Next goal: Stokes' Theorem

Recall Fundamental Theorem of calculus says:
\[ \int_{a}^{b} f'(x) \, dx = f(b) - f(a) \quad (\forall f \in C^0([a,b])) \]

We can rewrite this as
\[ \oint_{\partial \Omega} f \, \nu = \int_{\Omega} \nabla \times f \, d\omega \quad (f \in \Omega^0([a,b])) \]

1-dim manifold \( \Omega \) with boundary \( \partial \Omega \)
\( \Omega \) domain with boundary \( \partial \Omega \) in 1-dim manifold \( \mathbb{R} \)

Green's theorem: \( D \subseteq \mathbb{R}^2 \) a domain with smooth boundary
\( \partial D \) then \( \forall p, q \in C^1(D) \)
\[ \int_{D} (p(x,y) \, dx + q(x,y) \, dy) = \int_{\partial D} (- \frac{\partial q}{\partial y} + \frac{\partial p}{\partial x}) \, dx \, dy \]

If we set \( \alpha = p(x,y) \, dx + q(x,y) \, dy \), \( d\alpha = (- \frac{\partial q}{\partial y} + \frac{\partial p}{\partial x}) \, dx \times dy \)
Green's theorem takes the form
\[ \int_{D} d\alpha = \int_{\partial D} \alpha \quad (\forall \alpha \in \Omega^1(D)) \]

(Generalized) Stokes' theorem: \( M \) oriented manifold,
\( D \subseteq M \) a domain with smooth boundary \( \partial D \).
Then \( \forall \omega \in \Omega^{\dim M-1}(M) \)
\[ \int_{D} d\omega = \int_{\partial D} \omega \]
provided \( \partial D \) is given the correct orientation.

Our first step in proving Stokes' theorem is to construct/define the exterior derivative \( d : \Omega^i(M) \to \Omega^{i+1}(M) \)

Notation \( \Omega^* = \bigoplus \Omega^i(M) \)
\[ d_M : \Omega^*(M) \to \Omega^{*+1}(M) \quad (d_M = \bigoplus (d_M : \Omega^i(M) \to \Omega^{i+1}(M))) \]
Theorem. For any manifold \( M \) there is a unique \( \mathbb{R} \)-linear map \( d_m : \Omega^k(M) \to \Omega^{k+1}(M) \) called the exterior derivative so that:
1) \( \forall f \in C^0(M), \quad d_m f = df \)
2) \( \forall U \subseteq M \) open, \( \forall \omega \in \Omega^k(M) \)
   \[ d_m(\omega|_U) = (d_m\omega)|_U \]
3) \( \forall \omega \in \Omega^k(M), \forall \eta \in \Omega^l(M) \)
   \[ d_m(\omega \wedge \eta) = (d_m \omega) \wedge \eta + (-1)^k \omega \wedge (d_m \eta) \]
4) \( d_m \circ d_m = 0 \)

Once we prove that \( d_m \) exists and is unique, we drop the subscript \( m \) and write \( d \) for \( d_m \).

Proof (uniqueness). Suppose \( \forall \) manifold \( M \) we have \( d_m : \Omega^k(M) \to \Omega^{k+1}(M) \) satisfying (1)–(4).

Let \( (x_1, \ldots, x_m) : U \to \mathbb{R}^m \) be a coordinate chart.

Then \( \forall k, \forall \omega \in \Omega^k(M) \)
\[ d_m(\omega|_U) = \sum_{|I| = k} a_I \, dx_I = \sum_{|I| = k} a_I \, dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \quad a_I \in C^0(U) \]

Claim. \( d_U(\partial x_I) = 0 \).

Proof (induction on \( k = |I| \). If \( k = 1 \), \( d_U(\partial x_i) = d_U(\partial (x_i)) = (d_U \circ d_U)(x_i) = 0 \).

Suppose claim holds for all \( I \) with \( |I| = n \).

Then \( \forall i_1 < i_2 < \cdots < i_n < i_{n+1} = I \)
\[ d_U(\partial x_{i_1} \wedge \cdots \wedge dx_{i_{n+1}}) = d_U(\partial x_{i_1}) \wedge (dx_{i_2} \wedge \cdots \wedge dx_{i_{n+1}}) \]
\[ + (-1)^n \, dx_{i_1} \wedge d_U(\partial x_{i_2} \wedge \cdots \wedge dx_{i_{n+1}}) \]
\[ = 0 \quad \text{by inductive assumption} \]

\[ \Rightarrow d_{\partial x_I}(\omega|_U) = d_U(\partial x_I) = \partial x_I \sum_{a_I} a_I \, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \]
\[ = \sum a_I \, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \]
\[ = \sum a_I \, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \]
If \( d'_m : \Omega^i (M) \rightarrow \Omega^{i+1} (M) \) is another map with properties (1)-(4) then \( \forall \alpha \in \Omega^i (M) \), \( \forall \) coordinate chart \( (x_1, \ldots, x_m) : U \rightarrow \mathbb{R}^m \)
\[
(d'_m \alpha) | U = d'_U (2 \pi dx_1) = \sum_i d\alpha_i \wedge dx_i = (d_m \alpha) | U
\]
Since \( U \) is arbitrary, \( d'_m \alpha = d_m \alpha \) on all of \( M \).

**Existence** If \( M \) has a global coordinate chart \( (x_1, \ldots, x_m) : M \rightarrow \mathbb{R}^m \)
we have to define \( d_m \) by
\[
(\ast) \quad d_m (\sum a_i dx_i) = \sum d a_i \wedge dx_i
\]
We need to check that \( d_m \) defined by \( \ast \) has properties (1)-(4).

