Homework #10 Math 518 (corrected)
Due in class Wednesday, November 11, 2015

**Exercise 10.1.** In the previous homework you have defined the exponential map
\[ \exp : g \to G \]
for any Lie group \( G \). Prove that for any \( X \in g \) the curve \( \sigma(t) = \exp(tX) \) is the integral curve of the left invariant vector field \( \tilde{X} \) defined by \( X \). Hint: show that if \( \gamma(\tau) \) is an integral curve of the vector field \( Y \) then for any \( t \in \mathbb{R} \) the curve \( \tau \mapsto \gamma(t\tau) \) is an integral curve of the vector field \( tY \).

**Exercise 10.2.** Prove that the exponential map is natural. That is, given a map of Lie groups \( f : G \to H \), show that for any \( X \in g \) we have
\[ \exp(\delta f(X)) = f(\exp(X)), \]
where \( \exp \) on the left denotes the exponential map for the group \( H \), \( \exp \) on the right denotes the exponential map for \( G \) and \( \delta f : g \to h \) is the induced map of Lie algebras, i.e., \( \delta f = df_e \).

**Exercise 10.3.** Let \( \pi_E : E \to M \) and \( \pi_F : F \to M \) be vector bundles over \( M \).
(a) Show that \( E \times F \) is a vector bundle over \( M \times M \).
(b) Explain why the fiber product
\[ G = E \times_M F = \{(e, f) \in E \times F : \pi_E(e) = \pi_F(f)\} \]
can be considered a vector bundle over \( M \).
(c) Show that, as a vector bundle over \( M \), \( G \) is isomorphic to \( E \oplus F \). Hint: compare the transition maps.

**Exercise 10.4.** Prove that for any two differential forms \( \alpha \in \Omega^k(M) \), \( \beta \in \Omega^l(M) \) on a manifold \( M \),
\[ \alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha. \]

**Exercise 10.5.** The following exercise has been assigned before (sorry). You don’t need to do it. Awhile back you proved that the real projective space \( \mathbb{R}P^{n-1} \) is a manifold. Prove that the complex projective space \( \mathbb{C}P^{n-1} \) of complex lines in \( \mathbb{C}^n \) is a real manifold of dimension \( 2n - 2 \). Hint: it’s almost the same proof as in the case of \( \mathbb{R}P^{n-1} \). To keep the notation from getting out of control remember that the map
\[ \mathbb{C} \times (\mathbb{C} \setminus \{0\}) \to \mathbb{C}, (z, w) \mapsto z/w \]
is a perfectly nice \( C^\infty \) map.