Exercise 4.1. Prove that an embedded submanifold (as defined in class) is naturally a manifold if given the subspace topology.

Exercise 4.2. Prove that if $Q \subset M$ is an embedded submanifold and $F : M \to M'$ is a diffeomorphism then $F(Q) \subset M'$ is an embedded submanifold.

Exercise 4.3. Suppose $Q \subset M$ is an embedded submanifold and $F : N \to M$ is a smooth map with $F(N) \subset Q$. Prove that $F$ regarded as a map from $N$ to $Q$ is smooth (i.e., a map of manifolds).

Exercise 4.4. Prove that if $F : M \to N$ is a bijective map of manifolds such that the differential $dF_x$ is an isomorphism for all points $x \in M$ then $F$ is a diffeomorphism.

The following exercise has already appeared in homework #3. There is no need to turn in your solution twice, so don’t. But you need the exercise for the next problem.

Exercise 4.5. (a) Prove that for any manifold $M$ the diagonal

$$\Delta_M := \{(x, y) \in M \times M \mid x = y\}$$

is an embedded submanifold.

(b) Show that the map $\delta : M \to \Delta_M$, $\delta(x) := (x, x)$ is a diffeomorphism and that hence $\delta : M \to M \times M$ is an embedding.

Exercise 4.6. Suppose that $F : N_1 \to M$, $G : N_2 \to M$ are two maps of manifolds with $G$ being a submersion (this means that for any point $x \in N_2$ the differential $dG_x : T_xN_2 \to T_{G(x)}M$ is onto). Prove that the fiber product

$$N_1 \times_M N_2 := \{(x_1, x_2) \in N_1 \times N_2 \mid F(x_1) = G(x_2)\}$$

is an embedded submanifold of $N_1 \times N_2$. Hint: consider

$$F \times G : N_1 \times N_2 \to M \times M, \quad (F \times G)(x_1, x_2) := (F(x_1), G(x_2)).$$

Show that this map is transverse to the diagonal $\Delta_M$. 