Last time: Proved that if a map \( F: M \rightarrow N \) is transverse to an embedded submanifold \( Z \subset N \), then \( F^{-1}(Z) \) is an embedded submanifold of \( M \). Moreover, if \( F'(z) \neq 0 \), \( \dim M - \dim F'(z) = \dim N - \dim Z \).

- Defined embeddings and immersions.

**Definition** Let \( F: M \rightarrow N \) be a smooth map between two manifolds. The **rank** of \( F \) at \( x \in M \) is the rank of \( T_x F: T_x M \rightarrow T_{F(x)} N \), i.e.,

\[
\text{rank } F = \dim \left( T_x F \right) (T_x M)
\]

**Ex.** If \( F: M \rightarrow N \) is an immersion, then \( \forall x \in M \), \( \text{rank } F = \dim T_x M \).

**Def.** A map \( F: M \rightarrow N \) between two manifolds is a submersion if \( T_x F: T_x M \rightarrow T_{F(x)} N \) is onto for all \( x \in M \).

**Rank Theorem.** Suppose \( M, N \) are two connected manifolds and \( F: M \rightarrow N \) has rank \( k \) at all points of \( M \). Then \( \forall x \in M \) \( k \) coordinates \( \eta = (\eta_1, \ldots, \eta_k): U \rightarrow \mathbb{R}^k \), near \( x \), \( \eta = (\eta_1, \ldots, \eta_k): V \rightarrow \mathbb{R}^k \) near \( F(x) \) so that

\[
(\psi \circ F \circ \varphi^{-1}) (\eta_1, \ldots, \eta_k) = (\xi_1, \ldots, \xi_k, 0, \ldots, 0)
\]

for all \( \xi = (\xi_1, \ldots, \xi_k) \in \varphi(U \cap F^{-1}(V)) \subseteq \mathbb{R}^k \).

**Proof.** See Lee, 2nd edition, Thm 4.12. It's useful; but not central to the course.

**The tangent bundle** \( TM \) of a manifold \( M \)

**Plan.**

- Given a manifold \( M \), we define \( TM = \bigcup_{x \in M} T_x M \) as a set (\( \bigcup \) = disjoint union).

- Given a chart \( \varphi: U \rightarrow \mathbb{R}^m \) on \( M \) we manufacture a candidate coordinate chart \( \varphi: T_U = \bigcup_{x \in U} T_x M \rightarrow \varphi(U) \times \mathbb{R}^m \).

**Sanity check:** If \( \varphi: U \rightarrow \mathbb{R}^m \), \( \varphi: V \rightarrow \mathbb{R}^m \) are two charts.
In \( \varphi_0(\hat{\gamma}) : \varphi(UU'V) x R^m \to \varphi(V'U) x R^m \), \( \mathbb{C}^0 \)?

- We give \( TM = \text{lie}(TqM) \) the induced topology.

\[ \text{If } (\mathcal{O} \circ \mathcal{T}_M \circ \text{open} \circ U \circ \Psi) \circ U \subset R^m \text{, } \mathcal{O}(\mathcal{O} \circ \mathcal{T}_M \circ \text{open} \circ V) \subset U \times R^m \text{ open} \]

It's the smallest topology so that the candidate charts

\[ \varphi_i : T_U \to \varphi(U) x R^m \]

are homeomorphisms.

**Details:**

Given a chart \( \Psi = (x_1, \ldots, x_m) : U \to R^m \) on \( M \) and \( q \in U \), we have, for any \( U \subset T_qM \)

\[ U = \sum_{q \in U} \left( d\varphi_i \bigg|_q \right)^2 d\varphi_i \bigg|_q \]

ie \( (d\varphi_i)_q \to (d\varphi_i)_q) : T_qM \to R^m \) in an isomorphism of vector spaces.

Define

\[ \hat{\varphi} : \bigcup_{q \in U} T_qM \to R^m \times R^m \text{ by } \]

\[ \hat{\varphi}(q, v) = (x_i(q), \ldots, x_m(q), (d\varphi_i)_q(v), \ldots, (d\varphi_m)_q(v)) \]

Suppose \( \Psi = (y_1, \ldots, y_m) : V \to R^m \) is another chart, and \( \hat{\varphi} : TV \to R^m \times R^m \)

\[ \hat{\varphi}(q, v) = (y_i(q), \ldots, y_m(q), (d\varphi_i)_q(v), \ldots, (d\varphi_m)_q(v)) \]

is the associated candidate chart.

Now, take \( r \subset \varphi(UU'V) \subset R^m \text{. } \) We compute \((\hat{\varphi} \circ \varphi^{-1})(r, w)\).

\((\hat{\varphi})^{-1}(r, w) = (\varphi^{-1}(r), \sum w_i \frac{\partial}{\partial x_i} \bigg|_{\varphi^{-1}(r)}) \)

\[ \Rightarrow (\varphi_0(\hat{\gamma}))^{-1}(r, w) = \left( \varphi \left( \varphi^{-1}(r), \left( \sum y_i \frac{\partial}{\partial y_i} \right) \right), \right. \left. \sum w_i \frac{\partial}{\partial x_i} \bigg|_{\varphi^{-1}(r)} \right) \]

where \( q = \varphi^{-1}(r) \).

\[ (dy_i)_q \left( \sum w_i \frac{\partial}{\partial x_i} \bigg|_{q} \right) = \sum w_i \frac{\partial}{\partial x_i} \bigg|_{\varphi^{-1}(r)}(y_i) \]

Now \( \frac{\partial}{\partial x_i} \bigg|_{q}(y_i) = \frac{\partial}{\partial y_i} \bigg|_{r}(y_i \circ \varphi^{-1}) = \frac{\partial}{\partial r} \bigg|_{r}(r \circ \varphi \circ \varphi^{-1}) \)

The map \( \varphi(UU') \to R, r \to \frac{\partial}{\partial x_i} \bigg|_{r}(r \circ \varphi \circ \varphi^{-1}) = \frac{\partial}{\partial x_i} \bigg|_{r}(\varphi^{-1}(r)) \)
are $C^\infty$. They are entries of the Jacobian matrix of the $C^\infty$ map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \varphi(U \cap V)$.

