Last time: • Defined Lie derivatives of differential forms w.r.t. vector fields and proved Cartan's formula:
\[ L_x \alpha = d (x^*(x)) \alpha + x^*(\alpha) \wedge dx \]
• Defined volume forms: nowhere zero top degree forms.
Stated but didn't prove:

Prop 32.4: (1) A manifold \( M \) is orientable \( \iff \) \( \exists \) \( C^\infty \) form \( \mu \in \Omega^{\text{top}}(M) \)

with \( \mu_q \neq 0 \) \( \forall q \in M \).
(2) Two volume forms \( \mu, \nu \) define the same orientation:\
\[ \forall f \in C^\infty(M), f \geq 0 \implies \mu = f \nu. \]

Proof (1) \( \Rightarrow \): Suppose \( M \) is orientable. Then \( \exists \) a cover
\[ \{ \varphi_k : U_k \to \mathbb{R}^m \} \]

by coordinate charts with \( \det \left( D(\varphi_k \circ \varphi_k^{-1}) \right) > 0 \)

Let \( \{ \varphi_1, \ldots, \varphi_k \} \) be a partition of \( \mathbb{R}^m \) subordinate to \( U_k \).
\( \mathbb{R}^m \) has a volume form \( dr_1 \wedge \ldots \wedge dr_m \) (i.e. \( \mathbb{R}^m \to \mathbb{R} \) standard coordinates). Let
\[ \mu = \sum \varphi_k(\varphi_k^* (dr_1 \wedge \ldots \wedge dr_m)) \]

We need to show: \( \forall q \in M \), \( \mu_q \neq 0 \). Fix \( q \in M \).
Since \( \{ \text{Supp } \varphi_k \} \) is locally finite, \( \exists \alpha_1, \ldots, \alpha_k \) s.t.
\[ \text{for } q \neq \alpha_1, \ldots, \alpha_k \quad \varphi_k(q) = 0. \text{ ad } \varphi_k(q) \neq 0 \text{ i.e. } k. \]
We now compute:
\[ \left( \varphi_k^* (dr_1 \wedge \ldots \wedge dr_m) \right) \left( \varphi_k^* (dr_1 \wedge \ldots \wedge dr_m) \right) \]
\[ = \sum_{j=1}^k \varphi_j(q) \det \left( D(\varphi_j \circ \varphi_k^{-1})(\varphi_k(q)) \right) \cdot dr_1 \wedge \ldots \wedge dr_m \cdot \varphi_k(q) \]
\[ = \left( \sum \varphi_j(q) \det D(\varphi_j \circ \varphi_k^{-1})(\varphi_k(q)) \right) \cdot dr_1 \wedge \ldots \wedge dr_m \]
\[ \geq 0. \]

(\( \Leftarrow \)) Suppose \( \mu \in \Omega^{\text{top}}(M) \) is a volume form.

Let \( \{ \varphi_k : U_k \to \mathbb{R}^m \} \) be an atlas of charts on \( M \)

with \( U_k \) connected \( \forall k. \)
Then \((g^{-1}\tau)^{m}T = f_{\beta} d_{\alpha} - \partial_{\alpha} n\) for

Since \(f_{\beta} \in C^{\infty}(U_{\beta}(U_{\alpha}))\) with \(f_{\beta}(r) \neq 0 \forall r \in U_{\beta}(U_{\alpha})\).

Because \(U_{\beta}\) is connected, so \(U_{\beta}(U_{\alpha})\).

If \(f_{\beta} > 0\), keep \(U_{\beta} : U_{\beta} \rightarrow \mathbb{R}^{m}\).

If \(f_{\beta} < 0\), replace \(U_{\beta} : U_{\beta} \rightarrow \mathbb{R}^{m}\) by \(T : U_{\beta} \rightarrow \mathbb{R}^{m}\)

where \(T(r_{1}, r_{m}) = (-r_{1}, r_{m}, -r_{m})\)

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Remark (1).

Since \(\operatorname{rank}(\Lambda^{top}(T^{\ast}M) - M) = 1\),
\(\Lambda^{top}(T^{\ast}M) - M\) has a global nowhere zero section
\(\Leftrightarrow \Lambda^{top}(T^{\ast}M) - M\) is isomorphic to \(M \times \mathbb{R}^{1} - M\).

Explicitly if \(\mu \in \Lambda^{top}(M)\) is a volume form
\[ f: M \times \mathbb{R} \rightarrow \Lambda^{top}(T^{\ast}M) \quad \text{in} \quad f(q, t) = t^{n} \mu. \]

2. For any vector bundle \(E - M\), the zero section
\(z: M - E, \quad z(q) = (q, 0)\) embeds \(M\) into \(E\).

We identify \(M\) with its image in \(E\), (as zero section).

