Two topics have been requested for review:
1. Lie derivatives
2. Vector bundles.

**Lie derivatives**

Recall that there are two equivalent ways to view a vector field $X$ on a manifold $M$:
1) **as a derivation** $X: \mathcal{C}^0(M) \to \mathcal{C}^0(M)$
2) **as a section** $X: M \to TM$ of $\pi: TM \to M$.

If $f$ is a function and $X$ is a vector field, $(\gamma) \mapsto Xf$ is again a function. In terms of the (local) flow of $X$, $(Xf)(\phi(t, \gamma)) = \frac{d}{dt} \bigg|_0 f(\phi(t, \gamma)) = \lim_{t \to 0} \frac{1}{t} (f(\phi(t, \gamma)) - f(\gamma))$

We define the Lie derivative $L_X f$ of $f$ w.r.t. $X$ to be $Xf$.

(*) gives $L_X f$ a geometric interpretation.

(i) gives $L_X f$ an "algebraic" interpretation.

$\mathcal{C}^0(M) = \mathcal{O}^0(M)$.
$\Omega^0(M) = \bigoplus_{i=0}^{\infty} \Omega^i(M)$ is a graded commutative algebra.

(for $M$ fixed)

If $U \subset M$ open we have a restriction map $\Omega^0(M) \to \Omega^0(U)$.

A manifold $M$ we get a sheaf of graded commutative $\mathcal{C}^0(M)$-algebras.

Why do we care? If $U \subset \mathcal{O}^{\infty}(M)$ generated, as an algebra by

$\{t^i | f \in \mathcal{O}^0(M) \}$ and $\frac{\partial f}{\partial x_i} |_{x \in \mathcal{O}^0(M)}$
$\alpha \in \Omega^k(U), \quad \alpha' = \sum_{I=1}^{n-k} \frac{\partial \alpha}{\partial x_i} \wedge dx_i,$ \quad \alpha' \in \Omega^{k+1}(U)$

coordinates on the ball $U$.

$L_x : \Omega^0(U) \rightarrow \Omega^0(U)$ extends uniquely to a (graded) derivative $L_x : \Omega^0(U) \rightarrow \Omega^0(M)$ that commutes with $d$:

$L_x (d h) = d(L_x h) \quad \forall h \in \Omega^0(U)$

$L_x (\sum f_k df_k) = (L_x f_0) df_0 + \sum_{i=1}^{n-k} f_i d f_i \wedge df_k$

Since $\forall \alpha \in \Omega^0(M)$ is uniquely determined by its restriction to an open cover of $M$, this allows us to extend $L_x : \Omega^0(M) \rightarrow \Omega^0(M)$ to a unique derivative of $\Omega^0(M)$ (of deg $0$)

$L_x : \Omega^0(M) \rightarrow \Omega^0(M)$ which commutes w. restrictions to open subsets.

Question 1: What's the geometry?

A. Pretend $\alpha$ has a global flow $\Phi_t \alpha$. Then

\[ (L_x \alpha)_q = \frac{d}{dt} \left( \Phi^t \alpha \right)_{\Phi^t(q)} \quad \forall \alpha \in \Omega^0(M) \]

\[ \forall q \in M \]

Consequences:

If $M$ is compact orientable, $\mu \Omega^{top}(M)$ are from $L_x \mu$ (function) $M$

The function $\mu$ measures how the flow of $\alpha$ expands/contracts volume of $M$ (as measured by $\mu$).
We have also proved:
\[ L_x = d \circ \tau(x) + \tau(x) \circ d \]
\[ d : \Omega^*(M) \to \Omega^{*+1}(M) \text{ a graded der. of degree } 1 \]
\[ \tau(x) : \Omega^*(M) \to \Omega^{*-1}(M) \]
\[ \text{of degree } -1 \]
\[ \Rightarrow [d, \tau(x)] = d \circ \tau(x) - (-1)^{1 \cdot (-1)} \tau(x) \circ d \]
\[ \text{a graded derivation of degree } 0. \]
\[ [d, \tau(x)] = L_x \text{ on } \Omega^0(M) \]
d, \tau(x) commute w.r.t. restrictions to open sets.
\[ [d, \tau(x)] \]
\[ \Rightarrow \]
\[ [d, \tau(x)] = L_x \text{ on all of } \Omega^*(M). \]

A (k,l) tensor on a manifold M is a section of the bundle \((TM)^{\otimes k} \otimes (T^*M)^{\otimes l} \to M\).

A vector field is a (1,0) tensor
A 1-form is a (0,1) tensor
A Riemannian metric is a (symmetric, pos. definite) (0,2) tensor.

One can define Lie derivatives of tensor fields. There are two equivalent ways to do it:
- algebraically
- geometrically

Ex. If \( \gamma \) is a (1,0) tensor
we define \( L^Y X = [X, g Y] = X_0 Y - Y_0 X \) (algebraic def)

We can also define
\[
(+) \quad (L^X Y)_v = \frac{d}{d\tau} \bigg|_{\tau = 0} (\Phi_{\tau}(v)) Y(\Phi_{\tau}(v)) \quad \forall v \in \mathfrak{m}, \quad \Phi_{\tau} \quad \text{flow of} \quad X.
\]

Thus, The two definitions agree.

It is not hard to extend (++) and \( \Phi \) to arbitrary tensor fields.

Equivalently, once we know \( L^X \) on \((1,0)\) tensors and \(0,0\) -tensors (functions) we can extend it to \((0,1)\) tensors (1-forms) by requiring that
\[
L^X (\iota(Y) \alpha) = \iota(L^X Y) \alpha + \iota(Y) L^X \alpha \quad \forall X, Y, \alpha \in \Gamma(TM), \quad \forall \alpha \in \Omega^1(M).
\]

and then extend it to all tensors by requiring that \( L^X \) is a “tensor derivation” which
(a) commutes with contractions
(b) commutes with restrictions to open sets.