Last time: Given two vector spaces $V, W$ there is a vector space $V \otimes W$ and a bilinear map $\otimes : V \times W \to V \otimes W$ with the following universal property:

For any bilinear map $b : V \times W \to U$ there is a unique linear map $\tilde{b} : V \otimes W \to U$ such that $\tilde{b} \circ \otimes = b$.

**Remark:** $\otimes : V \times W \to V \otimes W$ is bilinear means:

1. $\lambda (v \otimes w) = \lambda v \otimes w$
2. $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$ for $v_i, w \in V, W$
3. $v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$ for $v, w_i \in V, W$.

**2. Universal property of $\otimes$ translated into:**

The map $	ext{Hom}(V \otimes W, U) \leftrightarrow \text{Mult}(V, W; U)$

$A \leftrightarrow A \otimes$

is a bijection: $\forall b \in \text{Mult}(V \times W, U)$ there is a unique linear map $A : V \otimes W \to U$ such that $A \otimes = b$.

**3.** $\tilde{b} \circ (V \otimes W)^* = \text{Hom}(V \otimes W, \text{IR}) \simeq \text{Mult}(V, W; \text{IR})$

We've seen that if $\dim V, \dim W, \dim U < \infty$ then

$$\dim \text{Mult}(V, W; U) = \dim V \cdot \dim W \cdot \dim U.$$

**Lemma 19.1** For any two finite dimensional vector spaces $V, W$

$$\dim (V \otimes W) = \dim V \cdot \dim W,$$

Proof: $\dim (V \otimes W) \leq \dim ((V \otimes W)^*)$ for $V, W$ and $\dim (V \otimes W)^* < \infty$

$\Rightarrow \dim (V \otimes W) = \dim ((V \otimes W)^*)$.

Now $i^* : (V \otimes W)^* \to \text{Hom}(V \otimes W, \text{IR}) \simeq \text{Mult}(V, W; \text{IR})$

Hence if $\dim V, \dim W < \infty$,

$$\dim (V \otimes W)^* = \dim \text{Mult}(V, W; \text{IR}) = \dim V \cdot \dim W \cdot \dim \text{IR} = \dim V \cdot \dim W.$$
Therefore, if \( \dim V, \dim W < \infty \), \( \dim V \otimes W = \dim V \cdot \dim W \). \( \square \)

Corollary 19.2 Suppose \( \dim V, \dim W < \infty \), \( \{ v_i \}_{i=1}^m \) a basis of \( V \), \( \{ w_j \}_{j=1}^n \) a basis of \( W \). Then \( \{ v_i \otimes w_j \}_{i,j} \) is a basis of \( V \otimes W \).

**Proof.** By construction \( \{ v \otimes w \mid v \in V, w \in W \} \) spans \( V \otimes W \).

Moreover, \( v \otimes w \), \( v \in V, w \in W \)

\[
(x \otimes y) = \left( \sum_i v_i^* (x) v_i \right) \otimes \left( \sum_j w_j^* (y) w_j \right)
\]

\[
= \sum_i v_i^* (x) w_j^* (y) (v \otimes w)
\]

\[
\Rightarrow \{ v \otimes w \}_{i,j} \text{ spans } V \otimes W.
\]

Since \( \dim V \otimes W = \dim V \cdot \dim W \), \( \{ v \otimes w \}_{i,j} \) is also a basis. \( \square \)

**Lemma 19.3** \( V \otimes W \) and \( W \otimes V \) are canonically isomorphic.

**Proof.** Consider \( \Phi: V \times W \rightarrow W \otimes V \), \( \Phi(v, w) = w \otimes v \)

\( \Phi \) is bilinear. \( \Rightarrow \exists \) linear map \( \overline{\Phi}: V \otimes W \rightarrow W \otimes V \)

with \( \overline{\Phi}(v \otimes w) = w \otimes v \). \( v \in V, w \in W \)

Similarly, \( \Phi: W \times V \rightarrow V \otimes W \), \( \Phi(w, v) = v \otimes w \) is bilinear.

\( \Rightarrow \exists \) linear map \( \overline{\Phi}: W \otimes V \rightarrow V \otimes W \) s.t.

\( \overline{\Phi}(w \otimes v) = v \otimes w \) \( \forall w \in W, v \in V \).

\( \Rightarrow (\overline{\Phi} \circ \Phi)(v \otimes w) = w \otimes v = \text{id} (v \otimes w) \) \( \forall w \in W, v \in V \).

\( \Rightarrow \overline{\Phi} \circ \Phi = \text{id} \).

Similarly \( \Phi \circ \overline{\Phi} = \text{id} \).

**Conclusion.** \( \exists \) unique \( \Phi: V \otimes W \rightarrow W \otimes V \) with \( \overline{\Phi}(v \otimes w) = w \otimes v \).

**Lemma 19.4** For any two vector spaces \( V, W \) with \( \dim V, \dim W < \infty \),

\( \text{Hom}(V, W) \cong V^* \otimes W. \)

**Proof.** Consider \( \Phi: V^* \times W \rightarrow \text{Hom}(V, W) \)

\( \Phi(l, w) = l(\cdot) w, \text{ i.e. } (\Phi(l, w))(v) = l(v) w \)

\( \forall v \in V. \)
$\phi$ is bilinear. $\Rightarrow$ 3! linear map

$\phi : V^* \otimes W \rightarrow \text{Hom}(V, W)$ with

$\phi(v \otimes w) = \theta(v)(w)$

Now pick a basis $\{v_i\}$ of $V$, $\{w_j\}$ of $W$. Denote the dual basis of $V$ by $v_i^*$.

