Exercise 13.1. Recall that for a finite dimensional vector space $V$ we have a canonical isomorphism $\text{Bilin}(V \times V; \mathbb{R}) \cong V^* \otimes V^*$, which we will now suppress.

Let $E \to M$ be a vector bundle. A metric on $E$ is a section $g$ of the vector bundle $E^* \otimes E^* \cong \text{Bilin}(E \times E; \mathbb{R})$ so that for each point $q \in M$ the corresponding bilinear map

$$ g_q : E_q \times E_q \to \mathbb{R} $$

is symmetric and positive definite: $g_q(v, v) \geq 0$ for all $v \in E_q$ and $g_q(v, v) = 0$ if and only if $v = 0$.

Prove that any vector bundle $E \to M$ has a metric.

Hints: prove this for product bundles first. This should give you a local existence of the metrics. Use a partition of unity to patch local metrics into a global metric.

Exercise 13.2. A Riemannian manifold is a manifold $M$ with a metric $g$ on its tangent bundle $TM$.

(a) Prove that if $(M, g)$ is a Riemannian manifold then the metric $g$ defines an isomorphism of vector bundles $g^\#: TM \to T^*M$ by

$$ g^\#(q, v) = g_q(v, -) \in T_q^*M $$

for all $(q, v) \in T_qM$.

(b) Show that for a smooth function $f$ on a Riemannian manifold $(M, g)$ there is a unique vector field $\nabla f$ with

$$ g_q(\nabla f(q), v) = df_q(v) $$

for all $q \in M$, $v \in T_qM$. This vector field $\nabla f$ is called the gradient vector field of $f$.

(c) Let $\gamma(t)$ be an integral curve of the gradient vector field $\nabla f$. Prove that

$$ \frac{d}{dt} f(\gamma(t)) \geq 0 $$

for all $t$ that $\gamma$ is defined. When is $\frac{d}{dt} f(\gamma(t)) = 0$?

Exercise 13.3. Consider the square $S = [0, 1] \times [0, 1] \subset \mathbb{R}^2$. Let $\alpha \in \Omega^1(\mathbb{R}^2)$ be a 1-form. Prove that Stokes’ theorem holds for $S$:

$$ \int_S d\alpha = \int_{\partial S} \alpha, $$

where the piece-wise smooth boundary $\partial S$ is oriented appropriately.

Exercise 13.4. Consider the 1-form $\alpha := dz + x \, dy \in \Omega^1(\mathbb{R}^3)$. Prove that $\alpha \wedge d\alpha$ is a volume form. ($\alpha$ is the standard contact form on $\mathbb{R}^3$).

Exercise 13.5. “Recall” that a map $F : M \to N$ of manifolds is a local diffeomorphism if at every point $q \in M$ its differential $T_qF$ is an isomorphism of vector spaces.

(a) Prove that the map $\pi : S^{n-1} \to \mathbb{R}P^{n-1}$ defined by sending a vector $v \in S^{n-1}$ to the line $[v] \in \mathbb{R}P^{n-1}$ through $v$ is a local diffeomorphism.
(b) Prove that $\mathbb{R}P^{n-1}$ is orientable if and only if $n$ is even.
Hints: (1) In 12.4 of previous homework you proved that the map $T : S^{n-1} \to S^{n-1}$, $T(x) = -x$ preserves the standard volume form $\nu := \left(\sum x_i \frac{\partial}{\partial x_i}\right) dx_1 \wedge \ldots \wedge dx_n |_{S^{n-1}}$ if and only if $n$ is even.
(2) If $\mu$ is a volume form on $\mathbb{R}P^{n-1}$ then $T^*(\pi^* \mu) = \pi^* \mu$, where $T : S^{n-1} \to S^{n-1}$ is the multiplication by $-1$:

$$T(x) = -x.$$ 

Conversely show that if $\nu$ is a volume form on $S^{n-1}$ with $T^* \nu = \nu$, then $\nu = \pi^* \mu$ for some volume form $\mu$ on $\mathbb{R}P^{n-1}$. 