Last Time.

1. Given a vector bundle \( E \xrightarrow{\pi} M \) we started constructing the dual bundle \( E^* \xrightarrow{\pi^*} M \) as a set \( E^* = \coprod_{q \in M}(E_q)^* \).

2. Out of trivializations \( \varphi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^k \) we constructed purported trivializations \( \varphi_\alpha^* : E^*|_{U_\alpha} \to U_\alpha \times (\mathbb{R}^k)^* \) bijections, linear on each fiber.

3. We checked \( (\varphi_\alpha^* \circ (\varphi_\beta^*)^{-1})(q,l) = (q, \varphi_\alpha^{-1}(\varphi_\beta(q))l) \) where \( \varphi_\alpha^*: U_\alpha \cap U_\beta \to \text{GL}(\mathbb{R}^k) \) are \( C^\infty \).

To prove that \( E^* \) is a manifold, that \( \varphi_\alpha^* \) are diffeomorphisms, and that \( \pi^*: E^* \to M \) is smooth we need a proposition.

**Proposition 27.1.** Suppose that we have a set \( X \), a cover \( \{U_\alpha\}_{\alpha \in A} \) of \( X \), a collection of bijections \( \{\psi_\alpha : V_\alpha \to W_\alpha\}_{\alpha \in A} \) where \( W_\alpha \) are manifolds such that for all \( \alpha, \beta \in A \)

(i) \( \psi_\alpha(V_\alpha \cap V_\beta) \) is open in \( W_\alpha \) and

(ii) \( \psi_\alpha \circ \psi_\beta^{-1} : \psi_\beta(V_\alpha \cap V_\beta) \to \psi_\alpha(V_\alpha \cap V_\beta) \) is \( C^\infty \),

then \( X \) is a manifold so that all \( \psi_\alpha \) are diffeomorphisms.

Note that the proposition implies that the total space \( E^* \) of the bundle dual to \( E \to M \) is a manifold. Moreover, for all \( U_\alpha \) the following diagram commutes

\[
\begin{array}{ccc}
E^*|_{U_\alpha} & \xrightarrow{\varphi_\alpha^*} & U_\alpha \times (\mathbb{R}^k)^* \\
\pi^* \downarrow & & \downarrow \text{pr}_1 \\
U_\alpha & \xrightarrow{\text{pr}_1} & U_\alpha
\end{array}
\]

Hence \( \pi^*|_{E^*_U} = \text{pr}_1 \circ \varphi_\alpha^* \) is \( C^\infty \). Therefore \( \pi^* : E^* \to M \) is \( C^\infty \). Not hard to check that \( \varphi_\alpha^* : E^*|_{E^*_U} \to U_\alpha \times (\mathbb{R}^k)^* \) are diffeomorphisms. Consequently \( E^* \xrightarrow{\pi^*} M \) is indeed a vector bundle.

**Sketch of proof.**

1. The sets \( \{\varphi_\alpha^{-1}(O) \mid \alpha \in A \text{ and } O \in W_\alpha \text{ is open}\} \) form a basis for a topology on \( X \) which make \( \psi_\alpha \) into homeomorphisms.

2. Each point \( x \in X \) lies in some \( V_\alpha \). \( \psi_\alpha(x) \) lies in a coordinate chart \( \varphi : U \to \mathbb{R}^m \) on \( W_\alpha \). Declare \( \varphi \circ \psi_\alpha : \psi_\alpha^{-1}(U) \to \mathbb{R}^m \) to be a coordinate chart. (ii) implies that the charts define an atlas.

\[ \square \]

Can we perform other operations? What do we need?

**Example 27.2.** Suppose given a vector bundle \( E \xrightarrow{\pi} M \) of rank \( k \) we want to construct the \( n \text{th} \) exterior power \( \wedge^n E \xrightarrow{\pi^n} M \) of a vector bundle \( E \to M \). We set

\[ \wedge^n E = \coprod_{q \in M} \wedge^n(E_q) \] (as a set).

Out of a collection \( \{\varphi_\alpha : E|_{U_\alpha} \to U_\alpha \times V\}_{\alpha \in A} \) of local trivializations with \( \bigcup U_\alpha = M \) (\( V \) is a fixed finite dimensional vector space) we get for all \( \alpha \) and all \( q \in U_\alpha \) linear isomorphisms

\[ \varphi_\alpha|_{E_q} : E_q \xrightarrow{\sim} \{q\} \times V. \]

Applying exterior power \( \wedge^n \) to everything above we get

\[ \wedge^n(\varphi_\alpha|_{E_q}) : \wedge^n E_q \to \{q\} \times \wedge^n(V), \]

whence

\[ \wedge^n(\varphi_\alpha) : \wedge^n E|_{U_\alpha} \to \{U_\alpha\} \times \wedge^n(V) \]

