Last time:  
- Constructed \( \otimes : V \times W \to V \otimes W \), \((v,w) \mapsto v \otimes w\)

By construction the set \( \{ v \otimes w \mid v \in V, w \in W \} \) spans \( V \otimes W \).

- Proved: if \( \{ v_i \} \) is a basis of \( V \), \( \{ w_j \} \) of \( W \) then
  \( \{ v_i \otimes w_j \} \) is a basis of \( V \otimes W \).

- \( \text{Hom}(V \otimes W, U) \to \text{Mult}(V, W; U), A \to A \otimes \)
in an iso of vector spaces.

Should have proved last time!

Lemma: For any two finite dimensional vector spaces \( V, W \)
there is an iso \( V^* \otimes W \to \text{Hom}(V, W) \) with
\[
\phi(l \otimes w) = l(w).
\]

Sketch of proof: The map \( V^* \otimes W \to \text{Hom}(V, W) \), \((l, w) \mapsto l(w)\)
in bilinear, hence induces linear map \( \phi : V^* \otimes W \to \text{Hom}(V, W) \).

Given a basis \( \{ v_i^* \} \) of \( V^* \), \( \{ w_j \} \) of \( W \), \( \{ v_i^* \otimes w_j \} \) is a basis
of \( \text{Hom}(V, W) \). Since \( \{ v_i \otimes w_j \} \) is a basis of \( V \otimes W \),
\( \phi \) is an iso.

QED

Remark: Similar argument shows: there is an iso
\( V^* \otimes W^* \to \text{Mult}(V, W; \mathbb{R}) \), with
\[
(\ell \otimes \mu) \mapsto \ell(\cdot) \mu(\cdot).
\]

Consequently
\[
V \otimes W \cong \text{Mult}(V^*, W^*; \mathbb{R}).
\]

Note: The canonical map \( V \times W \to \text{Mult}(V^*, W^*; \mathbb{R}) \) is
\[
(v, w) \mapsto \langle \cdot, v \rangle \ \cdot \ \langle \cdot, w \rangle
\]
where \( \langle \cdot, v \rangle : V^* \times V \to \mathbb{R} \), \( \langle \cdot, w \rangle : W^* \times W \to \mathbb{R} \) are
canonical pairings.

Recall: An algebra over \( \mathbb{R} \) is a vector space \( A \) together
with a bilinear map \( \cdot : A \times A \to A \).
The algebra \((A, \ast)\) is associative if
\[a \ast (b \ast c) = (a \ast b) \ast c \quad \forall a, b, c \in A.\]
Not all algebras are associative. For example Lie algebras are not associative.

Def. A graded vector space is a direct sum \(\bigoplus_{i=0}^{\infty} V_i\)
(gauging by \(\mathbb{Z}^2\); elements are sequences \((v_i)_{i \geq 0}\), \(v_i \in V_i\)
with \(v_i = 0\) for all but finitely many \(i\))

An algebra is graded if it is a graded vector space: \(A = \bigoplus A_i\)
and \(a_i \in A_i, a_j \in A_j\) \(a_i \ast a_j \in A_{i+j}\).

We'll be interested in several graded associative algebras:
The tensor algebra \(T(V) = \bigoplus_{i \geq 0} V \otimes^i\)
The Grassmann algebra \(\Lambda^*(V) = \bigoplus_{k \geq 0} \Lambda^k V\).

What's \(V \otimes^k\)? \(V \otimes^0 := IR, V \otimes^1 = V, V \otimes^2 = V \otimes V\).
For \(k \geq 2\) we can construct a vector space \(V \otimes^k\) together with
a \(k\)-linear map \(V^k = V \times \cdots \times V \xrightarrow{\otimes^k} V \otimes^k\) so that
\[\text{Hom} \left( V \otimes^k, U \right) \xrightarrow{\sim} \text{Mult} \left( V, \cdots, V; U \right)\]
We write \(v_1 \otimes \cdots \otimes v_k\) for \(\otimes^k(v_1, \cdots, v_k)\).

