Last time:

1. Defined an algebra $\Omega^i(M)$ of differential forms on a manifold $M$

$$\Omega^i(M) = \bigoplus_{q=0}^{\infty} \Lambda^q(T_q^* M),$$

Since $\Lambda^i(V) = \bigoplus_{q=0}^{\infty} \Lambda_i^q(V)$, for any vector space $V$

$$\Omega^i(M) = \bigoplus_{i=0}^{\infty} \Omega^i(M)$$

where $\Omega^i(M) := \text{sections of } \Lambda^i(T^* M) \to M$.

2. Differential forms pull back:

If $F: M \to N$ is a map, $\omega \in \Omega^i(N)$

Then we have $F^* \omega \in \Omega^i(M)$:

$$(F^* \omega)_q = \Lambda^i (\partial F_q)^* (\omega_{F(q)}) + q \epsilon_M.$$

Equivalently, $$(F^* \omega)_q (v_1, \ldots, v_i) = \omega_{F(q)} (df_q v_1, \ldots, df_q v_i) + v_i - v_i + T_q M.$$

For $f \in \Omega^0(N) = C^\infty(N)$, $F^* f = f \circ F$.

We proved: $\forall f \in C^\infty(N)$

$$F^* (df) = d (F^* f)$$

Note $\forall \alpha \in \Omega^i(N), \beta \in \Omega^j(N)$

$$F^* (\alpha \wedge \beta) = (F^* \alpha) \wedge (F^* \beta)$$

Goal for today: define integration of top degree compactly supported differential forms over oriented manifolds.
Remark. Suppose \([a, b] \subseteq \mathbb{R}, f \in C^\infty([a, b])\) (i.e. \(f \in C^\infty(a-e, b+e)\) with \(f|_{[a,b]} = f\)).

Then the integral (Riemann, Lebesgue) \(\int_{[a,b]} f\) makes sense.

Moreover,
\[
\int_{[a,b]} f = \int_a^b f(x) \, dx = -\int_b^a f(x) \, dx.
\]

Moral. In 1-variable calculus there are two types of integrals. One depends on the orientation of \([a,b]\) and the other doesn't: \(\int_a^b f(x) \, dx\) is secretly the integral of the 1-form \(f(x) \, dx\).

Remark. If \(M = \mathbb{R}^m, \mu \in \Omega^m_c(\mathbb{R}^m)\) (we suppose \(\mu\) is compact), then \(\mu = \int f \, dx_1 \wedge \cdots \wedge dx_m\) for some \(f \in C^\infty(\mathbb{R}^m)\).

So we define
\[
\int_{\mathbb{R}^m} f \, dx_1 \wedge \cdots \wedge dx_m := \int_{\mathbb{R}^m} f(\cdot) \, dx_1 \wedge \cdots \wedge dx_m.
\]

Now suppose \(M\) is a manifold, \(\phi : U \to \mathbb{R}^m\) a coordinate chart, \(\mu \in \Omega^m_c(M)\) with \(\text{supp} \, \mu \subseteq U\).

We could try to define
\[
\int_M \mu = \int_{\phi(U)} (\phi^{-1})^* \mu.
\]

Issue. Suppose \(\phi : U \to \mathbb{R}^m\) is another coordinate chart. Is it true that
\[
\int_{\phi(U)} (\phi^{-1})^* \mu = \int_{\phi(U)} (\phi^{-1})^* \mu?
\]
Easy fact: \( M \xrightarrow{F} N \xrightarrow{H} \Omega \) are two maps and \( \alpha \in \Omega^\ast\!(\Omega) \), \((H \circ F) \ast \alpha = F \ast (H \ast \alpha)\)

This is really the chain rule, in disguise.

