Exercise 8.1. Suppose that $V$ is an $n$-dimensional vector space. Given a linear map $A : V \to V$, we get a map $\Lambda^n(A) : \Lambda^n(V) \to \Lambda^n(V)$, and since $\dim \Lambda^n(V) = 1$, the map $\Lambda^n(A)$ is multiplication by a scalar. Show that this scalar is $\det A$.

Exercise 8.2. Prove that for any pair of linear maps $A : V \to W$ and $B : W \to U$ between finite dimensional vector spaces $\Lambda^*(BA) = \Lambda^*(B) \circ \Lambda^*(A)$.

Exercise 8.3. Let $G$ be a Lie group, $X$ a left invariant vector field of on $G$ and $\gamma : \mathbb{R} \to G$ an integral curve of $X$ with $\gamma(0) = 1$ (here and elsewhere 1 denotes the identity element of the group $G$). Prove that for all $s,t \in \mathbb{R}$

$$\gamma(s + t) = \gamma(s) \cdot \gamma(t),$$

where on the right $\cdot$ denotes the multiplication in $G$. In other words prove that integral curves through 1 of left invariant vector fields are Lie group homomorphisms. They are called 1-parameter subgroups.

Exercise 8.4. Recall that for a Lie group the tangent space at the identity has the structure of a Lie algebra. Prove: if $f : G \to H$ is a map of Lie groups (i.e., $f$ is smooth and preserves multiplication) then $df_1 : T_1G \to T_1H$ is a map of Lie algebras.

Exercise 8.5. Compute the pull-back of the form $\omega = xdy \wedge dz \wedge dw - ydx \wedge dz \wedge dw + zdx \wedge dy \wedge dw - wdx \wedge dy \wedge dz \in \Omega^*(\mathbb{R}^4)$ by the map $F : \mathbb{R}^3 \to \mathbb{R}^4$, $F(x, y, z) = (x, y, z, x^2 + y^2 + z^2)$

Exercise 8.6. Let $S = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_4 = x_1x_2x_3^2, \ 0 < x_1, x_2, x_3 < 1\}$. It’s a submanifold of $\mathbb{R}^4$ and the map $\varphi(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3)$ is a coordinate chart on $S$ giving it an orientation (don’t prove these two facts). Compute

$$\int_S x_3 dx_1 \wedge dx_2 \wedge dx_4,$$

where $S$ is oriented as above.