An integral curve of a vector field $X \in \Gamma(TM)$ through $q \in M$ in a $(C^\infty)$ map $\gamma: I \to M$ so that

$$\gamma(t) = X(\gamma(t)), \quad t \in I$$

$$\gamma(0) = q$$

Recall: $I$ is an open interval containing 0.

Remark 1: $U = (x_1, \ldots, x_m): U \to \mathbb{R}^m$ is a coordinate chart on $M$ with $q \in U$ (and $\gamma(I) \subseteq U$), then

(**) amounts to

$$\begin{align*}
\frac{d}{dt} x_i(t) &= X_i(\gamma(t)) \\
\frac{d}{dt} x_m(t) &= X_m(\gamma(t))
\end{align*}$$

where

$$\gamma(t) = (x_1 \circ \gamma)(t)$$

and $X = \sum X_i \frac{\partial}{\partial x_i}, \quad x_i \in C^\infty(U)$.

Fact/Theorem: Suppose $U \subseteq \mathbb{R}^m$ is an open set

$$X = (x_1, \ldots, x_m): U \to \mathbb{R}^m \subseteq C^\infty.$$

Then the ODE

$$\begin{cases}
\frac{d}{dt} x_i(t) = X_i(\gamma(t), \ldots, x_m(t)) & 1 \leq i \leq n \\
\frac{d}{dt} x_m(t) = X_m(\gamma(t), \ldots, x_m(t)) \\
\gamma(0) = q
\end{cases}$$

has a solution $\gamma^q(t)$ defined on some interval $I \ni 0$.

Moreover (1) if $T N \to U$ solves (**) then

$$\gamma(t) = T(t), \quad \forall t \in I \cap N$$

(2) $\gamma$ depends smoothly on $q$. 
There exists an open set $W \subseteq \mathbb{R} \times \mathbb{R}$, with $\mathbb{R} \times \{0\} \subseteq W$, and a $C^\infty$ map $\varphi : W \to \mathbb{R}$ so that

1. \[ \frac{d}{dt} \varphi(y, t) = X(\varphi(y, t)) \quad \forall (y, t) \in W \]
2. \[ \varphi(y, 0) = y \quad \forall y \in \mathbb{R} \]

Let $U = (-1, 1) \times (-1, 1)$. Define $F : U \to \mathbb{R}$ by $F(x) = 1$. Then

\[ \dot{y}(t) = 1 \quad y(0) = y \]

Then $\varphi(y, t) = t + y$.

So, $W = (-1, 1) \times (-1, 1)$.

Remark 14.1: Suppose $\gamma : (a, b) \to M$ is an integral curve of $X \in \Gamma(TM)$ with $\gamma(0) = q$. Then for $t \in (a, b)$

\[ \sigma(t) = \gamma(t + t_0), \quad a < t + t_0 < b \]

(i.e., $a - t_0 < t < b - t_0$)

is an integral curve of $X$ with $\sigma(0) = \gamma(t_0)$.

Reason:

\[ \frac{d}{dt} \sigma = \frac{d}{dt}(\gamma(t + t_0)) = \left. \frac{d\gamma}{dt} \right|_{t + t_0} = X(\gamma(t + t_0)) = X(\sigma(t)) \]
Lemma 14.2 Let $X \in \Gamma(TM)$ be a vector field. Then $\forall q \in M$ exists an integral curve $\gamma_q : I_q \to M$ of $X$ with $\gamma_q(0) = q$. Moreover $\gamma_q$ depends smoothly on $q$ and is locally unique: if $\sigma : J \to M$ is another integral curve of $X$ with $\sigma(0) = q$ then $\exists \varepsilon > 0$ st.

$\sigma \left|_{(-\varepsilon, \varepsilon)} \right. = Y \left|_{(-\varepsilon, \varepsilon)} \right.$.

Remark 14.3. It does not follow that $\sigma = \gamma$ on $J \cap I$.

Lemma 14.4 Suppose $\sigma : J \to M$, $\gamma : \Gamma \to M$ are two integral curves of $X \in \Gamma(TM)$ with $\sigma(0) = \gamma(0)$. Then $\forall t \in J \cap I \sigma(t) = \gamma(t)$ is an open in $I \cap J$.

Proof. Suppose $\sigma(t_0) = \gamma(t_0)$, let $\sigma(t) = \sigma(t + t_0)$ $\gamma(t) = \gamma(t + t_0)$. Then $\sigma$, $\gamma$ are two integral curves of $X$ with $\sigma(0) = \gamma(0)$. By 14.2 $\exists \varepsilon > 0$ st.

$\sigma(t) = \gamma(t)$ $\forall t$ with $-\varepsilon < t - t_0 < \varepsilon$

Lemma 14.5 Suppose $f, g : X \to Y$ are two continuous maps between topological spaces and $Y$ is Hausdorff. Then the set

$\{ x \in X | f(x) = g(x) \}$

is closed in $X$. 
Sketch of proof. Since $Y$ is Hausdorff, $A_Y = \{(y,y) \in Y \times Y \mid y_1 = y_2\}$ is closed in $Y$.

Let $x \in X$ such that $f(x) = g(x)$. Then $(f, g)^{-1}(A_Y)$ is the desired solution set.

Lemma 14.5

Suppose $M$ is Hausdorff, $X \in \Gamma(TM)$, $\sigma : I \to M$, $\tau : J \to M$ are two integral curves of $X$ with $\sigma(0) = \tau(0)$. Then

$$\sigma(t) = \tau(t) \quad \text{for } t \in I \cap J.$$  

Proof. $I \cap J$ is connected. The set

$$\{t \in I \cap J \mid \sigma(t) = \tau(t)\}$$

is open by 14.4 and closed by 14.5.

Corollary 14.7. Let $M$ be Hausdorff, $X \in \Gamma(TM)$.

(i) For each $y \in M$ with $y = X(0)$, there is a unique integral curve $\gamma_y$ of $X$ through $y$.

(ii) $\gamma_y(0) = y$.

$\gamma_y$ is maximal among all integral curves of $X$ through $y$.

Proof. Take the union of all integral curves of $X$ through $y$. By 14.6, this union is a well-defined curve.