Exercise 5.1. Show that if $F : M \to N$ is a (smooth) map between manifolds then so is the map $dF : TM \to TN$. The map $dF$ is defined by

$$dF(q, v) := dF_q(v) \quad \text{for all } q \in M, v \in T_q M.$$ 

Show that if $G : N \to Q$ is another map of manifolds then $d(G \circ F) = dG \circ dF$.

Exercise 5.2. Prove that if $Q \subset M$ is an embedded submanifold then $TQ$ naturally embeds in $TM$.

Exercise 5.3.
(a) Show that the tangent bundle $T\mathbb{R}^n$ is naturally diffeomorphic to $\mathbb{R}^n \times \mathbb{R}^n$.
(b) Show that there is a linear isomorphism between the space of vector fields $\Gamma(T\mathbb{R}^n)$ on $\mathbb{R}^n$ and the space $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ of smooth $\mathbb{R}^n$-valued functions.

Exercise 5.4. Let $A$ be an associative algebra over the reals. That is $A$ is a vector space over $\mathbb{R}$ with a bilinear associative multiplication

$$\circ : A \times A \to A.$$ 

Prove that the bracket $[\cdot, \cdot] : A \times A \to A$ defined by

$$[a, b] := a \circ b - b \circ a$$

satisfies the Jacobi identity.

Exercise 5.5. Suppose that $F : N_1 \to M$, $G : N_2 \to M$ are two maps of manifolds with $G$ being a surjective submersion. You have shown that the fiber product

$$N_1 \times_M N_2 := \{ (x_1, x_2) \in N_1 \times N_2 \mid F(x_1) = G(x_2) \}$$

is an embedded submanifold of $N_1 \times N_2$. It naturally comes with two projections

$$\pi_i : N_1 \times_M N_2 \to N_i \quad \pi_i(x_1, x_2) = x_i.$$ 

making the diagram

$$\begin{array}{ccc}
N_1 \times_M N_2 & \xrightarrow{\pi_2} & N_2 \\
\pi_1 \downarrow & & \downarrow G \\
N_1 & \xrightarrow{F} & M
\end{array}$$

commute.
(a) Prove that $\pi_1$ is a surjective submersion.
(b) Show that sections of $G : N_2 \to M$ pull back to sections of $\pi_1$. Namely given a section $s : M \to N_2$ of $G$ show that there a section $F^*s : N_1 \to N_1 \times_M N_2$ of $\pi_1$ making the diagram

$$
\begin{array}{c}
N_1 \times_M N_2 \xrightarrow{\pi_2} N_2 \\
F^*s \downarrow \quad \uparrow s \\
N_1 \xrightarrow{F} M
\end{array}
$$

commute.

**Exercise 5.6.** Let $G$ be a Lie group. Prove that the tangent bundle $TG$ of $G$ is diffeomorphic to $G \times g$, where $g := T_e G$ is the tangent space at the identity.

**Exercise 5.7.** Let $M, N$ be two manifolds. Prove that the tangent bundle of the product is naturally diffeomorphic to the product of the tangent bundles:

$$T(M \times N) \cong TM \times TN.$$