Recall: A section \( \pi: M \to N \) is a map \( q: N \to M \) so that \( \pi \circ q = \text{id}_N \).

Last time: We sketched/proved that for a manifold \( M \), there is an \( R \)-linear isomorphism
\[
\{ \xi: M \to TM \mid \pi \circ \xi = \text{id}_M \} = \Gamma(TM) \to \text{Der}(C^\infty(M))
\]
where \( \xi: C^\infty(M) \to C^\infty(M) \) is linear, \( \xi(fg) = f(x)g + f(x)\xi(g) \).

A section \( \xi \in \Gamma(TM) \) defines a derivation \( \xi \) by \( (\xi f)(q) = \xi f \cdot q \in \Gamma(M) \).

We checked: \( X, Y \in \text{Der}(C^\infty(M)) \)
\[ [X, Y] := X \circ Y - Y \circ X \in \text{Der}(C^\infty(M)) \]
\[ \text{Der}(C^\infty(M)) \text{ is a Lie algebra} \]
\[ \Gamma(TM) \text{ is a Lie algebra.} \]

Note: \( \dim_R (\Gamma(TM)) \) is infinite unless \( \text{dim}_M M = 0 \).

The next notion is essential for associating Lie algebras (of correct dimension) to Lie groups.

Let \( F: M \to N \) be a map between two manifolds. Vector fields \( X \in \Gamma(TM), Y \in \Gamma(TN) \) are \( F \)-related if
\[ Y(F(x)) = dF_{x*}(X(x)) \quad \forall x \in M \]

ie
\[
\begin{array}{c}
M \xrightarrow{X} TM \\
F \downarrow \quad \downarrow dF \\
N \xrightarrow{Y} TN \\
\end{array}
\]

commutes.

Aside: \( F \) defines a smooth map \( \text{d}F: TM \to TN \) by
\[ \text{d}F(q, v) = \text{d}F_q(v) \quad \forall q \in M, v \in \Gamma_M \]
Lemma 13.1. Suppose \( F : M \to N \) is a map of manifolds with \( x_1, x_2, y_1, y_2 \in \Gamma(TM), y_1, y_2 \in \Gamma(TN) \), \( x_i \) \( \text{F}-\)related to \( y_i \). Then \( [x_1, x_2] \) is \( \text{F}-\)related to \( [y_1, y_2] \).

Proof. 1) Note first that

\[ x_i \text{ is } \text{F}-\text{related to } y_i \]

\[ \forall q \in M, \forall h \in C^\infty(N) \]

\[ dF_q (x_i(q)) h = (y_i(F(q)))h \]

\[ \Rightarrow x_i(q) (h \circ F) = (y_i(h))(F(q)) \]

\[ \Rightarrow x_i (h \circ F) = y_i(h) \circ F \]

Where we used the identifications \( \Gamma(TM) \leftrightarrow \text{Der}(C^\infty(M)) \)

\( \Gamma(TN) \leftrightarrow \text{Der}(C^\infty(N)) \)

2) We now compute: \( \forall h \in C^\infty(N) \)

\[ [x_1, x_2] (h \circ F) = x_1 (x_2 (h \circ F)) - x_2 (x_1 (h \circ F)) \]

\[ = x_1 (x_2(h) \circ F) - x_2 (x_1(h) \circ F) \]

\[ = ([x_1, x_2](h)) \circ F - ([x_2, x_1](h)) \circ F \]

\[ = ([y_1, y_2](h)) \circ F \]

\[ \square \]

Def. Let \( G \) be a Lie group. Then for each \( a \in G \) we have a \( C^\infty \) map \( \lambda_a : G \to G \), \( \lambda_a(b) = ab \), (left multiplication by \( a \)), (i.e., multiplication on the left by \( a \in G \)).