Since \( \psi^{-1} \circ \psi \) commutes,

\[
\begin{align*}
(\psi^{-1})^\ast \mu &= (\psi^{-1} \circ \psi \circ \psi^{-1})^\ast \mu = (\psi \circ \psi^{-1})^\ast (\psi^{-1})^\ast \mu \\
\text{So the issue reduces to:} & \quad 0, 0' \subset \mathbb{R}^m \text{ open} \\
F: 0 \rightarrow 0' \text{ a diffeo, } & \quad \nabla \in \Omega^m_c (0'). \quad \text{Is} \\
\int_{0'} F^\ast \nu = \int_0 \nu \text{ ?} & \quad \text{(Two step answer)}
\end{align*}
\]

I. Proposition 24.2 (I meant to prove it last time)

Suppose \( 0, 0' \subset \mathbb{R}^m \) are open, \( F: 0 \rightarrow 0' \) is \( C^\infty \).

Then

\[
F^\ast (f(y) \, dy_1 \wedge \cdots \wedge dy_m) = f(F(x)) (\det DF(x)) \, dx_1 \wedge \cdots \wedge dx_m
\]

Proof: \( \forall x \in 0, \forall v_1 \ldots v_m \in T_x 0 \cong \mathbb{R}^m \)

\[
(F^\ast (f(y) \, dy_1 \wedge \cdots \wedge dy_m))(v_1, \ldots, v_m) = \left( f(dy_1 \wedge \cdots \wedge dy_m) \right)_{x=0} \left( DF_x (v_1), \ldots, DF_x (v_m) \right)
\]

\[
= f(F(x)) \, e_1^\ast \wedge \cdots \wedge e_m^\ast (DF(x) v_1, \ldots, DF(x) v_m)
\]

Where \( e_j^\ast \) is the standard basis of \((\mathbb{R}^m)\)

\[
= (f(F(x)) \cdot \det DF(x)) \, e_1^\ast \wedge \cdots \wedge e_m^\ast (v_1, \ldots, v_m)
\]

(by Homework)

\[
= f(F(x)) \cdot \det DF(x) \cdot (dx_1 \wedge \cdots \wedge dx_m)_{x=0} (v_1, \ldots, v_m)
\]

\[
= \text{For } P = f(y) \, dy_1 \wedge \cdots \wedge dy_m \text{, we are asking:}
\]

\[
\int_0 \frac{f(F(x)) \cdot \det DF(x) \cdot dx_1 \wedge \cdots \wedge dx_m}{\text{measure!}} = \int_{F(0)} \frac{f(y) \cdot dy_1 \wedge \cdots \wedge dy_m}{\text{measure not done!}}
\]
Change of variables formula says: if \( F : \Omega \rightarrow \Omega' \) is a diffeo (so that \( \Omega' = F(\Omega) \)) and \( f \in C^\infty_c(\Omega') \), then

\[
\int_{F(\Omega)} f(y) \, dy_1 \cdots dy_m = \int_{\Omega} f(F(x)) \cdot \det \text{DF}(x) \, dx_1 \cdots dx_m
\]

if \( \det \text{DF}(x) > 0 \) for \( x \in \Omega \).

**Definition** \( F : \Omega \rightarrow \Omega' \) is orientation-preserving diffeo if \( \det \text{DF}(x) > 0 \) for \( x \in \Omega \).

**Definition (first pass)** A manifold \( M \) is oriented if there's an atlas \( \{ \varphi_a : U_a \rightarrow \mathbb{R}^m \} \) so that for all \( a, b \)

\[
\varphi_b \circ \varphi_a^{-1} : \varphi_a(U_a \cap U_b) \rightarrow \varphi_b(U_a \cap U_b)
\]

are orientation preserving.

**Lemma 25.1** Above discussion shows: Suppose \( \psi, \varphi : U \rightarrow \mathbb{R}^m \) are two coordinate charts on \( M \) with \( \psi \circ \varphi^{-1} \) orientation preserving. Then \( \psi \circ \varphi^{-1} \) induces an orientation on \( \Omega^m_{\mathbb{R}}(U) \)

\[
\int_{\mathbb{R}^m} \psi^* \omega = \int_{\mathbb{R}^m} \varphi^* \omega
\]

Next time: Suppose \( M \) is an oriented manifold (we've chosen an atlas such that transition maps preserve orientation). Then \( \Omega^m_{\mathbb{R}}(M) \)

\[
\int_M \omega
\]

makes sense.