Consequently

$$(\psi \circ \varphi^{-1})^{-1} (r, w) = \bigl( (\psi \circ \varphi^{-1})(r), \left( \frac{\partial (\varphi^{-1})(w)}{\partial x_i} \right) \bigr)$$

and

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \times \mathbb{R}^m \to \varphi(U \cap V) \times \mathbb{R}^m$$

to a diffeomorphism (hence a homeomorphism).

Now define a topology on $TM = \bigsqcup_{q \in M} T_q M$ to be the smallest topology so that for any chart $\varphi : U \to \mathbb{R}^m$ and $\Phi : TU \to \varphi(U) \times \mathbb{R}^m$ in an atlas $\mathcal{A}$ on $M$, $\{ \Phi^{-1}(\varphi(U) \times \mathbb{R}^m) \}_{U \in \mathcal{A}}$ is a collection of homeomorphisms with smooth transition maps.

This gives $TM$ an atlas, hence a manifold structure.

The choice of an atlas on $M$ doesn't matter: any two compatible atlases on $M$ give rise to compatible atlases on $TM$.

Note also that $M$ is Hausdorff, so is $TM$. (Why?)

Remark. We have the canonical map $\pi : TM \to M$.

It's defined by $\pi(v) = q$, $v \in T_q M$.

Aside: one often writes $(q, v) \in TM$ to mean $q \in M, v \in T_q M$.

Exercise. $\pi : TM \to M$ is continuous.

Lemma 11.1. The canonical map $\pi : TM \to M$ is a $C^\infty$ map and a submersion.

Proof. Pick a chart $\varphi : U \to \mathbb{R}^m$ on $M$. Let $\Phi : TU \to \mathbb{R}^m \times \mathbb{R}^m$ be the induced chart on $TM$. For any $(r, w) \in \Phi(U \times \mathbb{R}^m)$
\[
(\phi \circ \pi \circ (\psi)^{-1}) (r, w) = \psi \left( \pi \left( \phi^{-1} (r), \sum w: \frac{\partial}{\partial x_i} \big|_{\phi^{-1}(r)} \right) \right)
\]
\[
\Rightarrow \phi \left( \phi^{-1} (r) \right) = r
\]

ie \( \phi \circ \pi \circ (\psi)^{-1} : (\psi(w) \times \mathbb{R}^m) \rightarrow \psi(w) \) is
\[(r, w) \mapsto r, \text{ which is } \in \mathcal{C}^\infty.
\]

If \( \psi : V \rightarrow \mathbb{R}^m \) is another coordinate chart on \( M \), then
\[
\phi \circ \pi \circ (\psi)^{-1} = \frac{\phi \circ \pi \circ (\psi)^{-1} \circ \phi \circ \psi}{\phi \circ \psi}
\]
\[
\Rightarrow \Pi \in \mathcal{C}^\infty. \text{ Also } \phi \circ \pi \circ (\psi)^{-1} : \psi(w) \times \mathbb{R}^m \rightarrow \mathbb{R}^m
\]
in a submersion. \( \phi \circ (\psi)^{-1} \) are diffeomorphisms
\[
\Rightarrow \Pi \text{ is a submersion.}
\]

**Definition (algebraic).** A vector field on a manifold \( M \) is a linear map \( \mathcal{X} : \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M) \) so that
\[
\mathcal{X}(fg) = \mathcal{X}(f) \cdot g + f \mathcal{X}(g) \quad \forall f, g \in \mathcal{C}^\infty(M),
\]
in a derivation \( \mathcal{X} : \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M) \).

**Definition (geometric).** A vector field \( \mathcal{X} \) on a manifold \( M \) is a section of the tangent bundle \( \pi : TM \rightarrow M \), i.e.
\[
\mathcal{X} : M \rightarrow TM \in \mathcal{C}^\infty
\]
and \( \pi \circ \mathcal{X} = \text{Id}_M. \)

**Notation** \( \text{Der}(\mathcal{C}^\infty(M)) = \{ \mathcal{X} : \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M) \mid \mathcal{X} \text{ is a derivation} \} \)
\[
\mathcal{X}(M) = \Gamma(TM) = \{ \mathcal{X} : M \rightarrow TM \mid \pi \circ \mathcal{X} = \text{Id}_M \}.
\]

**Proposition** There is a linear isomorphism
\[
\Gamma(TM) \overset{\cong}{\rightarrow} \text{Der}(\mathcal{C}^\infty(M)).
\]

It is given by \( \mathcal{X} \mapsto D \mathcal{X} \)
\[
(D \mathcal{X} f)(q) = \mathcal{X}(q) f \quad \forall f \in \mathcal{C}^\infty(M), q \in M.
\]

**Proof** Next time.