Exercise 14.1. Consider the open subset $M$ of $\mathbb{R}^3$ consisting of $\mathbb{R}^3$ with the points $(1, 0, 0)$ and $(-1, 0, 0)$ deleted. That is $M = \mathbb{R}^3 \setminus \{(1, 0, 0), (-1, 0, 0)\}$. Compute the de Rham cohomology of $M$.

Exercise 14.2. Let $U$ be some open subset of $\mathbb{R}^3$. Prove that $H^1(U) = 0$ if and only if for any vector field $F$ on $U$ with curl$F = 0$ there is $f \in C^\infty(U)$ so that $F = \nabla f$.

Exercise 14.3. Let $U$ be some open subset of $\mathbb{R}^3$. Prove that $H^2(U) = 0$ if and only if for any vector field $F$ on $U$ with div$F = 0$ there is a vector field $G$ on $U$ so that $F = \nabla \times G$.

Exercise 14.4. What is
\[ \int_{S^2} \omega, \]
where $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ is the standard unit sphere in $\mathbb{R}^3$ oriented as the boundary of the unit ball (i.e., outward normal) and
\[
\omega = (3x^2 \cos(y) + e^{xy}) \, dx \wedge dy + 17x^3 \, dx \wedge dz + (x + yz^3 + \sin(z)) \, dy \wedge dz
\]
You can assume the standard facts about areas of spheres, volumes of balls etc.

Exercise 14.5. Let $D$ be an $m$-dimensional oriented domain with boundary $\partial D$ (equivalently, a manifold with boundary). Suppose $\alpha, \beta$ are two compactly supported differential forms on $D$ such that
1. $\beta|_{\partial D} = 0$ and
2. $|\alpha| + |\beta| = m - 1 (= \dim D - 1)$.

Prove that
\[ \int_D d\alpha \wedge \beta = (-1)^{|\alpha|} \int_D \alpha \wedge d\beta. \]

Exercise 14.6. Let $D$ be a compact $m$-dimensional oriented domain with boundary $\partial D$ (equivalently, a manifold with boundary), $F : \partial D \to N$ a smooth map and $\omega \in \Omega^{m-1}(N)$ a closed form. Prove that if $F$ can be extended to a smooth map $\tilde{F} : D \to N$ then
\[ \int_{\partial D} F^* \omega = 0. \]