Homework #11 Math 518  
Due in class  Wednesday, November 13, 2019

Exercise 11.1 (Lagrange multipliers).  Let $N$ be a manifold, $F = (f_1, \ldots, f_k) : N \to \mathbb{R}^k$ a smooth map, $c \in \mathbb{R}^k$ a regular value of $F$, $M = F^{-1}(c)$, $h : N \to \mathbb{R}$ a smooth function. Suppose $p \in M$ is a local maximum of the restriction $h|_M$. Prove that there are unique scalars $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ so that

$$dh_p = \sum_{j=1}^k \lambda_j (df_j)_p.$$ 

Hints:  
1. Let $U$ be a subspace of a finite dimensional vector space $V$. Define the annihilator of $U$ to be 

$$U^\circ := \{ \ell \in V^* \mid \ell(u) = 0 \text{ for all } u \in U \}.$$ 

Prove that if $\{v_1, \ldots, v_n\}$ is a basis of $V$ so that $\{v_1, \ldots, v_k\}$ is a basis of $U$ and $\{v_1^*, \ldots, v_n^*\}$ is the dual basis, then $\{v_{k+1}^*, \ldots, v_n^*\}$ is a basis of $U^\circ$.  

2. Prove that $dh_p \in (T_p M)^\circ$.  

3. Prove that $\{(df_1)_p, \ldots, (df_k)_p\}$ is a basis of $(T_p M)^\circ$.  

Exercise 11.2. In the previous homework you have defined the exponential map 

$$\exp : \mathfrak{g} \to G$$ 

for any Lie group $G$. Prove that for any $X \in \mathfrak{g}$ the curve $\sigma(t) = \exp(tX)$ is the integral curve of the left invariant vector field $\bar{X}$ defined by $X$. Hint: show that if $\gamma(\tau)$ is an integral curve of the vector field $Y$ then for any $t \in \mathbb{R}$ the curve $\tau \mapsto \gamma(t\tau)$ is an integral curve of the vector field $tY$.  

Exercise 11.3. Prove that the exponential map is natural. That is, given a map of Lie groups $f : G \to H$, show that for any $X \in \mathfrak{g}$ we have 

$$\exp(\delta f(X)) = f(\exp(X)),$$ 

where exp on the left denotes the exponential map for the group $H$, exp on the right denotes the exponential map for $G$ and $\delta f : \mathfrak{g} \to \mathfrak{h}$ is the induced map of Lie algebras, i.e., $\delta f = df_e$.  

Exercise 11.4. Let $\pi_E : E \to M$ and $\pi_F : F \to M$ be vector bundles over $M$.  

(a) Show that $E \times F$ is a vector bundle over $M \times M$.  

(b) Explain why the fiber product 

$$G = E \times_M F = \{(e, f) \in E \times F : \pi_E(e) = \pi_F(f)\}$$ 

can be considered a vector bundle over $M$. Hint: consider the map $\pi : G \to M$ defined by $\pi(e, f) = \pi_E(e)$.  

(c) Show that, as a vector bundle over $M$, $G$ is isomorphic to $E \oplus F$. Hint: compare the transition maps.  

Exercise 11.5. Prove that for any two differential forms $\alpha \in \Omega^k(M)$, $\beta \in \Omega^l(M)$ on a manifold $M$, 

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha.$$ 

Hint: Suppose $V$ is a finite dimensional vector space, $v_1, \ldots, v_k, w_1, \ldots, w_l \in V$. Let $\alpha = v_1 \wedge \ldots \wedge v_k$, $\beta = w_1 \wedge \ldots \wedge w_l$. Show that $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$.  

1