definition** A family of maps \( \{ \psi_{x,y} : U_{x,y} \to \text{GL}(k, \mathbb{R}) \} \) satisfying

\( 1 \) - \( 3 \) is called a Čech cocycle with values in \( \text{GL}(k, \mathbb{R}) \)

associated to the cover \( \{ U_x \} \) of \( M \).

**Lemma 28.2** says: vector bundles give rise to Čech cocycles.

A converse is true as well: given a Čech cocycle there is a vector bundle \( E \) over \( M \) whose Čech cocycle \( \psi \) (isomorphic to)

the given one.

Two cocycles \( \{ \psi_{x,y} : U_{x,y} \to \text{GL}(k, \mathbb{R}) \} \) and \( \{ \psi'_{x,y} : U_{x,y} \to \text{GL}(k, \mathbb{R}) \} \)

are isomorphic ("differ by a coboundary") if there is a family of maps \( \{ f_{x,y} : U_x \to \text{GL}(k, \mathbb{R}) \} \) s.t.

\[ \psi_{x,y}(q) = f_{x,y} \psi_{x,y}(q) f_{x,y}(q)^{-1} \quad \forall q \in U_{x,y} \quad \forall x, y, \beta. \]
The reason why cocycles give rise to vector bundles is that vector bundles glue together ("vector bundles satisfy descent"). We'll prove that.

The reason why we want Čech cocycles is that one can operate on them. For example, given a cocycle \((\varphi_{\alpha} : U_{\alpha} \to GL(k, \mathbb{R}))\), we can compose it with the \(C^\infty\) map \( (A^{-1})^* : GL(k, \mathbb{R}) \to GL((\mathbb{R}^k)^*) \)

\[ A \mapsto (A^{-1})^*. \]

We get a new cocycle because \((A \cdot B)^{-1} = (A^{-1})^* (B^{-1})^*\).

If \(E^\alpha\) comes from \(E \to M\), \((E^\alpha)^*\) corresponds to the dual bundle \(E^*\) with \((E^*)_q = (E_q)^* \quad \forall q \in M\).

This is how we constructed \(T^* M \to M\) out of \(TM \to M\).

**Proof of 28.2** Since \(E|_{U_\alpha} \to U_\alpha\) are trivial \(F\) bundles, isomorphisms \(\varphi_{\alpha} : E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^k\). For each \(\alpha, \beta \in A\) consider

\[ \varphi_{\alpha} \circ \varphi_{\beta}^{-1} : U_{\beta} \times \mathbb{R}^k \to U_{\alpha} \times \mathbb{R}^k.\]

By 28.1
\[ (\varphi_{\alpha} \cdot \varphi_{\beta}^{-1})(q, v) = (q, \varphi_{\beta}(q) v) \]

for some \(C^\infty\) map \(\varphi_{\beta} : U_{\beta} \to GL(k, \mathbb{R})\).

Since \(\varphi_{\alpha} \circ \varphi_{\alpha}^{-1} = id\), \(\varphi_{\beta}(q) = id \quad \forall q \in U_{\alpha} + U_{\alpha} = U_\alpha\).

Since \((q, v) = ((\varphi_{\alpha} \circ \varphi_{\beta}^{-1}) \circ (\varphi_{\alpha} \circ \varphi_{\alpha}^{-1}))(q, v) = (q, \varphi_{\beta}(q) \varphi_{\alpha}(q) v)\)

\(\varphi_{\beta}(q) \varphi_{\alpha}(q) = id \quad \forall q \in U_{\alpha} \).

Similarly \(\varphi_{\alpha} \circ \varphi_{\alpha}(q) = id\) \(\forall q \in U_{\alpha}\).