A vector field \( X \in \Gamma(TG) \) is \textit{left-invariant} if \( X \) is \( \lambda_a \)-related to itself for every \( a \in G \):

\[ X(ag) = (d\lambda_a)g X(g) \quad \forall a \in G. \]

Lemma 13.2. Left invariant vector fields on a Lie group \( G \) form a Lie algebra, \( \text{Lie}(G) = g. \).
Remark: If $X \in \Gamma(TG)$ is left invariant then $X$ is uniquely determined by $X(e)$, the identity in $G$:

$$\forall a \in G$$

$$X(a) = X(\lambda_a(e)) = (d\lambda_a)_e X(e).$$

This gives us a injective (linear!) map

$$\text{left-inv. vector fields on } G \rightarrow TG$$

$$X \mapsto X(e).$$

In fact the map is onto.

Reason: Given $v \in TG$ define the corresponding left-invariant vector field $V$ by

$$V(a) = (d\lambda_a)_e v$$

Need to check: $V : G \rightarrow TG \subset C^\infty$.

It is because $v$)

$$m : G \times G \rightarrow G \quad m(a, b) = ab \in C^\infty$$

$$\Rightarrow \partial_2 m : G \times TG \rightarrow TG$$

$$(\partial_2 m)(a, (b, i(0))) = \frac{d}{dt} m(a, \gamma(t)) = \frac{d}{dt} \lambda_a(\gamma(t)) \Rightarrow \partial_2 m \quad G \times TeG \rightarrow TG \subset C^\infty$$

Here we used the fact that $Y$ manifold $M$, $V \in TM$

The inclusion $TeG \rightarrow TM$ is $C^\infty$.

(in fact it's an embedding)

---

Proof of 13.2 Suppose $X, Y \in \Gamma(TG)$ are left invariant.

Then $\forall a \in G$ $X$ is $La$-related to $X$

$Y$ is $La$-related to $Y$

By 13.1 $[X, Y]$ is $La$-related to $[X, Y]$ \forall a

$\Rightarrow [X, Y]$ is left-invariant.
Integral curves of vector fields

**Notation**
I = open interval containing 0,
\[ \text{i.e. } I = (a, b), \quad a < 0 < b, \quad -\infty \leq a, b < \infty, \]
(\text{so } I = (-\infty, b), \text{ or } I = (a, +\infty) \text{ or } I = \mathbb{R})
\quad \text{or } I = (a, b) \quad \text{if } a, b \in \mathbb{R}

**Definition**
Let \( X \) be a vector field on a manifold \( M \).
A curve \( \gamma: I \rightarrow M \) is an integral curve of \( X \) through \( q \) \( \Rightarrow \gamma(0) \) if
\[ \dot{\gamma}(t) = X(\gamma(t)) \quad \forall t \in I \]

**Aside**
If \( \phi = (x_1, \ldots, x_m): U \rightarrow \mathbb{R}^m \) is a coordinate chart
then \[ \dot{y}(t) = \sum (dx_i)_x(t) \left( x_i(\gamma(t)) \frac{d}{dx_i} \right) y(t) \]
and \[ (dx_i)_x(t) \left( \dot{y}(t) \right) = \dot{y}(t) \cdot x_i = \frac{d}{dt} \left( x_i(\gamma(t)) \right) \]

So if we set \[ \dot{y}_i(t) = (x_i \circ \gamma)(t) \]
then \[ \dot{y}(t) = \sum \dot{y}_i(t) \frac{d}{dx_i} y(t) \]
where \( \dot{y}_i(t) \) is the derivative of the function \( y_i: I \rightarrow \mathbb{R} \).

**Example**
\( M = \mathbb{R}^m \), \( X = \sum \frac{\partial}{\partial x_0} X_i \in \mathbb{C}^r(M) \)
\( y = (x_1, \ldots, x_m): I \rightarrow \mathbb{R}^m \) is an integral curve of \( X \) iff
\[ \sum \dot{y}_i(t) \frac{d}{dx_0} y(t) = \sum X_i(\gamma(t)) \frac{d}{dx_0} y(t) \]
Since \( \frac{\partial}{\partial x_i} \bigg|_{q_0} \) is a basis of \( \mathcal{T}_q \mathbb{R}^m \), \( q \)

* amounts to

\[
\begin{align*}
\dot{q}_i &= X_i (q, \mathbf{v}(t), - \mathbf{y}(t)) \\
\vdots \\
\dot{y}_m(t) &= X_m (q, \mathbf{v}(t), - \mathbf{v}(t))
\end{align*}
\]

a system of ODEs.