Last time, I finished proving that if $M$ is an oriented manifold, there is a well-defined integration map:

$$f_m : \Omega^c_\text{top}(M) \to \mathbb{R}, \quad M \to f_m$$

2) Defined vector bundles, maps between two vector bundles over the same manifold $M$, product bundles.

**Def.** A vector bundle $\pi : E \to M$ is trivial if it is isomorphic to a product bundle $p_1 : M \times V \to M$.

($V$ = vector space of dimension = rank $E$)

**Ex.** For any Lie group $G$, the tangent bundle $T_G \to G$ is trivial.

**Remarks.** If $\pi : E \to M$ is a vector bundle and $W \subseteq M$ is open, then we can restrict $E$ to $W$. Namely:

$$E \mid W := \pi^{-1}(W) \subseteq W$$

It's again a vector bundle.

2) We can rephrase the definition of the vector bundle as saying: vector bundles are locally trivial.

Namely if $\pi : E \to M$ is a vector bundle, then $U \times \mathbb{R}^k$ open and $U \times \mathbb{R}^k \subseteq E$ is trivial. (Local trivializations are isomorphisms with product bundles.)

3) If $\pi : E \to M$ is a vector bundle then $\pi$ is a surjective submersion.

Reason: true for product bundles $\Rightarrow$ true for trivial bundles.
true locally for any vector bundle. \(\Rightarrow\) true.

**Definition** A section of a vector bundle \(\pi: E \to M\) is a map \(s: M \to E\) with \(\pi \circ s = \text{id}_E\).

**Notation** \(\Gamma(E)\) = space of sections of \(E \to M\).

**Example**

\[
\begin{align*}
\Gamma(TM) &= \text{vector fields} \\
\Gamma(T^*M) &= 1\text{-forms} \\
\Gamma(A^k(T^*M)) &= k\text{-forms} \\
\Gamma(M \times V \to M) &= \mathcal{C}^\infty(M, V) = \text{smooth maps from } M \text{ to } V.
\end{align*}
\]

**Exercise** Let \(E \to M\) be a vector bundle. A map of sets \(s: M \to E\) with \(\pi \circ s = \text{id}\) is \(\mathcal{C}^\infty\).

A local trivialization \(\psi: E|_U \to U \times \mathbb{R}^k\) of \(E\)

\(\psi s: U \to U \times \mathbb{R}^k\) is \(\mathcal{C}^\infty\).

**Corollary** The space of sections \(\Gamma(E)\) of a vector bundle \(E \to M\) is a \(\mathcal{C}^\infty(M)\) module. In particular \(\forall f \in \mathcal{C}^\infty(M)\), \(f \circ \Gamma(E)\) is again a \(\mathcal{C}^\infty\) defined by \((f \circ s)(q) = f(q) \circ s(q)\) is again a smooth section of \(E \to M\).

**Definition** A local section of a vector bundle \(E \to M\) is a section of the restriction \(E|_U\) for some \(U \subset M\) open.
$X \cong \text{a manifold}$

For each candidate chart of $X$.

Condition (iii) guarantees that the transition maps $g_{ij} = g(x_i^{-1} \circ x_j) : \mathbb{U}_i \cap \mathbb{U}_j \to \mathbb{N}$ are homeomorphisms.

Show $\mathbb{R}^n \times X$ is a manifold & coordinate charts.

Given $x \in X$, $\mathbb{R}^n \times x \subset \mathbb{R}^n \times X$.

**Theorem (topology)**

Proof. The set $\{ U_i \}$ is an open cover.

First step. Suppose we have a set $X$, a cover $\{ \mathbb{U}_i \}$ of $X$, and a collection of bijections $g_i : \mathbb{U}_i \to \mathbb{N}^n$. Then, to construct $\mathbb{R}^n \times X$.

Given two vector bundles $E \oplus M \to \mathbb{F}$, define $\text{Hom}(E, \mathbb{F})$ by:

- $\mathbb{E} \cong \mathbb{M}$
- $\text{Hom}(E, \mathbb{F}) = \{ f : \mathbb{E} \to \mathbb{F} \}$
- $\text{Hom}(\mathbb{E}, \mathbb{F}) = \mathbb{F}$

Conclude $\mathbb{E} \cong \mathbb{M}$ with $\mathbb{F}$. Then $\text{Hom}(E, \mathbb{F})$ is a vector bundle $E \oplus M \to \mathbb{F}$, with $\text{Hom}(E, \mathbb{F})$ being the fiber.
Categories, functors and "smooth" functors.

Definition. A category \( \mathcal{C} \) is

- a collection of objects \( \mathcal{C}_0 \),
- for each pair of objects \( X, Y \in \mathcal{C}_0 \) a set \( \text{Hom}(X,Y) \) of morphisms/arrow [which can be empty]
- for each triple of objects \( X, Y, Z \) a composition
  
  \[ \circ : \text{Hom}(Y,Z) \times \text{Hom}(X,Y) \to \text{Hom}(X,Z) \]

  \[ ((z \overset{g}{\leftarrow} Y), (y \overset{f}{\leftarrow} X)) \mapsto (z \overset{g \circ f}{\leftarrow} X) \]

- for each object \( X \) of \( \mathcal{C} \) a morphism \( 1_X \in \text{Hom}(X,X) \)

such that

1) \( \forall f \in \text{Hom}(X,Y) \quad 1_Y \circ f = f = f \circ 1_X \)

2) \( \circ \) is associative

\[ h \circ (g \circ f) = (h \circ g) \circ f \quad \forall \ W \overset{h}{\leftarrow} Z \overset{g}{\leftarrow} Y \overset{f}{\leftarrow} X \]

We write \( \mathcal{C}_1 := \coprod_{X,Y \in \mathcal{C}_0} \text{Hom}(X,Y) \) = collection of morphisms of \( \mathcal{C} \).

\( \text{Ex} \quad \mathcal{C} = \text{Man}, \) the category of manifolds

objects = manifolds

morphisms = smooth maps, i.e. maps of manifolds

\( \text{Ex} \quad \text{Vect} \ (\equiv \text{Vector}) \) objects = vector spaces over \( \mathbb{R} \)

morphisms = linear maps

\( \text{Ex} \quad \text{Set} \) objects are sets, morphisms are functions.

**WARNING** There are plenty of categories where objects are not "sets with extra structure."
Cheapest examples are preorders

**Def.** Let $P$ be a set. A binary relation $\preceq \subseteq P \times P$ is a preorder if:
1. $\forall a \in P \quad a \preceq a$ (reflexivity)
2. $(a \preceq b \land b \preceq c) \Rightarrow a \preceq c$ (transitivity)

A preorder $(P, \preceq)$ is a category:
- objects are elements of $P$
- $\text{Hom}(a, b) = \{ * \}$ if $a \preceq b$
- $\emptyset$ otherwise

"Better" example: Rel.
- objects are sets
- $\text{Hom}(X, Y) = \text{relations from } X \text{ to } Y$
  - arbitrary subset of $Y \times X$
- composition is defined by:
  $$(Z \preceq Y) \circ (Y \preceq X) = \{ (z, x) \in Z \times X \mid \exists y \in Y : (z, y) \in S, (y, x) \in R \}.$$