3. If \(E - M\) is isomorphic to \(M \times \mathbb{R}^{k}\), then
\(E - M\) is diffeomorphic to \(M \times (\mathbb{R}^{k} - 0, 0)\).

Hence: (for \(M\) connected) \(M\) is orientable \(\Leftrightarrow \Lambda^{top}(T^{\ast}M) - M\) has exactly two connected components.

\(C^{\infty}\) manifolds with boundary and domains with regular boundary.

Key definition
\[ H^{m} = \{ x \in \mathbb{R}^{m} \mid x_{1} \leq 0 \} \quad \text{closed half-space} \]
with subspace topology.

\(W = \mathbb{R}^{m}\) open and \(f \in C^{\infty}(\mathbb{R}^{m}, \mathbb{R}^{k})\)
\[ \tilde{W} \cap H = W \quad \text{and} \quad \tilde{f}|_{W} = f. \]
\[ f(x) = \begin{cases} \sqrt{x} & \text{if } x < 0 \\ 0 & \text{if } x = 0 \end{cases} \in \mathcal{C}^\infty \text{ on } (-\infty, 0] \subseteq \mathbb{R}. \]

\[ g(x) = \sqrt{1-x^2} \text{ is not } \mathcal{C}^\infty \text{ on } (-\infty, 0]. \]

**Def.** Let \( U, W \subseteq \mathbb{R}^m \) be two open sets.

\[ F : U \rightarrow W \in \mathcal{C}^\infty \text{ if } i \circ F : U \rightarrow \mathbb{R}^m \in \mathcal{C}^\infty \]

Here \( i : W \subset \mathbb{R}^m \) is the inclusion. (read: manifold-with-boundary)

**Def.** A topological space \( M \) is a manifold with boundary if \( V \subseteq M \) open and \( U \subseteq M \) and \( \phi \) an homeomorphism (coordinate chart)

\[ \phi : U \rightarrow W \subseteq \mathbb{R}^m \]

for some open \( W \subseteq \mathbb{R}^m \). We require that if \( \psi : U \rightarrow \mathbb{R}^m, \psi : V \rightarrow \mathbb{R}^m \) are two charts then \( \psi \circ \phi^{-1} : \varphi(U \cap V) \rightarrow \psi(U \cap V) \in \mathcal{C}^\infty \).

**Ex.** \( M = \{ x \in \mathbb{R}^m \mid \sum x_i^2 \leq 1 \} \) is a manifold with boundary.

The square \( [0,1] \times [0,1] \subseteq \mathbb{R}^2 \) is not; it has corners and \( [0,\infty) \times [0,\infty) \) is not diffeomorphic to \( (-\infty, 0] \times \mathbb{R} \).

Stokes theorem is often stated for manifolds with boundary.

To make the statement rigorous we'd need to define tangent bundle of a manifold-with-boundary, decide if it's a vector bundle, develop calculus of vector fields and differential forms etc.

Alternatively we have the following fact:
Fact: Given a manifold with boundary $M$ exist a manifold $\overline{M}$ containing $M$ so that $\forall q \in M \exists$ a coordinate chart $\varphi : U \rightarrow \mathbb{R}^m$, $U \subseteq \overline{M}$ open, with

$$\varphi(U \cap M) = \varphi(U) \cap x_1 \leq 0^*$; \quad \varphi(U) \cap x_1 > 0^*.$$

We call $\varphi : U \rightarrow \mathbb{R}^m$, a coordinate chart adapted to $M$.

Def: (regular domain) A subset $D$ of a manifold $M$ is a regular domain (i.e., a domain with smooth boundary) if $\forall q \in D$ there exists a coordinate chart $\varphi : U \rightarrow \mathbb{R}^m$ on $M$ with $q \in U$ and

$$\varphi(U \cap D) = \varphi(U) \cap x_1 < 0^*. $$

Easy to prove: (1) if $D \subseteq M$ is a regular domain then $D$ is a manifold with boundary.

(2) For a regular domain $D \subseteq M$

$$\partial D := \{ q \in D \mid \exists \text{ open nbd } U \text{ of } q \text{ in } M \text{ with } U \cap (M \setminus D) \neq \emptyset \}$$

is a codimension 1 embedded submanifold of $M$.

In an adapted chart $\varphi : U \rightarrow \mathbb{R}^m$

$$\varphi(\partial D) = \{ x \in \varphi(U) \mid x_1 = 0^* \}.$$

Orientation of $M$ should induce an orientation of $\partial D$.

To do this properly we need:

Lemma 33.1: Let $D \subseteq M$ be a regular domain. Then there is a vector field $\bar{n}$ defined on a nbd of $\partial D$ which points out of $D$.