Exercise 6.1. Let $M$ be a manifold with $\dim M > 1$, $q \in M$ and $V \subset T_q M$ a proper subspace of the tangent space (thus $V \neq 0, T_q M$). Prove that there is an embedded submanifold $Q \subset M$ with $q \in Q$ and $T_q Q = V$.

Exercise 6.2. Compute the flow of the vector field $x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}$ on $\mathbb{R}^2$.

Exercise 6.3. Suppose a vector field $X$ on a manifold $M$ is zero at a point $p \in M$. Show that there is a neighborhood $U$ of $p$ in $M$ so that for each point $q \in U$ the integral curve through $q$ exists for all $t \in [-1, 1]$. Hint: this is really an exercise in understanding product topology on $\mathbb{R} \times M$.

Exercise 6.4. The point of this problem is to show that nothing interesting happens to the flow of a vector field away from the points where the vector field is 0. You may be able to find a solution to this problem in any number of books. If you do, read it, understand it and write up the solution in your own words.

Let $X$ be a vector field on a manifold $M$. Suppose $X(q) \neq 0$. Show that there is a coordinate chart $\phi = (x_1, \ldots, x_m) : U \to \mathbb{R}^m$ with $q \in U$ so that

$$X(p) = \frac{\partial}{\partial x_1} \bigg|_p$$ for all $p \in U$.

Hints: (i) Show first that there is a coordinate chart $\psi = (y_1, \ldots, y_m) : V \to \mathbb{R}^m$ so that $\psi(q) = 0$ and

$$X(q) = \frac{\partial}{\partial y_1} \bigg|_q$$

(just one point $q$, not all the points in $V$).

(ii) We know that there is $\epsilon > 0$ so that flow $\Phi$ of $X$ exists for time $t \in (-\epsilon, \epsilon)$ for all points $p$ in some open set $W \subset V$. Define the map

$$f : (-\epsilon, \epsilon) \times (W \cap \{0\} \times \mathbb{R}^{m-1}) \to M$$

by

$$f(t, (0, r)) = \Phi(t, \psi^{-1}(0, r)).$$

Compute the differential $df_{(0,(0,0))}$ and show that it is onto.

Exercise 6.5. Let $V$ and $W$ be two finite dimensional vector spaces over $\mathbb{R}$. Show that if $\{v_i\}$ is a basis of $V$, $\{v^*_i\}$ the dual basis of $V^*$ and $\{w_j\}$ a basis of $W$, then $\{v^*_i(\cdot)w_j\}$ is a basis of the vector space $\text{Hom}(V, W)$ of linear maps from $V$ to $W$. 

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