Last time - Defined the product category $C \times D$; $C, D$ two categories.

\[ C^n = C \times \ldots \times C \text{ makes sense.} \]

- Defined functors $F: C \to D$ as maps that preserve identities $(F(1_C) = 1_{F(C)})$

and composition:

\[ F(g \circ f) = F(g) \circ F(f). \]

Introduced $\text{Vect}^{\text{iso}}$. Objects are finite dim. vector spaces over $\mathbb{R}$.

Morphisms = isomorphisms of vector spaces.

**Purpose:** to define smooth functors.

These are functors $F: (\text{Vect}^{\text{iso}})^n \to \text{Vect}^{\text{iso}}$.

So that $\forall (V_i; V_n) \in \text{Objects of } (\text{Vect}^{\text{iso}})^n$

\[ F: \text{Hom}(V_1, V_n), (V_i, V_n)) \to \text{Hom}(F(V_1, V_n), F(V_i, V_n)) \]

\[ \text{GL}(V_1)^n \times \ldots \times \text{GL}(V_n) \]

in a $C^\infty$ map, hence a map of Lie groups.

**Aside:** A morphism $f: X \to Y$ in a category $C$ is an iso if $f \circ g: Y \to X$ is in $C$ and $g \circ f: X \to Y$ is in $C$.

If $F: C \to D$ is a functor and $f: X \to Y \in C$, $f$ is an iso, then $F(f)$ in an iso. (prove it!)

Also proved/noted: if $f: M \times V \to M \times V$ is an iso of product bundles, then $f$ is of the form

\[ f(m, v) = (m, \phi(m, v)) \]

where $\phi: M \to \text{GL}(V)$ is $C^\infty$.

Why is this important/useful?
Vector bundles and cocycles.

Let $E \rightarrow M$ be a vector bundle of rank $k$, with local trivializations $\{ \varphi_x : E|U_x \rightarrow U_x \times \mathbb{R}^k \}$. Set $U_{x\beta} = U_x \cap U_\beta$, $U_{x\beta\gamma} = U_x \cap U_\beta \cap U_\gamma$. Then

$$\varphi_x \circ \varphi_\beta^{-1} : U_{x\beta} \times \mathbb{R}^k \rightarrow U_{x\beta} \times \mathbb{R}^k$$

is of the form

$$(\varphi_x \circ \varphi_\beta^{-1})(q, v) = (q, \varphi_{x\beta}(q)v)$$

for some $\varphi_{x\beta} : U_{x\beta} \rightarrow GL(k, \mathbb{R})$, a $C^\infty$ map. Easy to see (cocycle conditions).

Lemma 29.1 Given a manifold $M$, an open cover $\{ U_\alpha \}$ and a collection of $C^\infty$ maps $\{ \varphi_{x\beta} : U_{x\beta} \rightarrow GL(k, \mathbb{R}) \}$ satisfying cocycle conditions, $\exists$ a vector bundle $E \rightarrow M$ of rank $k$ with transition maps $\{ \varphi_{x\beta} \}$.

Sketch of proof Let $E = \bigcup (U_x \times \mathbb{R}^k)$. Define a relation $\sim$ on $E$ by

$$U_x \times \mathbb{R}^k \ni (q, v) \sim (q', v') \in U_\beta \times \mathbb{R}^k \iff q = q' \text{ and } \varphi_{x\beta}(q)v' = v.$$  

Cocycle condition $\Rightarrow \sim$ is an equivalence relation. Let $E = E/\sim$. It's the desired vector bundle. $\blacksquare$

Lemma 19.2 Suppose $E' \rightarrow M$, $E^2 \rightarrow M$ are two vector bundles with local trivializations $\{ \varphi_x^i : E^i|U_x \rightarrow U_x \times \mathbb{R}^k \}$ so that $\varphi_{x\beta}^1 = \varphi_{x\beta}^2 \forall x, \beta$. Then $E'$ is isomorphic to $E^2$ (as v. bundles over $M$).

Proof $\varphi_{x\beta}(q, v) = U_{x\beta} \times \mathbb{R}^k$
\[(\varphi^1_\alpha \circ (\varphi^2_\beta)^{-1})(q, v) = \varphi^1_\alpha(q) v = \varphi^2_\alpha(q) v = (\varphi^2_\alpha \circ (\varphi^2_\beta)^{-1})(q, v).\]

Define an iso \( f : E^1 \rightarrow E^2 \) by setting
\[f(E^1|_{U_{\alpha}}) = (\varphi^2_\alpha)^{-1} \circ \varphi^1_\alpha.\]

Since \( \varphi^1_\alpha = \varphi^2_\alpha \circ (\varphi^1_\beta)^{-1} \)
\[\varphi^1_\alpha \circ (\varphi^1_\beta)^{-1} = \varphi^2_\alpha \circ (\varphi^2_\beta)^{-1}.\]

