Last time: Proved that $H^k_c(\mathbb{R}^n) = \{0\}$ for $k < n$.

Con: \( i: H^1_c(\mathbb{R}^n) \to \mathbb{R} \) is a map $i: \sigma \mapsto \int_{\mathbb{R}^n} \sigma$.

Hence: \( \forall \sigma \in \Omega^1_c(\mathbb{R}^n), \int_{\mathbb{R}^n} \sigma = 0 \Rightarrow \sigma = d\eta \text{ for some } \eta \in \Omega^{n-1}_c(\mathbb{R}^n) \).

Note $B_R(0) \subset \mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^n$.

\[ \Rightarrow \quad \forall \text{ manifold } M, \forall U \subset M \text{ open, set } U \text{ is diffeomorphic to } \mathbb{R}^m, \forall \omega \in \Omega^m_c(M), \int_M \omega = 0 \Rightarrow \omega = d\eta \text{ for some } \eta \in \Omega^{m-1}_c(M). \]

**Theorem 41.1:** Let $M$ be a connected orientable manifold of dimension $m$. Then $H^m_c(M) \cong \mathbb{R}$.

In particular, \( \forall \omega \in \Omega^m_c(M), \int_M \omega = 0 \Rightarrow \omega = d\eta \text{ for some } \eta \in \Omega^{m-1}_c(M). \)

**Proof** Fix $U \subset M$ open, $U$ an open ball in $\mathbb{R}^m \subset \mathbb{R}^n$.

*By (*) above, $H^m_c(U) \cong \mathbb{R}$. We argue: the map $*: H^m_c(U) \to H^m_c(M)$, $[\omega] \mapsto [\omega]$ extended by 0 to all of $M$ is an isomorphism.

(1) amounts to showing:

(2) $\forall \omega \in \Omega^m_c(M), \exists \omega' \in \Omega^m_c(U)$ s.t. $\omega - \omega' = dK$ for some $K \in \Omega^{m-1}_c(M)$

(2) a equivalent to:
Now fix \( w \in \Omega^m_c (M) \). Since \( \text{supp } w \) is compact
\[ \exists V_1, \ldots, V_n \subset M \text{ open, } V_i \ni \text{open ball in } IR^m \approx IR^m \]
with \( V_i \cup \cdots \cup V_n \supset \text{supp } w \).
\( V_0 = M \setminus \text{supp } w \), \( V_i \supset \text{V.i} \) is an open cover of \( M \). Choose a partition \( \{p_i\} \subset \{V_i\} \) of \( 1 \) with \( \text{supp } p_i \subset U_i \).
\[ \frac{1}{n} = \left( p_1 + \cdots + p_n \right) \mid \text{supp } w. \]
\[ w = \frac{1}{n} \sum_{i=1}^{n} (p_i \circ w) \text{ and } \text{supp } p_i \subset V_i. \]

Suppose we can show that for each \( p_i \circ w \in \Omega^m_c (U) \) s.t.
\[ \{w_i\} = \{p_i \circ w\} \text{ in } H^m_c (M). \]
Then \( \{w\} = \sum_{i=1}^{n} (p_i \circ w) = \sum_{i=1}^{n} w_i \}
\[ \text{and } \frac{1}{n} \sum_{i=1}^{n} w_i \in \Omega^m_c (U), \text{ i.e. we're done.} \]

We are therefore reduced to proving:

Lemma 4.1.1. Suppose \( M \) is connected orientable, \( U, V \subset M \) open
\[ U \cap V = \text{open ball}. \] Then \( \forall w \in \Omega^m_c (V) \exists \omega \in \Omega^m_c (U) \text{ s.t.} \]
\[ \{w\} = \{\omega\} \text{ in } H^m_c (M). \]

