Last time: Defined the tangent bundle $TM$ of a manifold $M$. As a set

$$TM = \bigcup_{q \in M} T_q M = \bigsqcup_{q \in M} T_q M$$

Coordinates (and topology) on $TM$ come from coordinates on $M$. If $\varphi = (x_1, \ldots, x_m) : U \to \mathbb{R}^m$ is a coordinate chart on $M$, then $\tilde{\varphi} : TM \to \mathbb{R}^m \times \mathbb{R}^m$

$$\tilde{\varphi}(q, v) = (\varphi(q), (dx_1)_q(v), \ldots, (dx_m)_q(v))$$

is a chart on $TM$.

The canonical map $\pi : TM \to M$, $\pi(q, v) = q$ is a surjective submersion.

Def ("geometric") A vector field $X$ on a manifold $M$ is a section of $\pi : TM \to M$:

$$X : M \to TM, \quad \text{st.} \quad \pi X = \text{id}_M,$$

i.e., $X(q) \in T_q M, \forall q \in M$.

Def ("algebraic") A vector field $X$ on a manifold $M$ is a derivation

$$X : C^\infty(M) \to C^\infty(M),$$

i.e., $X$ is $\mathbb{R}$-linear and $f, g \in C^\infty(M)$

$$X(fg) = X(f)g + fX(g).$$

Notation $\text{Der}(C^\infty(M)) = \text{space of derivations of } C^\infty(M)$

$\mathcal{X}(M) = \Gamma(TM) = \text{space of sections of } \pi : TM \to M.$

Proposition 12.1 There is a linear isomorphism

$$\Gamma(TM) \to \text{Der}(C^\infty(M)).$$
Lemma 12.2. Derivations are local: if $f \in \mathcal{C}^\infty(M)$
$U \subseteq M$ open and $f|_U \equiv 0$, then $\forall X \in \text{Der}(\mathcal{C}^\infty(M))$
$Xf|_U \equiv 0$. That is, $\text{supp}(Xf) \subseteq \text{supp}(f)$.

Proof: same as the proof that tangent vectors are local:
$\forall q \in U \exists p \in \mathcal{C}^\infty(M)$, $\text{supp } p \subseteq U$, $p \equiv 1$ near $q$.
Then $0 = pf. \
0 = X(0) = X(pf) = X(p) f + p \cdot X(f).
\Rightarrow 0 = \left(\frac{d}{dt}X(t)f(q)\right)|_{t=0} = (Xf)(q)$.

Corollary 12.3. \forall U \subseteq M open, $\forall X \in \text{Der}(\mathcal{C}^\infty(M))$
we have a well-defined derivation
$X|_U : \mathcal{C}^\infty(U) \to \mathcal{C}^\infty(U)$
so that

$\mathcal{C}^\infty(M) \xrightarrow{X} \mathcal{C}^\infty(M)$
$
\downarrow \text{lin} \quad \downarrow \text{lin}
$

$\mathcal{C}^\infty(U) \xrightarrow{X|_U} \mathcal{C}^\infty(U)$ commutes.

Sketch of proof: $\forall f \in \mathcal{C}^\infty(U)$, $\forall q \in U$ define
$(X|_U f)(q) = (X(p f))(q)$ when
$p \in \mathcal{C}^\infty(U)$, $\text{supp } p \subseteq U$, $p \equiv 1$ near $q$.

12.2 $\Rightarrow$ $(X|_U f)(q)$ doesn't depend on the choice of $p$. 

\end{proof}
Proof of 12.1 Given a section $\pi : M \to TM$, $f \in C^\infty(M)$

define $D_x f : M \to \mathbb{R}$ by $(D_x f)(q) := X(q)f$.