We define (graded) multiplication
\[\cdot_{k, l}: V \otimes^k \times V \otimes^l \to V \otimes^{k+l}\]
As follows: the map \(V^k \times V^l \to V^k \otimes^l\)
\((v_1, \cdots, v_k, v_{k+1}, \cdots, v_{k+l}) \mapsto v_1 \otimes \cdots \otimes v_k \otimes v_{k+1} \otimes \cdots \otimes v_{k+l}\)
in \(k+l\) linear.

Fix \((v_1^k, \cdots, v_l^k) \in V^k\). We get a \(k\)-linear map
\[ V^k \otimes (V_1, \ldots, V_k) \rightarrow V \otimes \cdots \otimes V_k \otimes V_{k+1} \cdots \otimes V_{k+l} = V \otimes (k+l) \]

\[ \sim \Phi: V^k \otimes V^l \rightarrow V^{(k+l)} \]

with \((V \otimes \cdots \otimes V_k, (V_{k+1}, \ldots, V_{k+l})) \rightarrow V \otimes \cdots \otimes V_k \otimes V_{k+1} \cdots \otimes V_{k+l}\).

For each fixed \( t \in V^k \), the map \( \Phi(t, -): V^l \rightarrow V^{(k+l)} \) is \( k \)-linear.

And it's linear at \( t \), so we get a bilinear map

\[ \circ_{k+l} : V^k \otimes V^l \rightarrow V^{(k+l)} \]

with \( \circ_{k+l} (V \otimes \cdots \otimes V_k, V_{k+1} \otimes \cdots \otimes V_{k+l}) = V \otimes \cdots \otimes V_k \otimes V_{k+1} \cdots \otimes V_{k+l} \).

This family of maps \( \circ_{k+l} \) for \( k, l \geq 0 \) define a bilinear map

\[ \circ: \mathcal{T}(V) \times \mathcal{T}(V) \rightarrow \mathcal{T}(V) \]

\[ ((t_i), (s_j)) \mapsto (- \sum_{k+l=n} \circ_{k+l} (t_k, s_l)) \]

**Issue:** Associativity of \( \circ \).

**Two solutions:**

I. Let \( \text{Mult}^k(V^*, R) := \text{Mult}(V^* \otimes \cdots \otimes V^* \otimes R) \) space of \( k \)-linear maps.

Prove: \( V^k \otimes \text{Mult}^k(V^*, R) \)

\[ \otimes \text{Mult}^k(V^*, R) \]

and \( V^k \otimes V^l \rightarrow \text{Mult}^k(V^*, R) \times \text{Mult}^l(V^*, R) \)

\[ \circ_{k+l} : \text{Mult}^k(V^*, R) \times \text{Mult}^l(V^*, R) \rightarrow \text{Mult}^{k+l}(V^*, R) \]

commutes.

Since multiplication of multilinear maps is associative, we're done.

II. We need to check

\[ \circ_{k+l, m} : \text{Mult}^k(V^*, R) \times \text{Mult}^l(V^*, R) \rightarrow \text{Mult}^{k+l+m}(V^*, R) \]

\[ \circ_{k+l, m} \text{ commutes} \]
By construction of $\rho_{k+l}$, 
\[ \rho_{k+l}(\otimes^k, \otimes^l) \rightarrow \otimes^{k+l} \]
commutes.

Hence 
\[ \otimes^{k+l+m} \rightarrow \otimes^{k+l} \times \otimes^m \]
commutes.

On the other hand, 
\[ \otimes^{k+l+m} \rightarrow \otimes^{k+l+m} \]
commutes.

Similarly, 
\[ \otimes^{k+l+m} \rightarrow \otimes^{k+l} \times \otimes^m \]
commutes.

\[ \Rightarrow (A) \] commutes on triples of the form
\[ (\otimes^{v_{k+l}} \otimes^{v_{k+l+1}} \otimes^{v_{k+l+m}}) \]
These generate 
\[ \otimes^{v_k} \times \otimes^l \times \otimes^m \]
\[ \Rightarrow (A) \] commutes.