Hence for all indices \( \alpha \) and \( \beta \) with \( U_\alpha \cap U_\beta \neq \emptyset \) we have

\[ (\wedge^n(\varphi_\alpha) \circ (\wedge^n(\varphi_\beta))^{-1})(q,\eta) = (q, \wedge^n(\varphi_\alpha(q))\eta). \]
For any finite dimensional vector space $V$ over $\mathbb{R}$ we have a map

$$\Lambda^n : \text{GL}(V) \to \text{GL}(\Lambda^n V), \quad A \mapsto A^n,$$

which is a group homomorphism and is polynomial in $A$. That is to say, $\Lambda^n((a_{ij}))$ has entries which are polynomials in $a_{ij}$’s. Hence $\Lambda^n$ is $C^\infty$. Therefore the purported transition maps $\Lambda^n(\varphi_{\alpha\beta}) : U_\alpha \cap U_\beta \to \text{GL}(\Lambda^n V)$ are $C^\infty$. Now Proposition 27.1 implies that $\Lambda^n E$ is a manifold and the local trivializations $\{\Lambda^n \varphi_\alpha : \Lambda^n E|_{U_\alpha} \to \{U_\alpha\} \times \Lambda^n(V)\}$ are smooth. Proceeding as in the case of the dual bundle we get that $\Lambda^n E \to M$ is a vector bundle of rank $\binom{n}{k}$.

Note that at this point we have constructed, for any manifold $M$, the bundles $\Lambda^n(T^*M) \to M$ and hence differential forms.

**Example 27.3.** Suppose that $E \xrightarrow{\pi_E} M$ and $F \xrightarrow{\pi_F} M$ are two vector bundles. Let’s try and construct the Whitney sum $E \oplus F \to M$. We choose a cover $U_\alpha$ of $M$ such that $E|_{U_\alpha}$ and $F|_{U_\alpha}$ are both trivial for all $\alpha$. We have trivializations

$$\varphi^E_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^k \quad \varphi^F_\alpha : F|_{U_\alpha} \to U_\alpha \times \mathbb{R}^l$$

We set $E \oplus F = \coprod_{\alpha \in M} E_\alpha \oplus F_\alpha$ (as a set). The purported trivializations are

$$\varphi^{E \oplus F}_\alpha : (E|_{U_\alpha} \oplus F|_{U_\alpha}) \to U_\alpha \times (\mathbb{R}^k \oplus \mathbb{R}^l)$$

The corresponding transition maps are

$$\varphi^{E \oplus F}_{\alpha \beta}(q) = \varphi^E_{\alpha \beta}(q) \oplus \varphi^F_{\alpha \beta}(q)$$

and the map $\text{GL}(\mathbb{R}^k) \times \text{GL}(\mathbb{R}^l) \to \text{GL}(\mathbb{R}^{k+l})$ with $(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ is $C^\infty$. Proceeding as in the case of exterior powers we get that $E \oplus F \to M$ is a vector bundle.

**Question.** What’s the general principle?

**Answer.** $C^\infty$ functors.

To define functors we must first define categories.

**Definition 27.4.** A category $C$ consists of

- A collection of objects $C_0$.
- For each pair of objects $X, Y \in C_0$ a set $\text{Hom}_C(X, Y)$ of arrows/morphisms.
- For each triple of objects $X, Y, Z \in C_0$ a composition $\circ : \text{Hom}_C(Y, Z) \times \text{Hom}_C(X, Y) \to \text{Hom}_C(X, Z)$

$$\begin{pmatrix} Z \leftarrow^g Y \leftarrow^f X \end{pmatrix} \mapsto Z \xrightarrow{g \circ f} X$$

- For each object $X \in C_0$ a morphism $1_X : \text{Hom}_C(X, Y)$ such that
  (i) For all $f \in \text{Hom}_C(X, Y)$ we have $1_Y \circ f = f \circ 1_X$; and
  (ii) $\circ$ is associative: for all $W \xrightarrow{h} Z \xrightarrow{g} Y \xleftarrow{f} X$ we have $h \circ (g \circ f) = (h \circ g) \circ f$.

We set $C_1 = \coprod_{X, Y \in C_0} \text{Hom}_C(X, Y)$. This is a collection of all morphisms.

**Example 27.5.** $C = \text{Set}$, the category of all sets and maps of sets is a category. $C_0$ is the collection of all sets and $C_1$ is the collection of all maps.

**Example 27.6.** $C = \text{Top}$, the category of topological spaces and continuous maps.

**Example 27.7.** $C = \text{Man}$, the category of manifolds. $C_0$ is usually taken as the collection of all finite dimensional, Hausdorff, paracompact manifolds and the morphisms are $C^\infty$ maps.

**Example 27.8.** $C = \text{Lie}$, the category of Lie groups.

**Example 27.9.** $C = \text{Vec}$, the category of finite dimensional vector spaces over $\mathbb{R}$ where the morphisms are linear maps.
Example 27.10. $\mathcal{C} = \text{Vec}^{\text{iso}}$, the category of finite dimensional vector spaces over $\mathbb{R}$ where the morphisms are linear isomorphisms.

Next Time. Functors, smooth functors, and differential forms.