Last time: defined what it means for a map \( F: M \to N \) to be transverse to a submanifold \( Z \subset N \):

- proved: if \( F \nabla Z \), then \( F^{-1}(Z) \) is an embedded submanifold and
  \[ \text{codim}_M F^{-1}(Z) = \text{codim}_N Z \]
  (provided \( F^{-1}(Z) \neq \emptyset \)).
- Defined embeddings and immersions.

Today: rank of a map at a point
- Constant rank theorem
- Tangent bundles.

**Definition:** Let \( F: M \to N \) be a smooth map between two manifolds. The rank of \( F \) at \( x \in M \) is
  \[ \text{rank}_x(F) = \dim \left( \text{df}_x(T_x M) \right). \]

**Example:** Suppose \( F: M \to N \) is an immersion (as \( M \) is connected). Then \( \forall x \in M \), \( \text{rank}_x F = \dim M \).

**Definition:** A map \( F: M \to N \) is a submersion if
  \[ \text{d}F_x: T_x M \to T_{F(x)} N \text{ is onto} \quad \forall x \in M \]
  i.e., if \( \text{rank}_x F = \dim N \quad \forall x \) (again, assuming connectivity).

**Useful fact:** Rank theorem

Suppose \((M, N) \) are connected and \( F: M \to N \) has rank \( k \) at all points of \( M \). Then \( \exists p \in M \)

3 coordinates \((y_1, \ldots, y_m): U \to \mathbb{R}^m \) near \( p \)

\( y^i: (y_1, \ldots, y_m): V \to \mathbb{R}^n \) near \( F(p) \) s.t.

\[ y \circ F \circ F^{-1}(u_1, \ldots, u_m) = (u_1, \ldots, u_k, 0, \ldots, 0) + u_{k+1} \mathcal{U}. \]

**Proof:** Linear algebra + inverse function theorem.
Tangent bundle $TM$ of a manifold $M$.

Plan:
- We define first what $TM$ is a set.
- We then manufacture candidate coordinate charts on $TM$ out of charts on $M$
- and check that transition maps are $C^\infty$.
- We give $TM$ a topology.

We set $TM := \bigcup_{q \in M} T_q M$ (as sets)

($\bigcup_{q \in M} T_q M = \bigcup_{q \in M} T_q M \times \mathbb{R}^n$)

Suppose $\psi = (x_1, \ldots, x_m) : U \rightarrow \mathbb{R}^m$ is a coordinate chart on $M$. Recall $u \in T_q M$

$\varphi = \sum_{i=1}^n (dx_i)_q(u) \frac{\partial}{\partial x_i}|_q$

i.e. $( (dx_1)_q(u), \ldots, (dx_m)_q(u)) : T_q M \rightarrow \mathbb{R}^m$ is an iso.

Define $\bar{\psi} : \bigcup_{q \in U} T_q M (= TU) \rightarrow \psi(U) \times \mathbb{R}^m$ by

$\bar{\psi} (q, u) = (x_1(q), \ldots, x_m(q), (dx_1)_q(u), \ldots, (dx_m)_q(u))$

for all $q \in U$, $u \in T_q M$.

We'll refer to $\bar{\psi}$ as the associated (candidate) coordinate chart on $TU \subset TM$.

Sanity check

Suppose $\psi = (y_1, \ldots, y_m) : U \rightarrow \mathbb{R}^m$

$\varphi = (y_1, \ldots, y_m) : V \rightarrow \mathbb{R}^m$

are two coordinate charts on $M$.