Then $\{v_i^* \otimes w_j\}$ is a basis of $V^* \otimes W$ and

$\phi(v_i^* \otimes w_j) = v_i^*(w)$.

Since $\{v_i^*(w)\}_{i,j}$ is a basis of $\text{Hom}(V, W)$, $\phi$ is an isomorphism.

Exercise: Similar argument shows: if $\dim V, \dim W < \infty$,

$V^* \otimes W^* \cong (V \otimes W)^* = \text{Mult}(V, W; \mathbb{R})$.

Exercise: For any three vector spaces $V, W, U$, 3! canonical map

$\alpha : (V \otimes W) \otimes U \rightarrow V \otimes (W \otimes U)$ with

$\alpha(v \otimes w \otimes u) = v \otimes (w \otimes u) \forall (v, w, u) \in V \times W \times U$

In what follows we will often suppress $\alpha$.

Recall: An algebra over $\mathbb{R}$ is a real vector space $A$ together

with a bilinear map $\cdot : A \times A \rightarrow A$, $(a, b) \mapsto a \cdot b$.

An algebra $A$ is associative if $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ \forall $a, b, c \in A$.

Definition: A graded vector space is a direct sum $\bigoplus_{i=0}^{\infty} V_i$ where $V_i$'s are vector spaces.

Recall that $\bigoplus_{i=0}^{\infty} V_i = \{ \sum_{i=0}^{\infty} v_i | v_i \in V_i, \forall i \text{ but finitely many } i \text{'s} \}$.

Definition: A graded algebra is a graded vector space $A = \bigoplus_{i=0}^{\infty} A_i$

which is an algebra with the property that

$a \in A_i, b \in A_j : a \cdot b \in A_{i+j}$.
We are interested in two graded associative algebras:

- the tensor algebra $T(V) := \bigoplus_{k=0}^{\infty} V^\otimes k$ of a vector space $V$.
- the Grassmann (or exterior) algebra $\bigwedge(V) := \bigoplus_{k=0}^{\infty} \bigwedge^k(V)$ of a vector space $V$.

What's $V^\otimes k$?

By definition $V^\otimes 0 = K$, $V^\otimes 1 = V$, $V^\otimes 2 = V \otimes V$.

For $k \geq 2$ there is a unique (up to isomorphism) vector space $V^\otimes k$ together with a $k$-linear map $\otimes^k : V \otimes \cdots \otimes V \to V^\otimes k$ such that

$$\text{Hom}(V^\otimes k, U) \cong \text{Mult}(V \otimes \cdots \otimes V; U) \quad \forall U.$$

For example we may take $V^\otimes k = \bigotimes_{k-1} V^\otimes 1$.

Or we can construct $V^\otimes k$ as a quotient:

$$\text{End}(V^\otimes 1, \ldots, V^\otimes 1, V^\otimes 1) \cong V^\otimes k.$$

We write $v_{i} \otimes \cdots \otimes v_{k}$ for $\otimes^k(v_{i} \otimes \cdots \otimes v_{k}) \in V^\otimes k$.

The graded multiplication

$$ \cdot_{k,l} : V^\otimes k \times V^\otimes l \to V^\otimes (k+l) \quad (k,l \geq 1)$$

is defined as follows:

$$\cdot_{k,l} : V^\otimes k \times V^\otimes l \to V^\otimes (k+l)$$

$$\cdot_{k,l}(v_{i} \otimes \cdots \otimes v_{k}, v_{j} \otimes \cdots \otimes v_{l}) = v_{i} \otimes \cdots \otimes v_{k} \otimes v_{j} \otimes \cdots \otimes v_{l} \otimes v_{j} \otimes \cdots \otimes v_{k} \otimes v_{j} \otimes \cdots \otimes v_{l}$$

is $k+l$-linear by definition.

Fix $(v_{i}, \ldots, v_{k}) \in V^\otimes k$. We get a $k$-linear map

$$\cdot_{k} : V^\otimes k \otimes V \to V^\otimes (k+1)$$

$$\cdot_{k}: \cdot_{k,1} : V^\otimes k \otimes V \to V^\otimes (k+1)$$

with

$$\cdot_{k}(v_{i} \otimes \cdots \otimes v_{k}, v_{l}) = v_{i} \otimes \cdots \otimes v_{k} \otimes v_{l} \otimes v_{i} \otimes \cdots \otimes v_{k} \otimes v_{l} \otimes v_{i} \otimes \cdots \otimes v_{k} \otimes v_{l}.$$

For each fixed $t \in V^\otimes k$ the $V^\otimes l$-linear map $\cdot_{k} : V^\otimes l \to V^\otimes (k+l)$ is

$$\cdot_{k, l} : V^\otimes k \otimes V^\otimes l \to V^\otimes (k+l)$$

with

$$(\cdot_{k, l})(v_{i} \otimes \cdots \otimes v_{k}, v_{l} \otimes \cdots \otimes v_{l}) = v_{i} \otimes \cdots \otimes v_{k} \otimes v_{l} \otimes v_{l} \otimes \cdots \otimes v_{k} \otimes v_{l}.$$