Last time: Defined a manifold as a topological space with an equivalence class of $C^2$ atlases.

Most of the time we want our manifolds to be \textit{Hausdorff}.

Recall what that means:

A topological space $X$ is \textit{Hausdorff} if $\forall x, y \in X, x \neq y$, $\exists U, V \subseteq X$ open with $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Example of a non-Hausdorff manifold:

Let $\gamma = \mathbb{R} \times \{ -1, 1 \}$ and $X = \gamma / \sim$

where $(x, 1) \sim (y, -1) \iff (x = y$ and $x \neq 0)$:

\[
\begin{array}{c}
\bullet (0, 1) \\
\bullet (0, -1) \\
\bullet (2, 1) \\
\end{array}
\]

$X$ is a "line with two origins".

The two inclusions $\iota_1 : \mathbb{R} \rightarrow \mathbb{R} \times \{ \pm 1 \}$, $\iota_1(x) = (x, 1)$

$\iota_{-1}(x) = (x, -1)$

followed by projection/quotient map

$\pi : \mathbb{R} \times \{ \pm 1 \} \rightarrow (\mathbb{R} \times \{ \pm 1 \}) / \sim$

are injective. So let $U_i = \pi(\mathbb{R} \times \{ \pm 1 \})$ $i = \pm 1$.

Then $\pi\iota_i : \mathbb{R} \rightarrow U_i$ are bijections

$X = U_1 \cup U_2$

$\varphi_i = (\pi\iota_i)^{-1} : U_i \rightarrow \mathbb{R}$ are coordinate charts.

$(\varphi_k \circ \varphi_i^{-1})(x) = x$ $\forall x \in \mathbb{R}$ $\forall i, k = \pm 1$.

$\varphi_i : U_i \rightarrow \mathbb{R}$ $i = \pm 1$ is an atlas.

$X$ is \textit{not} Hausdorff.
Ex 1: two inequivalent atlases on $X = \mathbb{R}$

$X = \mathbb{R}$ \\
$\psi_1 : \mathbb{R} \to \mathbb{R}^4$, $\psi_1(x) = x$ one atlas \\
$\psi_2 : \mathbb{R} \to \mathbb{R}^4$, $\psi_2(x) = x^3$ another atlas.

(+$\psi$ continuous and $\psi^{-1}(y) = y^{1/3}$ so also continuous)

The two atlases are not equivalent:

$(\psi \circ \psi^{-1})(x) = x^{1/3}$ which is not differentiable at 0!

Q: So the manifolds $(\mathbb{R}, (\psi_1 : \mathbb{R} \to \mathbb{R}^4))$ and $(\mathbb{R}, (\psi_2 : \mathbb{R} \to \mathbb{R}^4))$

are not identical. But how different are they?

We'd like to have a way of saying that two manifolds are "isomorphic"

Def: Let $M$ and $N$ be two manifolds (topological spaces w. equivalent classes of atlases). A map of manifolds (aka a smooth map) is a map $f : M \to N$ of underlying sets which is "smooth in coordinates":

A chart $\psi : U \to \mathbb{R}^m$ on $M$, A chart $\varphi : V \to \mathbb{R}^n$ on $N$

$\varphi \circ f \circ \psi^{-1} : \psi(U \cap f^{-1}(V)) \to \mathbb{R}^n$

is $C^\infty$.

Ex: The inclusion map $i : S^1 \to \mathbb{R}^2$ is a map of manifolds.

Reason: $\varphi^{-1} : (0, 2\pi) \to S^1$, $\varphi^{-1}(\theta) = (\cos \theta, \sin \theta)$ is the inverse of a coord. chart on $S^1$; $id : \mathbb{R}^2 \to \mathbb{R}^2$ is the coord chart on $\mathbb{R}^2$. $id \circ 2 \circ \varphi^{-1}(\theta) = (\cos \theta, \sin \theta)$ which is $C^\infty$. 
\( \psi^{-1} : (-\pi, \pi) \to S^1, \quad \psi^{-1}(\theta) = (\cos \theta, \sin \theta) \) is another coord chart for \( S^1 \).

(\( \text{id}_{\mathbb{R}^2} \circ \psi \circ \psi^{-1} \) ) \( (x) = (\cos x, \sin x) \), which is \( C^\infty \).

Since \( \{ \psi : S^1 \times (0,1) \to (-\pi, \pi) \} \)

is an atlas on \( S^1 \), we are done: \( \pi : S^1 \to \mathbb{R}^2 \)

is a smooth map.

**Exercise:** Check that the notion of a \( C^\infty \) map / map of manifolds is well-defined and doesn’t depend on a choice of an atlas.