Is $\bar{\psi} \circ \bar{\varphi}^{-1} : \psi(U \cap V) \times \mathbb{R}^m \rightarrow \psi(U \cap V) \times \mathbb{R}^m$ $C^\infty$?
We compute: \( \psi \in \psi(U \cap V) \in W = (w_1, \ldots, w_m) \in \mathbb{R}^m \)

\[
(\psi^{-1})^{-1}(r, w) = (\psi^{-1}(r), \sum_{i=1}^{m} w_i \frac{\partial}{\partial x_i}(\psi^{-1}(r)))
\]

\[
\Rightarrow (\psi \circ (\psi^{-1})^{-1})(r, w) = (\psi(\psi^{-1}(r)), \sum_{i=1}^{m} w_i \frac{\partial}{\partial x_i}(\psi^{-1}(r))) = \sum_{i=1}^{m} w_i \frac{\partial y_i}{\partial x_i}(\psi^{-1}(r))
\]

The map \( \psi(U \cap V) \to \mathbb{R}^m, r \mapsto \frac{\partial y_i}{\partial x_i}(\psi^{-1}(r)) = \frac{1}{(c_0 \cdot \psi^{-1})(r)} \frac{\partial y_i}{\partial x_i}(\psi^{-1}(r)) \in C^\infty \)

\[
(\psi \circ \phi^{-1})(r, w) = \left((\psi \circ \phi^{-1})(r), \left(\frac{\partial y_i}{\partial x_i}(\psi^{-1}(r))\right)(w)\right)
\]

\[\Rightarrow \hat{\psi} \circ (\phi^{-1})^{-1} : \psi(U \cap V) \times \mathbb{R}^m \to \psi(U \cap V) \times \mathbb{R}^m \text{ is a homeomorphism.}\]

Now define a topology on \( TM \) by

\[ \circ (TM \text{ is open } \iff \forall \psi : U \to \mathbb{R}^m, \phi(U \cap Tu) \text{ is open in } \psi(U) \times \mathbb{R}^m.\]

Then \( \phi : Tu \to \psi(U) \times \mathbb{R}^m \) is a collection of homeomorphisms with smooth transition maps.

This gives \( TM \) a manifold structure.

Note: If \( M \) is Hausdorff, then so is \( TM \). Why?

Note: We have a canonical projection \( \pi : TM \to M \)

\[ \pi(q, v) = q \quad q \in M, \forall v \in T_q M\]

Claim \( \pi : TM \to M \) is a \( C^\infty \) map and a submersion.
Proof. Let \( \varphi : U \to \mathbb{R}^m \) be a coordinate chart on \( M \). Then \( \varphi : TU \to \mathbb{R}^m \times \mathbb{R}^m \) is a coordinate chart on \( TM \).

Then \( \varphi (r, w) = (\varphi(U)) \times \mathbb{R}^m \): 

\[
(\varphi \circ \pi \circ \psi^{-1})(r, w) = \varphi (\pi(\varphi^{-1}(r)), \Sigma W \cdot \frac{d}{dx} \mid \psi^{-1}(w))
\]

\[
= \varphi (\varphi^{-1}(r)) = r
\]

\[
\Rightarrow \varphi \circ \pi \circ \psi^{-1} : \varphi(U) \times \mathbb{R}^m \to \mathbb{R}^m \cap C^0.
\]

If \( \psi : \Omega \to \mathbb{R}^m \times \mathbb{R}^m \) is another coordinate chart on \( TM \), then:

\[
\varphi \circ \pi \circ \psi^{-1} = (\psi \circ \pi \circ \psi^{-1})^0 \circ (\varphi \circ \psi^{-1})
\]

\[
\Rightarrow \pi \cap C^0.
\]

Also, \( \varphi \circ \pi \circ \psi^{-1}(r, w) = r \)

\[
\Rightarrow d\varphi \circ d\pi \circ (\psi^{-1})' \cap \text{onto}
\]

\[
\Rightarrow d\varphi \circ d\pi \cap \text{onto (since } d\pi, \text{ } d(\varphi^{-1})'\text{ are one-to-one).}
\]

Vector fields.

Def. A vector field \( X \) on a manifold \( M \) is a smooth map \( X : M \to TM \) with \( \pi \circ X = \text{id}_M \), i.e., \( X(q) \in T_q M \).

Next time vector fields are derivations \( \mathcal{C}^0(M) \to \mathcal{C}^0(M) \).