Exercise 2.1.

1. Let \( \{ \varphi_\alpha : U_\alpha \to \mathbb{R}^m \} \) be an atlas on a manifold \( M \) and \( \{ \psi_\beta : V_\beta \to \mathbb{R}^n \} \) an atlas on a manifold \( N \). Prove that
\[
\{ \varphi_\alpha \times \psi_\beta : U_\alpha \times V_\beta \to \mathbb{R}^m \times \mathbb{R}^n \}
\]
is an atlas on the topological space \( M \times N \) (product topology).

2. Check that the projection map \( p_1 : M \times N \to M \), which defined by
\[
p_1(x, y) = x
\]
is smooth. \( p_2 : M \times N \to N \) is defined similarly and is also smooth; the proof is the same.

3. Prove that for any manifold \( Q \) a map \( f : Q \to M \times N \) is smooth if and only if the two composites \( p_1 \circ f, p_2 \circ f \) are smooth.

Exercise 2.2. Prove that the manifolds \( M \times N \) and \( N \times M \) are diffeomorphic. Hint: write down a (potential) diffeomorphism from \( M \times N \) to \( N \times M \) and check that it is, indeed, a diffeomorphism.

Exercise 2.3. Check that for any manifold \( M \) and for any point \( p \in M \) the tangent space \( T_p M \) is a vector space over the reals.

Exercise 2.4. Let \( f : M \to N \) be a smooth map between two manifolds, \( p \in M \) a point and \( v \in T_p M \) a tangent vector. Show that the map
\[
C^\infty(N) \to \mathbb{R}, \quad h \mapsto v(h \circ f),
\]
is a tangent vector to \( N \) at \( f(p) \). This tangent vector is variously denoted by \( df_p(v) \) and by \( T_{f(p)}(v) \).