**Example:** Real projective space \( \mathbb{RP}^{n-1} \), the space of lines (through 0) in \( \mathbb{R}^n \).

As a set \( \mathbb{RP}^{n-1} = (\mathbb{R}^n \setminus \{0\}) / \sim \)

where \( \sim \) \( v \sim v' \Leftrightarrow \exists t \neq 0 \text{ s.t. } v = tv' \) \( (t+1) \).

\( \pi : \mathbb{R}^n \setminus \{0\} \to \mathbb{RP}^{n-1}, \quad \pi(v) = [v] = \text{the class of } v \)

Topology on \( \mathbb{RP}^{n-1} \): \( U \subseteq \mathbb{RP}^{n-1} \text{ is open } \Leftrightarrow \pi^{-1}(U) \subseteq \mathbb{R}^n \setminus \{0\} \text{ is open.} \)

**Claim:** \( \mathbb{RP}^{n-1} \) is a manifold of dimension \( n-1 \)

**Claim:** \( \pi : \mathbb{R}^n \setminus \{0\} \to \mathbb{RP}^{n-1} \) is a smooth map.

We need charts.

Set \( U_i = \{ [v_i, -v_i] \in (\mathbb{R}^n \setminus \{0\}) / \sim \mid v_i \neq 0 \} \)

\( \pi^{-1}(U_i) = \{ (v_i, -v_i) \in \mathbb{R}^n \setminus \{0\} \mid v_i \neq 0 \} \), which is open.

\( \psi_i : U_i \to \mathbb{R}^{n-1}, \quad \psi_i(\mathbf{v}) = (\mathbf{v}/v_i, \mathbf{v}/v_i, \ldots, v_{i-1}/v_i, v_{i+1}/v_i, \ldots, v_n) \)

\( \psi_i^{-1}(x_1, \ldots, x_{n-1}) = [x_1, \ldots, x_{i-1}, 1, x_i, -x_{i+1}] \)

**Transition maps:**

\( (\psi_j \circ \psi_i^{-1})(x_1, \ldots, x_{n-1}) = \psi_j(\mathbf{v}) = (\mathbf{v}/x_j, \mathbf{v}/x_j, \ldots, x_{n-1}/x_j) \)

which is \( C^\infty \) on \( \psi_i(U_i \cap U_j) \).
The manifold structure on $\mathbb{R}^{n-1} \times \mathbb{R}$ is given by one chart $\psi: \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{N}, \psi(x) = y$.

Finally,

$$(\psi \circ \pi \circ \psi^{-1})(y_1, ..., y_n) = \psi_i (y_1, ..., y_n) = (y_1, ..., y_n)$$

which is $C^\infty$ on $\pi^{-1}(U_i) = \{y \in \mathbb{R}^n \mid y_i \neq 0\}$.

$\pi$ is smooth.

Example $M = (\mathbb{R}, \mathbb{R}^2; \mathbb{R} \to \mathbb{R}^2)$ $N = (\mathbb{R}, (x^3: \mathbb{R} \to \mathbb{R}^2))$

Claim $f: M \to N$ $f(y) = y^{1/3}$ and $g: N \to M, g(x) = x^3$ are $C^\infty$.

Check: $(\psi \circ f \circ \psi^{-1})(y) = \psi \left( \left( y^{1/3} \right)^3 \right) = \psi \left( \psi^{-1}(y) \right) = y$

$(\psi \circ g \circ \psi^{-1})(x) = \psi \left( \psi^{-1}(x^3) \right) = \psi \left( \psi^{-1}(x^{1/3})^3 \right) = \psi \left( \psi^{-1}(x) \right) = x$

Note: $f \circ g = \text{id}_N$, $g \circ f = \text{id}_M$.

Definition A map $f: M \to N$ of manifolds is a diffeomorphism if $f \circ g: N \to M$, a smooth map, so that $f \circ g = \text{id}_N$, $g \circ f = \text{id}_M$.

Conclusion $(\mathbb{R}, \text{id})$ and $(\mathbb{R}, x^3)$ are not the same manifolds, but they are diffeomorphic ("isomorphic manifolds") so not that different in some sense.

Definition A smooth function on a manifold $M$ is a smooth map $f: M \to \mathbb{R}$ ($\mathbb{R}$ is a manifold with chart $\text{id}: \mathbb{R} \to \mathbb{R}$):

A chart $\psi: U \to \mathbb{R}^n$ on $M$, $f \circ \psi^{-1}: \psi(U) \to \mathbb{R}$ is $C^\infty$. 