We argue that $D_x f \in C^\infty(M)$. Enough to show:

A coord. chart $\psi = (\psi_1, \ldots, \psi_m) : U \to \mathbb{R}^m$ on $M$, $D_x f |_U \in C^\infty(U)$

Since $\pi : M \to \mathbb{R}^m$ is $C^\infty$, $\pi \circ X : U \to \psi(U) \times \mathbb{R}^m$

Let

$(\pi \circ X)(q) = (\psi(q), (d\xi_1)_q(X(q)), \ldots, (d\xi_m)_q(X(q)))$

so $q \mapsto (d\xi_i)_q(X(q))$ are $C^\infty$ functions.

$\Rightarrow \forall f \in C^\infty(M)$, $q \in U$

$(D_x f)(q) = \left( \sum (d\xi_i)_q(X(q)) \frac{\partial}{\partial \psi_i} \right) f(q)$

$= \sum (d\xi_i)_q(X(q)) \frac{\partial f}{\partial \psi_i}(q)$

$\Rightarrow D_x (C^\infty(M)) \subseteq C^\infty(M)$.

$D_x$ is a derivation since $\forall f, g \in C^\infty(M)$, $q \in M$

$D_x (fg)(q) = \left( X(q)(fg) - (X(q)f) g(q) + f(q)(X(q)g) \right)$

$= (D_x f) g + f D_x g)(q)$

$\Rightarrow D_x \in \text{Der}(C^\infty(M))$.

The fact that $D_x : C^\infty(M) \to C^\infty(M)$ is $\mathbb{R}$-linear is even easier.

We now construct the inverse map. Given $X \in \text{Der}(C^\infty(M))$

define $S_x : M \to TM$ by

$(S_x)(q) f = (Xf)(q)$ for $f \in C^\infty(M)$.

Since $X \in \text{Der}(C^\infty(M))$

$S_x(q) f \in T_q M$, $q \in M$
We need to check that \( \mathfrak{X} \) is \( \mathcal{C}^\infty \).

If a global coordinate chart \( \varphi = (x_1, \ldots, x_m) : M \to \mathbb{R}^m \)

then \( x_1, \ldots, x_m \in \mathcal{C}^\infty (M) \).

\[
\mathfrak{X} (x_1), \ldots, \mathfrak{X} (x_m) \in \mathcal{C}^\infty (M)
\]

\[
\mathfrak{X}_x = \sum \frac{\partial}{\partial x_i} \bigg| _x \mathfrak{X}_i = \sum \left( \mathfrak{X}_i (x_j) \right) \frac{\partial}{\partial x_j} | _x
\]

\[
\mathfrak{X}_x \in \mathcal{C}^\infty.
\]

In general case we cover \( M \) by coordinate charts.

On each such chart \( \varphi = (x_1, \ldots, x_m) : \mathcal{U} \to \mathbb{R}^m \)

\[
\mathfrak{X}_x | _\mathcal{U} = \frac{1}{\varphi} (\mathfrak{X}_n | _\mathcal{U}).
\]

Since \( \mathfrak{X}_x | _\mathcal{U} \) makes sense by 12.3, the general case follows from the special case.

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**Lemma 12.4**

\( \text{Der} (\mathcal{C}^\infty (M)) \) is a Lie algebra with the bracket \([\mathfrak{X}, \mathfrak{Y}]\) defined by

\[
[X, Y] f = (X \circ Y - Y \circ X)(f), \quad \forall f \in \mathcal{C}^\infty (M).
\]

**Proof** Clearly since \( X, Y \) are linear, so is \([X, Y] \).

Need to check: \([X, Y] \in \text{Der} (\mathcal{C}^\infty (M)) \). It's a computation:

\[
[X, Y] (fg) = X(Y(fg)) - Y(X(fg)) =
\]

\[
= X(Y(f)g + fY(g)) - Y(X(f)g + fX(g))
\]

\[
= X(Y(f)) \cdot g + Y(f) X(g) + X(f) Y(g) + f X(Y(g)) - Y(X(f)) \cdot g - X(f) Y(g) - Y(f) X(g) - f \cdot Y(X(g))
\]

\[
= (X(Y(f)) - Y(X(f))) \cdot g + f \cdot (X(Y(g)) - Y(X(g)))
\]

\( \square \)