Last Time. We started discussing Stokes’ theorem:
\[ \int_{\partial D} \alpha = \int_D d\alpha \]
for a regular domain \( D \) with boundary \( \partial D \) in an oriented manifold \( M \) and a compactly supported form \( \alpha \) of degree \( \dim M - 1 \).

Today we will make sense of the "\( d \)" in "do" in the statement of Stokes’ theorem above. Later we will define regular domains, their boundaries and orientation of boundaries induced by the orientation of the ambient manifold.

**Theorem 29.1** (Exterior Derivative). For every manifold \( M \) there is a unique \( \mathbb{R} \)-linear map
\[ d_M : \Omega^* (M) \to \Omega^{*+1} (M) \]
(i.e. \( \forall \omega \in \Omega^k (M) \) we have \( d_M \omega \in \Omega^{k+1} (M) \)) called exterior derivative such that
1. For all \( f \in C^\infty = \Omega^0 (M) \) we have \( d_M f = df \).
2. For all open \( W \subseteq M \) and all \( \omega \in \Omega^k (M) \) we have \( (d_M \omega)_W = d_W (\omega_W) \).
3. For all \( \omega \in \Omega^k (M) \) and all \( \eta \in \Omega^l (M) \) we have \( d_M (\omega \wedge \eta) = d_M (\omega) \wedge \eta + (-1)^k \omega \wedge d_M \eta \).
4. For all \( \omega \in \Omega^k (M) \) we have \( d_M (d_M \omega) = 0 \) (i.e. \( d_M^3 = 0 \)).

**Proof of Uniqueness.** Suppose that there exists \( d_M : \Omega^* (M) \to \Omega^{*+1} (M) \) with properties (1)-(4) above. Let \((x_1, \ldots, x_m) : U \to \mathbb{R}^m\) be a coordinate chart. Then for all \( \alpha \in \Omega^k (M) \)
\[ \alpha|_U = \sum_{|I|=k} a_I \, dx_I = \sum_{I=i_1 < \cdots < i_k} a_I \, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \]

**Claim 1.** If \( d \) exists then we must have \( d_U (dx_I) = 0 \)

**Proof.** We will proceed by induction on \( k \). If \( k = 1 \) then
\[ d_U (dx_i) \overset{by(1)}{=} d_U d_U x_i \overset{by(4)}{=} 0. \]
Suppose that \( d_U (dx_{i_1} \wedge \cdots \wedge dx_{i_n}) = 0 \), then
\[ d_U (dx_{i_1} \wedge \cdots \wedge dx_{i_n} \wedge dx_{i_{n+1}}) \overset{by(3)}{=} d_U (dx_{i_1} \wedge \cdots \wedge dx_{i_n}) \wedge dx_{i_{n+1}} + (-1)^n dx_{i_1} \wedge \cdots \wedge dx_{i_n} \wedge d_U (dx_{i_{n+1}}). \]
Hence
\[
(d_M \alpha)|_U = d_U (\alpha|_U) = d_U \left( \sum_{|I|=k} a_I \, dx_I \right) = \sum_{|I|=k} (d_U a_I \wedge dx_I + a_I \, d_U (dx_I)) = \sum_{|I|=k} da_I \wedge dx_I. 
\]
Therefore if \( d'_M : \Omega^* (M) \to \Omega^{*+1} (M) \) is another \( \mathbb{R} \)-linear map with properties (1)-(4), then for all \( k \), all \( \alpha \in \Omega^k (M) \), and all coordinate charts \( U \) we have
\[ (d'_M \alpha)|_U = d'_U (\alpha|_U) = d'_U \left( \sum a_I \, dx_I \right) = \sum da_I \wedge dx_I = (d_M \alpha)|_U. \]
This proves uniqueness of the exterior derivative \( d_M \). \( \square \)

**Proof of Existence.** For each coordinate chart \((x_1, \ldots, x_m) : U \to \mathbb{R}^m\) on \( M \) define for all \( k \) the map \( d_U : \Omega^k (M) \to \Omega^{k+1} (M) \) by
\[
(29.1) \quad d_U \left( \sum_{|I|=k} a_I \, dx_I \right) \overset{def}{=} \begin{cases} 
  da_I & \text{if } |I| = 0 \\
  \sum_{|I|=k} da_I \wedge dx_I & \text{if } |I| > 0 
\end{cases}
\]
Assume for the moment:

**Claim 2.** $d_U$ defined by (29.1) has properties (1)-(4).

Now define $d_M : \Omega^*(M) \to \Omega^{*+1}(M)$ as follows: for any coordinate chart $U$ set

\[(29.2) \quad (d_M \alpha)|_U = d_U(\alpha|_U)\]

Does equation 29.2 make sense? Suppose that $U$ and $V$ are two compatible coordinate charts. Then

\[
(d_U(\alpha|_U)|_{U\cap V}) = d_{U\cap V}(\alpha|_{U\cap V}) = d_{U\cap V}(\alpha|_V|_{U\cap V}) = (d_U(\alpha|_V)|_{U\cap V})
\]

Next we prove claim 2. We check conditions (1)-(4).

