Exercise 12.1. Recall that Riemannian metric $g$ on a manifold $M$ is a smooth section of the bundle $T^*M \otimes T^*M \to M$ so that for all $q \in M$, the bilinear map $g_q : T_qM \times T_qM \to \mathbb{R}$ is an inner product. If $g$ is a Riemannian metric on a manifold $M$, the pair $(M, g)$ is called a Riemannian manifold. Prove that every (Hausdorff paracompact) manifold $M$ has a Riemannian metric. Hint: Construct the metric locally in coordinates first.

Exercise 12.2. Prove that if $(M, g)$ is a Riemannian manifold then the metric $g$ defines an isomorphism of vector bundles $g^\# : TM \to T^*M$ by $g^\#(q, v) = g_q(v, -) \in T_q^*M$ for all $(q, v) \in T_qM$.

Exercise 12.3. Recall that for a smooth function $f$ on a Riemannian manifold $(M, g)$ there is a unique vector field $\nabla f$ with $g_q(\nabla f(q), v) = df_q(v)$ for all $q \in M, v \in T_qM$. This vector field $\nabla f$ is called the gradient vector field of $f$.

Let $\gamma(t)$ be an integral curve of the gradient vector field $\nabla f$. Prove that $\frac{d}{dt}f(\gamma(t)) \geq 0$ for all $t$ that $\gamma$ is defined. When is $\frac{d}{dt}f(\gamma(t)) = 0$?

Exercise 12.4. Consider the square $S = [0, 1] \times [0, 1] \subset \mathbb{R}^2$. Let $\alpha \in \Omega^1(\mathbb{R}^2)$ be a 1-form. Prove that Stokes’ theorem holds for $S$: $\int_S d\alpha = \int_{\partial S} \alpha$, where the piece-wise smooth boundary $\partial S$ is oriented appropriately.

Exercise 12.5. Let $A = \bigoplus_{k=0}^\infty A_k$ be a graded commutative algebra over $\mathbb{R}$. That is, for any $a \in A_k$, $b \in A_n$ we have $ab \in A_{k+n}$ and $ba = (-1)^{kn}ab$.

For example the algebra of differential forms on a manifold $M$ is a graded commutative algebra. A graded derivation of degree $m$ of the algebra $A$ is an $\mathbb{R}$-linear map $D : A \to A$ so that $D(A_k) \subset A_{k+m}$ for all $k$ and $D(ab) = (Da)b + (-1)^{km}a(Db)$ for all $a \in A_k, b \in A$. This is better written as $D(ab) = (Da)b + (-1)^{|a||D|}a(Db)$.

The graded commutator $[D_1, D_2]$ of two derivations $D_1$, $D_2$ of degrees $k_1$, $k_2$ respectively is defined by $[D_1, D_2] = D_1 \circ D_2 - (-1)^{k_1k_2}D_2 \circ D_1$,
or equivalently,

\[ [D_1, D_2] = D_1 \circ D_2 - (-1)^{|D_1||D_2|} D_2 \circ D_1. \]

Prove that \([D_1, D_2]\) is a graded derivation of degree \(k_1 + k_2 = |D_1| + |D_2|\).

**Exercise 12.6.** A map \(F : M \to N\) of manifolds is a local diffeomorphism if at every point \(q \in M\) its differential \(dF_q\) is an isomorphism of vector spaces.

(a) Prove that the map \(\pi : S^{n-1} \to \mathbb{R}P^{n-1}\) defined by sending a vector \(v \in S^{n-1}\) to the line \([v] \in \mathbb{R}P^{n-1}\) through \(v\) is a local diffeomorphism.

(b) Prove that \(\mathbb{R}P^n\) is orientable if and only if \(n\) is odd.

Hints: (1) Problems 11.5 and 11.6 of previous homework may be very useful.
(2) If \(\mu\) is a volume form on \(\mathbb{R}P^n\) then \(T^*(\pi^*\mu) = \pi^*\mu\), where \(T : S^n \to S^n\) is the multiplication by \(-1\):

\[ T(x) = -x. \]

Conversely show that if \(\nu\) is a volume form on \(S^n\) with \(T^*\nu = \nu\), then \(\nu = \pi^*\mu\) for some volume form \(\mu\) on \(\mathbb{R}P^n\).