\textbf{Proof}
\[ \to \text{ If } \int_M w = 0, \text{ then, since supp } w \subset V, \int_V w = 0. \]
\[ \Rightarrow \text{ } w = \eta w \text{ for some } \eta \in \Omega^m_c (V) \subset \Omega^m_c (M). \]
\[ \Rightarrow \text{ } \{w\} = 0 \text{ in } H^m_c (M). \] Take \( \omega' = 0. \)
\[ \text{ (2) Suppose } \int_M w = c \neq 0 \text{ and } U \cap V \neq \emptyset. \]
Then \( \exists w \in U \cap V \text{ open and a coordinate chart} \]
\[ \varphi = (x_1, \ldots, x_n) : W \to BR(0) \text{ for some } R > 0. \]
\( f \in C^\infty_c (BR(0)) \text{ s.t. } \int_{BR(0)} f = 0, \]
Consider \( w' = (f \circ \varphi) \text{ d}x_1 \ldots \text{ d}x_n. \)
By construction \( \int_{W'} W' = \int_{W} f = c \) and

\[ \text{Supp } W' \subseteq W \subseteq U. \quad \text{But } \text{Supp } W' \subseteq V \text{ as well.} \]

Since \( \int_{V} \omega' = \int_{V} \omega' \) \text{ and } \( \omega' = \int_{V} \omega' \) \text{ in } \( H^{m}_{c}(V) \).

\[ \Rightarrow \exists \, c \Omega_{c}^{m-1}(V) \subseteq \Omega_{c}^{m-1}(W) \quad \text{st. } \omega - \omega' = dK. \]

(3) Now suppose \( U \cap V = \emptyset \).

Pick \( p \in V, q \in U, \) and a path \( \gamma : [0,1] \to M \) s.t. \( \gamma(0) = p, \gamma(1) = q \). Since \( \gamma([0,1]) \) is compact, \( \exists \, W_{n}, -W_{n} \subseteq M \) open \( W_{n} \subseteq \) open ball, \( W_{0} = V, W_{k} = U \) and \( W_{i} \cap W_{j} \neq \emptyset \) \text{ if } \( i \neq j \).

Given \( \omega \in \Omega_{c}^{m}(V) \) \( \exists \, \omega^{(0)} \in \Omega_{c}^{m}(W_{1}) \) \text{ s.t. } \[ \omega = \omega^{(0)} \] \text{ in } \( H^{m}_{c}(M) \) \text{ by (1) and (2).}

Given \( \omega^{(0)} \), \( \exists \, \omega^{(1)} \in \Omega_{c}^{m}(W_{2}) \) \text{ s.t. } \[ \omega^{(0)} = \omega^{(1)} \] \text{ in } \( H^{m}_{c}(M) \)

\[ \therefore \exists \omega^{(k)} \in \Omega_{c}^{m}(W_{k}) \subseteq \Omega_{c}^{m}(U) \quad \text{s.t.} \]

\[ [\omega] = [\omega^{(0)}] = [\omega^{(1)}] = \cdots = [\omega^{(k)}] \quad \text{in } \( H^{m}_{c}(M) \). \]

\[ \square \]

**Definition:** A continuous map \( f : X \to Y \) between two topological spaces is **proper** if \( \forall \, K \subseteq Y \text{ compact, } f^{-1}(K) \) is compact

**Ex:** Any homeomorphism \( f : X \to Y \) is proper since

\[ f^{-1}(K) = \text{image of } K \text{ under } f^{-1} \text{, and} \]

continuous maps take compact sets to compact sets.

**Ex:** \( M, N \) compact manifolds. Then any map of manifolds \( f : M \to N \) is proper.
Lemma 41.3: Suppose \( f : M \to N \) is a proper map of manifolds. Then \( \forall \omega \in \Omega^\bullet_c(N), \ f^* \omega \in \Omega^\bullet_c(M) \).

Proof:

\[
\text{supp}(f^* \omega) = \{ x \in M \mid (f^* \omega)_x \neq 0 \} = \{ x \in M \mid \omega_{f(x)} \neq 0 \} = f^{-1} \{ y \in N \mid \omega_y \neq 0 \}
\]

\( \forall U \subset N \), \( f^{-1}(U) \subset f^{-1}(\text{supp} \omega) \) (check it)

\( \Rightarrow \text{supp}(f^* \omega) \subset f^{-1}(\text{supp} \omega) \)

Since \( \text{supp} \omega \) is compact, and \( f \) is proper, \( f^{-1}(\text{supp} \omega) \) is compact. Since \( \text{supp}(f^* \omega) \) is closed and since \( \text{supp}(f^* \omega) \subset f^{-1}(\text{supp} \omega) \) it is compact as well.

Consequence: A proper map \( f : M \to N \) gives rise to \( H^\bullet_c(f) : H^\bullet_c(N) \to H^\bullet_c(M) \).