(1) holds by construction
(2) is easy because \( \forall W \subseteq M \) open
\[
(\sum a_i dx_i) | W = \sum (a_i | W) (dx_i | W)
\]
and
\[
d (a_i | W) = (da_i | W)
\]
(3) is a computation. Since \( d_m \) is linear and \( \wedge \) is bilinear it's enough to consider the case \( \omega = a_i dx_i , \eta = b_j dx_j \) for some multi-indices \( I, J \). Then
\[
a_i dx_i \wedge b_j dx_j = a_i b_j dx_i \wedge dx_j,
\]
\[
d_m (a_i dx_i \wedge b_j dx_j) = d_m (a_i b_j dx_i \wedge dx_j) = d (a_i b_j) \wedge dx_i \wedge dx_j
\]
Now
\[
d (a_i b_j) = da_i \cdot b_j + a_i db_j \quad \text{(product rule)}
\]
\[
\Rightarrow d_m (a_i dx_i \wedge b_j dx_j) = (da_i \cdot b_j) \wedge dx_i \wedge dx_j + a_i d b_j dx_i \wedge dx_j + a_i d b_j dx_i \wedge dx_j = (da_i \wedge dx_i) \wedge (b_j \wedge dx_j) + (-1)^{|I|} a_i dx_i \wedge (db_j \wedge dx_j) = d_m (\omega \wedge \eta + (-1)^{|I|} \omega \wedge \eta)
\]
Here \( |\omega| = |I| = \text{degree of } \omega \).

(4) \( d_m^2 = 0 \) because mixed partials commute:
for any \( \alpha = a_i dx_i \in \Omega^i (M) \)
\[
d_m (d_m \alpha) = d_m (d a_i \wedge dx_i) = d_m \left( \sum_i \frac{\partial a_i}{\partial x_i} dx_i \wedge dx_i \right)
\]
\[
= \left( \sum_i \left( \sum_j \frac{\partial^2 a_i}{\partial x_j \partial x_i} dx_j \wedge dx_i \right) \right) \wedge dx_j
\]
For \( i=j \) \( \text{d}x_i \wedge \text{d}x_j = 0 \).

For each \( i \neq j \), we have two terms: \( \frac{\partial^2 q_i}{\partial x_j \partial x_i} \text{d}x_j \wedge \text{d}x_i \) and \( \frac{\partial^2 q_i}{\partial x_i \partial x_j} \text{d}x_i \wedge \text{d}x_j \).

Since \( \text{d}x_i \wedge \text{d}x_j = -\text{d}x_j \wedge \text{d}x_i \),

\[
\frac{\partial^2 q_i}{\partial x_j \partial x_i} \text{d}x_j \wedge \text{d}x_i + \frac{\partial^2 q_i}{\partial x_i \partial x_j} \text{d}x_i \wedge \text{d}x_j = 0.
\]

\[\Rightarrow \quad d_M (\text{d}M \alpha) = 0, \quad \text{i.e.} \quad dM \circ dM = 0.\]

**General case:** Given a manifold \( M \) and \( \alpha \in \Omega^i (M) \),

cover \( M \) by coordinate charts \((U_i; U_x \rightarrow \mathbb{R}^m)\).

For \( q \in U_x \) define \( (dM \alpha)_q = (dU_x (\alpha | U_x))_q \).

For \( q \in U_x \cap U_y \)

\[
(dU_x (\alpha | U_x))_q = (dU_y (\alpha | U_y)(x | U_y))_q = (dU_x (\alpha | U_x)(x | U_x))_q,
\]

\[
= (dU_y (\alpha | U_y))_q = \ldots = (dU_y (\alpha | U_y))_q. \Rightarrow dM \alpha \text{ is well-defined}.
\]

Remains to check: \( dM \) defined this way has properties (1)-(4).

This is not hard, but takes time.

**Example:** \( \alpha = x \, dy - y \, dx \in \Omega^1 (\mathbb{R}^2) \)

\[d\alpha = d(x \wedge dy - y \wedge dx) = 2 \, dx \wedge dy \]

\[\beta = \frac{1}{x^2 + y^2} (x \, dy - y \, dx) \in \Omega^1 (\mathbb{R}^2 \setminus \{0\}) \]

\[d\beta = d \left( \frac{x}{x^2 + y^2} \right) \wedge dy - d \left( \frac{y}{x^2 + y^2} \right) \wedge dx = \frac{2}{x^2 + y^2} \left( \frac{x}{x^2 + y^2} \right) \wedge dx \wedge dy - \frac{2}{x^2 + y^2} \left( \frac{y}{x^2 + y^2} \right) \wedge dx \wedge dy = 0.\]
Remarks. There are several other ways to define/construct the exterior derivative:

1) One can show: if \( \alpha \in \Omega^n(M) \), \( X_1 \cdots X_{n+1} \in \Gamma(TM) \) (and \( n \geq 0 \))

\[
\mathsf{d}\alpha (X_1, \cdots, X_{n+1}) = \sum_i (-1)^{i+1} X_i (\alpha (X_1, \cdots, \widehat{X}_i, \cdots, X_{n+1})) + \\
\sum_{i<j} (-1)^{i+j} \alpha ([X_i, X_j], X_1, \cdots, \widehat{X}_i, \cdots, \widehat{X}_j, \cdots, X_{n+1})
\]

\[
\text{If } n = 0, \text{ this is } \mathsf{d}
\]

\[
(\mathsf{d}f)(x) = X(f),
\]

2) If \( \alpha \in \Omega^n(M) \), \( \psi : M \to \Lambda^n(T^*M) \)

One can manufacture \( \mathsf{d}\alpha : M \to \Lambda^{n+1}(T^*M) \) out of

\( T\alpha : TM \to T(\Lambda^n(T^*M)) \) . . .