Homework #3 Math 518
Due in class Wednesday, September 18, 2019

Note that all manifolds are Hausdorff and paracompact unless noted otherwise. In particular you have bump functions as needed.

Exercise 3.1. 1. Prove that for any manifold $M$ the diagonal
\[ \Delta_M := \{(x,y) \in M \times M \mid x = y\} \]
is an embedded submanifold.

2. Show that the map $\delta : M \rightarrow \Delta_M$, $\delta(x) := (x,x)$ is a diffeomorphism.

Exercise 3.2. Let $M$ and $N$ be two manifolds. Prove that for any point $(a,b) \in M \times N$ the tangent space $T_{(a,b)}(M \times N)$ is naturally isomorphic to the product of tangent spaces $T_aM \times T_bN$. Hint: the product $M \times N$ comes with two projections $p_1 : M \times N \rightarrow M$ and $p_2 : M \times N \rightarrow N$. Their derivatives give you maps $T_{(a,b)}p_1 : T_{(a,b)}(M \times N) \rightarrow T_aM$, $T_{(a,b)}p_2 : T_{(a,b)}(M \times N) \rightarrow T_bN$...

Exercise 3.3. The goal of the exercise is to prove that 1 is a regular value of the determinant map $\det : \text{GL}(n,\mathbb{R}) \rightarrow \mathbb{R}$.

1. We proved in class that for a finite dimensional vector space $V$, the tangent space $T_pV$ at some point $p \in V$ is canonically isomorphic to $V$. Prove that for any open subset $U$ of $V$, and any $p \in U$, $T_pU$ is naturally isomorphic to $V$. In particular if $V = M_n(\mathbb{R})$ the space of $n \times n$ matrices and $U = \text{GL}(n,\mathbb{R})$ then $T_A\text{GL}(n,\mathbb{R})$ is naturally isomorphic to $M_n(\mathbb{R})$ for all $A \in \text{GL}(n,\mathbb{R})$.

2. Prove that the derivative $T_I\det : T_I\text{GL}(n,\mathbb{R}) = M_n(\mathbb{R}) \rightarrow T_1\mathbb{R} = \mathbb{R}$ is trace. Hint: let $E_{ij}$ be the matrix with 1 in the $ij$th slot and 0 elsewhere. Compute $\frac{d}{dt}|_0 (\det(I + tE_{ij}))$.

3. In a previous homework you proved that $L_A : \text{GL}(n,\mathbb{R}) \rightarrow \text{GL}(n,\mathbb{R})$ is a diffeomorphism for any $A \in \text{GL}(n,\mathbb{R})$. Use it to prove that $T_A\det : T_A\text{GL}(n,\mathbb{R}) \rightarrow T_1\mathbb{R}$ is onto for $A \in \det^{-1}(1)$ if and only if $T_I\det : T_I\text{GL}(n,\mathbb{R}) \rightarrow T_1\mathbb{R}$ is onto.

4. Now prove that 1 is a regular value of $\det : \text{GL}(n,\mathbb{R}) \rightarrow \mathbb{R}$.

Exercise 3.4. Let $M$ be a manifold. Define a vector field $v$ on $M$ to be a derivation $v : C^\infty(M) \rightarrow C^\infty(M)$. In other words $v$ is linear and for all $f,g \in C^\infty(M)$ we have
\[ v(fg) = fv(g) + v(f)g. \]

1. Prove that vector fields are local: if two functions $f_1, f_2 \in C^\infty(M)$ agree on some open set $U \subset M$ then $v(f_1)|_U = v(f_2)|_U$ as well.
   Hint: for any $p \in U$ there is a smooth function $\tau$ with $\tau = 1$ on some neighborhood $V$ of $p$ and $\text{supp} \tau \subset U$. 

1
2. Prove that for any function \( f \in C^\infty(M) \) and for any vector field \( v \) on \( M \)

\[
\text{supp}(v(f)) \subset \text{supp}(f).
\]

Hint: suppose \( f|_U = 0 \) on some open set \( U \). What is \( v(f)|_U \)?

**Exercise 3.5.** Let \( p \) be a point in a manifold \( M \). The point of this exercise is to define the space of germs of functions at \( p \), check that it is an \( \mathbb{R} \)-algebra and show that the tangent space \( T_p M \) is isomorphic to the dual of the space \( I_p/I_p^2 \) where \( I_p \) is the ideal of germs of functions that are 0 at \( p \).

1. Consider the set \( \mathcal{G} \) of pairs \((f,U)\) where \( U \subset M \) is an open neighborhood of \( p \) and \( f \in C^\infty(U) \). Define a relation \( \sim \) on \( \mathcal{G} \) by

\[
(U, f) \sim (W, g) \quad \text{if and only if} \quad \text{there is an open set } V \subset U \cap W \text{ with } f|_V = g|_V.
\]

The **set of germs** \( C^\infty(M)_p \) of functions at \( p \) is defined to be the quotient set \( \mathcal{G}/\sim \). We write \([f]\) for the equivalence class of \((f,U)\). Sketch a proof that \( C^\infty(M)_p \) is an \( \mathbb{R} \)-algebra.

2. Prove that there the **evaluation map** \( ev_p : C^\infty(M)_p \to \mathbb{R} \) given by \( ev_p([f]) := f(p) \) is well-defined.

3. Prove that a tangent vector \( v \in T_p M \) gives rise to a well-defined linear map \( \bar{v} \) from the algebra of germs to \( \mathbb{R} \). It is given by \( \bar{v}([f]) := v(f) \). Prove that \( \bar{v} \) is a derivation.

4. Prove that if \( ev_p([f]) \neq 0 \) then there is a germ \( [g] \in C^\infty(M)_p \) so that \( [g][f] = [1] \). Conclude that the ideal

\[
I_p = \{ [f] \in C^\infty(M)_p \mid ev_p([f]) = 0 \}
\]

is maximal.

5. Prove that for any \([h]\) in the square \( I_p^2 \) of the ideal \( I_p \) and for any tangent vector \( v \in T_p M \)

\[
\bar{v}([h]) = 0.
\]

Use it to prove that the map

\[
\bar{v} : I_p/I_p^2 \to \mathbb{R}; \quad \bar{v}([f] + I_p^2) = \bar{v}([f])
\]

is a well-defined linear map.

6. Prove that the vector space \( I_p/I_p^2 \) is isomorphic to the dual of the tangent space \( T_p M \). Hence we could have defined the tangent space to a manifold \( M \) at a point \( p \) as \((I_p/I_p^2)^*\). Hint: Hadamard’s lemma, perhaps used twice.