Then \( \varphi^1_\alpha \circ E^1|_{U_{\alpha}} = (E^1|_{U_{\alpha}}) \cap (E^1|_{U_{\beta}}) \)
\[(\varphi^2_\alpha)^{-1} \circ \varphi^1_\alpha \circ (\varphi^1_\beta)^{-1} \circ \varphi^2_\beta = \]
\[= (\varphi^2_\alpha)^{-1} \circ \varphi^2_\alpha \circ (\varphi^2_\beta)^{-1} \circ \varphi^2_\beta = \text{id}\]
\[\Rightarrow f \text{ is well-defined.}\]

\( f \) is an iso by construction. \( \square \)

**Theorem 29.3** Let \( \mathcal{F} : \text{Vect}^{\text{iso}} \rightarrow \text{Vect}^{\text{iso}} \) be a \( C^0 \) functor.

Then, \( n \)-tuple of vector bundles \( (E^i) \rightarrow M \)
with transition maps \( \{ \varphi^i_\alpha : U_{\alpha} \rightarrow GL(R^k, IR^n) \} \), \( i = 1, \ldots, n. \)

\( \exists \) vector bundle \( \mathcal{F}(E^1, \ldots, E^n) \rightarrow M \) with transition maps
\( \{ \mathcal{F}_0(\varphi^1_\alpha, \ldots, \varphi^n_\alpha) : U_{\alpha} \rightarrow GL(F(R^k, FR^n)) \} \)

The fiber \( \mathcal{F}(E^1, \ldots, E^n) \) at \( q \in M \) is (isomorphic to ) \( \mathcal{F}(E^q, \ldots, E^q). \)

**Proof**. There is a cover \( \{ U_{\alpha} \} \) of \( M \) of \( E^i|_{U_{\alpha}} \) is trivial.

Let \( \varphi^i_\alpha : E^i|_{U_{\alpha}} \rightarrow U_{\alpha} \times IR^n \) be the corresponding trivialization maps and \( \{ \varphi^i_\alpha : U_{\alpha} \rightarrow GL(IR^k) \} \)
the corresponding transition maps.

They satisfy the cocycle conditions.

Since \( \mathcal{F} \) is a functor
\( \mathcal{F}_0(\varphi^1_\alpha, \ldots, \varphi^n_\alpha) : U_{\alpha} \rightarrow GL(F(IR^k, IR^n)) \)
also satisfy the cocycle conditions.
Since $F$ is a $C^\infty$-functor, the maps
\[ F_0 \left( y^{\alpha_0}, \ldots, y^{\alpha_p} \right) : U_\alpha \to GL(\mathbb{R}^n) \]
are $C^\infty$. By Lemma 29.1, they define a vector bundle. Note that by Lemma 29.2, this bundle is unique up to an isomorphism. \( \square \)

Proof 2. Consider the set $E = \bigcup_{\alpha \in \mathcal{M}} F(E_\alpha, \ldots, E_\alpha)$.

For each $\alpha$, $\forall \beta \in U_\alpha$, we have isomorphisms
\[ \psi_\alpha(\beta) = F \left( y^{\alpha_0}(\beta), \ldots, y^{\alpha_p}(\beta) \right) : \mathbb{R}^n \to \mathbb{R}^n \]
\[ E(E_\alpha, \ldots, E_\alpha) \to \mathbb{R}^n \times F(\mathbb{R}^{k_1}, \ldots, \mathbb{R}^{k_n}) \]
This gives us candidate trivializations
\[ \phi_\alpha : E \mid U_\alpha \to \mathbb{R}^n \times F(\mathbb{R}^{k_1}, \ldots, \mathbb{R}^{k_n}) \]

Since $F$ is a functor,
\[ \left( \psi_\alpha \circ \psi^{-1}_\beta \right) (\alpha, \beta) = F \left( y^{\alpha_0}(\alpha), \ldots, y^{\alpha_p}(\alpha) \right) \cdot \left( y^{\alpha_0}(\beta), \ldots, y^{\alpha_p}(\beta) \right) \]
Since $F$ is $C^\infty$, this shows that
\[ \psi_\alpha \circ \psi^{-1}_\beta : U_\alpha \times F(\mathbb{R}^{k_1}, \ldots, \mathbb{R}^{k_n}) \to U_\beta \times F(\mathbb{R}^{k_1}, \ldots, \mathbb{R}^{k_n}) \]
are $C^\infty$.

Technical result from lecture 27 \( \Rightarrow \) $E$ is a manifold, and $\phi_\alpha : E \mid U_\alpha \to \mathbb{R}^n \times F(\alpha)$ are all diffeos.

\( \Rightarrow \) $E$ is a vector bundle with transition maps
\[ \phi_\beta \circ \phi^{-1}_\alpha = F \left( y^{\alpha_0}(\beta), \ldots, y^{\alpha_p}(\beta) \right) \ldots F \left( y^{\alpha_0}(\alpha), \ldots, y^{\alpha_p}(\alpha) \right) \]
\( \square \)