(1). For all $f \in C^\infty(U)$ we have $d_U f = df$ by definition of $d_U$.

(2). Recall that we proved that $d$ commutes with pullbacks and restriction $|_W$ to $W$ is the pullback by the inclusion $W \hookrightarrow U$. Hence for any $f \in C^\infty(U)$ and any open $W \subseteq U$ we have

\[
(d_U f)|_W = df|_W = d(f|_W) = d_W(f|_W)
\]

Now

\[
d_U(a_I dx_I)|_W = (da_I \wedge dx_I)|_W = da_I|_W \wedge (dx_I)|_W = d(a_I|_W) \wedge (dx_I|_W) = d_W((a_I dx_I)|_W)
\]

Hence $d_U$ has property (2).

(3). It is no loss of generality to assume that $\omega = a_I dx_I$ and $\eta = b_J dx_J$ for some indices $I$ and $J$ and some functions $a_I, b_J$. Then

\[
d_U(\omega \wedge \eta) = d_U(a_I dx_I \wedge b_J dx_J) = d_U(a_I b_J dx_I \wedge b_J dx_J) = da_I b_J \wedge dx_I \wedge dx_J = \frac{\partial^2 a_I}{\partial x_i \partial x_j} dx_j \wedge dx_i + (-1)^k \left(\frac{\partial a_I}{\partial x_i} \right) \wedge \frac{\partial b_J}{\partial \omega} \wedge dx_J
\]

This proves that (3) holds for $d_U$.

(4).

\[
d_U(d_U(a_I dx_I)) = d_U(da_I \wedge dx_I) = d_U(da_I) \wedge dx_I + (-1)^k da_I \wedge d_U(dx_I)
\]

\[
= d_U \left(\sum \frac{\partial a_I}{\partial x_i} dx_i \right) \wedge dx_I
\]

Now

\[
d_U \left(\sum \frac{\partial a_I}{\partial x_i} dx_i \right) = \sum \frac{\partial^2 a_I}{\partial x_i \partial x_j} \wedge dx_j \wedge dx_i
\]

hence is 0. Therefore $d_U(d_U(a_I dx_I)) = 0$ and consequently $d_U \circ d_U = 0$. This proves claim 2. $\square$
It remains to show that \( d_M \) defined by equation 29.2 has properties (1)-(4).

(1). For all \( f \in C^\infty(M) \) and any coordinate chart \( U \)

\[
(d_M f) \bigg|_U = \frac{\partial}{\partial x^a} \bigg|_x \cdot f(x^a),
\]

by our definition of \( d_M \).

\[
d_U (f \big|_U) = \frac{\partial}{\partial x^a} \bigg|_x \cdot f(x^a) = df \bigg|_U.
\]

Since \( U \) is arbitrary, \( d_M f = df \).

(2). For all open \( W \subseteq M \) and all coordinate charts \( U \subseteq M \) we know that \( U \cap W \) is a coordinate chart on \( W \). Since \( d_U \) has property (3), we know that for all \( \mu \in \Omega^*(U) \)

\[
(d_U \mu) \bigg|_{U \cap W} = d_{U \cap W} (\mu \big|_{U \cap W})
\]

Therefore for any \( \omega \in \Omega^*(M) \)

\[
((d_M \omega) \big|_{U \cap W}) = d_{U \cap W} (\omega \big|_{U \cap W})
\]

Hence \( d_M \omega \big|_{W} = d_{W} (\omega \big|_{W}) \) if \( d_M \) and \( d_W \) are defined by equation 29.2.

(3). Say \( \omega \in \Omega^k(M), \eta \in \Omega^l(M) \) and \( U \) a coordinate chart, then

\[
d_M (\omega \wedge \eta) \bigg|_U \overset{29.2}{=} d_U ((\omega \wedge \eta) \bigg|_U)
\]

\[
= d_U (\omega \big|_U \wedge \eta \big|_U)
\]

\[
= d_U (\omega_U \wedge (\eta \big|_U)) + (-1)^k (\omega \big|_U) \wedge d_U (\eta \big|_U)
\]

\[
= (\omega \wedge (d_M \eta) \bigg|_U + (-1)^k (\omega \wedge (d_M \eta)) \bigg|_U
\]

(4).

\[
d_M (d_M \omega) \bigg|_U = d_U ((d_M \omega) \bigg|_U) = d_U (d_U (\omega \big|_U)) = 0
\]

since \( d_U \circ d_U = 0 \).