Sketch of solutions of the take-home midterm

1) It is enough to consider the case where \( M = \mathbb{R}^m \) and \( N = \mathbb{R}^n \). If \( F^{-1}(q) = \emptyset \), \( (dF)^{-1}(q, 0) = \emptyset \).

Suppose \( x \in F^{-1}(q) \), \( v \in T_x M \cong \mathbb{R}^m \) and \( dF_x(v) = 0 \).

\[ dF : TIR^m \cong \mathbb{R}^m \times \mathbb{R}^m \rightarrow TIR^n \cong \mathbb{R}^n \times \mathbb{R}^n \]

of the form

\[ dF(x, w) = (F(x), DF(x)w) \]

Consequently

\[ d\left( dF \right)(x, v) = D(dF)(x, v) = \begin{pmatrix} \frac{\partial \text{DF}(x)}{\partial x} & 0 \\ \frac{\partial \text{DF}(x)}{\partial y} & \text{DF}(x) \end{pmatrix} \]

Since \( x, w \mapsto \text{DF}(x, w) \) is linear in \( w \).

By assumption \( \text{DF}(x) : \mathbb{R}^m \rightarrow \mathbb{R}^n \) is onto

\[ \Rightarrow d\left( dF \right)(x, v) \text{ is onto as well.} \]

2) The easiest way to solve the problem is to observe that a vector field \( X \in \Gamma(TM) \) is tangent to \( Q \) if and only if \( X|_Q \in \Gamma(TQ) \) and \( X \) are \( i \)-related, where \( i : Q \rightarrow M \) is the inclusion. Since \( X|_Q \) is \( i \)-related to \( X \) and \( Y|_Q \) is \( i \)-related to \( Y \),

\[ [X|_Q, Y|_Q] \text{ is } i \text{-related to } [X, Y] \]

i.e.

\[ [X|_Q, Y|_Q] \in \mathfrak{X}(Q) \]

3) By definition of \( \gamma \), \( \dot{\gamma}(t) = X(\gamma(t)) \forall t \).

\[ \Rightarrow \forall f \in \mathcal{C}^\infty(M) \]

\[ \dot{\gamma}(t) f = X(\gamma(t)) f = 0 \]

since \( X(p) f = 0 \) \( \forall p \in M \) by assumption.

Now

\[ \dot{\delta}(t) f = \frac{d}{dt} (f \circ \gamma) = (f \circ \gamma)'(t) \]

\[ \Rightarrow \quad \text{Since } (f \circ \gamma)'(t) = 0 \quad \forall t, (f \circ \gamma)(t) \text{ is} \]

\[ \text{constant.} \]
a constant function of $t$ (we implicitly assume that the domain of $Y$ is connected).

$\Rightarrow f(\gamma(t)) = f(\gamma(0)) \neq t$.

4 a) Since $d \gamma_e : T_e G \times T_e G \rightarrow T_e G$ is linear and since $\forall x, y \in T_e G$,

$\langle x, y \rangle = \langle x, 0 \rangle + \langle 0, y \rangle$

it's enough to prove that $d \gamma_e \langle x, 0 \rangle = x$ (and that $d \gamma_e \langle 0, y \rangle = y$).

Pick a curve $\gamma : (-\varepsilon, \varepsilon) \rightarrow G$ with $\gamma(0) = e$, $\gamma'(0) = x$.

Then $\bar{x} = (\gamma(t), e)$ is a curve through $(e, e) \in G \times G$ with $\bar{x}(0) = (x, 0)$.

$\Rightarrow d \gamma_e (x, 0) = \left. \frac{d}{dt} \right|_0 \gamma(t) 
= \left. \frac{d}{dt} \right|_0 \gamma(t) 
= \left. \frac{d}{dt} \right|_0 \gamma(t) = x$.

A proof that $d \gamma_e (0, y) = y$ is similar.

b) Given $v \in T_e G$, let $Y$ be a curve in $G$ with $\dot{\gamma}(0) = e$, $\gamma'(0) = v$.

Then $e = \gamma(t) \cdot (\gamma(t))^{-1}$.

$\Rightarrow 0 = \left. \frac{d}{dt} \right|_0 (\gamma(t) \cdot (\gamma(t))^{-1}) = \left. \frac{d}{dt} \right|_0 (m \circ (id, inv)) (\gamma(t))$

$= dm_{e,e} \circ (d(id)_e \cdot v, d(inv)_e (v)) = v + d(inv)_e v$

$\Rightarrow d(inv)_e (v) = -v$