Definition 25.4. We still need to define differential forms. As a first step we define vector bundles.

Proof. Fix $q \in M$, $v \in T_q M$. Recall that $\forall h \in C^\infty(M)$ we have $(dh_q)(v) = v(h)$. Therefore 
\[d(F^*f)_q(v) = v(f \circ F) = (dF_q v)(f) = d_f F(q)(dF_q v) = (F^*d f)_q(v)\]

□

Fact 25.2. If $N \subseteq M$ is a codimension 1 submanifold, then $\forall \mu \in \Omega^c_{\dim M}(M)$ we have $\int_M \mu = \int_{M \setminus N} \mu$.

Example 25.3. Compute 
\[\int_{S^1} -\frac{y}{x^2 + y^2} \, dx + \frac{x}{x^2 + y^2} \, dy\]

for $S^1 \triangleq \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \}$ oriented by the parametrization $f : \theta \mapsto (\cos \theta, \sin \theta)$ for $0 < \theta < 2\pi$. Then 
\[\int_{S^1} -\frac{y}{x^2 + y^2} \, dx + \frac{x}{x^2 + y^2} \, dy = \int_{(0, 2\pi)} f^* \left( -\frac{y}{x^2 + y^2} \, dx + \frac{x}{x^2 + y^2} \, dy \right) \]
\[= \int_{(0, 2\pi)} -\frac{\sin \theta}{\cos^2 \theta + \sin^2 \theta} \, d(\cos \theta) + \frac{\cos \theta}{\cos^2 \theta + \sin^2 \theta} \, d(\sin \theta) \]
\[= \int_{(0, 2\pi)} \sin^2 \theta \, d\theta + \cos^2 \theta \, d\theta \]
\[= \int_{(0, 2\pi)} d\theta = 2\pi\]

We still need to define differential forms. As a first step we define vector bundles.

Definition 25.4. A (real) vector bundle $E$ of rank $k$ over a manifold $M$ is a manifold $E$ together with a smooth map $\pi = \pi_E : E \to M$ such that the following two conditions hold.

1. For each $q \in M$ the fiber $E_q \triangleq \pi^{-1}(q)$ is a real vector space of dimension $k$.
2. For each $q \in M$ there exists an open neighborhood $U$ of $q$ and a diffeomorphism $\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ so that
   (a) $\forall p \in U$, $\varphi(E_p) = \{ p \} \times \mathbb{R}^k$. I.e. the following diagram commutes:

      \[\begin{array}{ccc}
      \pi^{-1}(U) & \xrightarrow{\varphi} & U \times \mathbb{R}^k \\
      \pi \downarrow & & \downarrow \text{Pr}_1 \\
      U & \xrightarrow{\varphi|_{E_p}} & \{ p \} \times \mathbb{R}^k
      \end{array}\]

   (b) $\forall p \in U$, $\varphi|_{E_p} : E_p \to \{ p \} \times \mathbb{R}^k$ are $\mathbb{R}$-linear isomorphisms.

Terminology.

- $E$ is called the total space of the vector bundle.
- $M$ is called the base.
- $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ is called a local trivialization.
- $E_p = \pi^{-1}(p)$ is called the fiber above $p \in M$.
- $\mathbb{R}^k$ is called the typical fiber.

Remark 25.5. Replacing “real” by “complex” in the definition above gives a definition of a complex vector bundle.

Remark 25.6. Instead of $\mathbb{R}^k$ in the definition of the vector bundle we could have a fixed finite dimensional vector space $V$ (i.e., we don’t really need to choose a basis).

Example 25.7. $M \times \mathbb{R}^k \to M$, $(p, v) \mapsto p$ is a vector bundle of rank $k$. $\varphi = \text{id}$ and $U = M$. 

Example 25.8. \( M \) any manifold. \( TM \stackrel{\pi}{\to} M \) is a vector bundle of rank equal to \( \dim M \). Local trivializations? If \( \varphi = (x_1, \ldots, x_n) : U \to \mathbb{R}^n \) is a coordinate chart on \( M \), then \( \pi^{-1}(U) = TU = U \times \mathbb{R}^m \), \((p, v) \mapsto (p, (dx_1)_p v, \ldots, (dx_m)_p)\) is a local trivialization.

Definition 25.9. Let \( \pi_E : E \to M \), \( \pi_F : F \to M \) be two vector bundles over \( M \). A map of vector bundles \( f : E \to F \) is a map of manifolds \( f : E \to F \) so that

1. \( \forall q \in M, f(E_q) \subseteq F_q \). i.e. the following diagram commutes:

\[
\begin{array}{ccc}
E & \xrightarrow{f} & F \\
\pi_E \downarrow & & \downarrow \pi_F \\
M & & 
\end{array}
\]

2. \( \forall q \in M, f|_{E_q} : E_q \to F_q \) is linear.

Definition 25.10. A map of vector bundles (over \( M \)) \( f : E \to F \) is an isomorphism if there exists a map of vector bundles \( g : F \to E \) such that \( g \circ f = \text{id}_E \) and \( f \circ g = \text{id}_F \).

Exercise 25.1. Any bijective map of vector bundles is an isomorphism. Hint: Do it for trivial bundles first.

Definition 25.11. A vector bundle \( \pi : E \to M \) is trivial if it is isomorphic to a bundle \( M \times V \to M \) for some vector space \( V \).

Example 25.12. For any Lie group \( \mathfrak{g} \), \( T\mathfrak{g} \to \mathfrak{g} \) is trivial. The map \( \mathfrak{g} \times T \text{id}\mathfrak{g} \to T\mathfrak{g}, (g, v) \mapsto (g, (dL_g)\text{id}v) \) is an isomorphism. The inverse is given by \( h(g, w) = (g, (dL_g)_g w) \).

Fact 25.13. \( T\mathbb{S}^2 \to \mathbb{S}^2 \) is not trivial.

Remark 25.14. If \( E \stackrel{\pi}{\to} M \) is a vector bundle and \( W \subseteq M \) is open, then \( E|_W = \pi^{-1}(W) \hookrightarrow W \) is also a vector bundle. Note that if \( E \to M \) is a vector bundle, then for all \( p \in M \) there exists a neighborhood \( U \) of \( p \) such that \( E|_U \) is trivial. “Vector bundles are locally trivial.”

Definition 25.15. A section \( s \) of a vector bundle \( \pi : E \to M \) is a smooth map \( s : M \to E \) such that \( \pi \circ s = \text{id}_M \).

Example 25.16. A section of \( TM \to M \) is a vector field.

Example 25.17. A section of \( T^* M \to M \) is a 1-form.

Example 25.18. A section of \( M \times V \to M \), where \( V \) is a vector space, is a \( V \)-valued function on \( M \).

Remark 25.19. The space of section \( \Gamma(E) \) of a vector bundle \( E \to M \) is a \( C^\infty(M) \)-module: we can add two smooth section and get a smooth section; we can multiply a smooth section by a smooth function and get a smooth section.

Definition 25.20. A local section of a vector bundle is a section of \( E|_W \to W \) for some open \( W \subseteq M \).

Example 25.21. Consider \( \mathbb{S}^2 \equiv \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\} \subseteq \mathbb{R}^3 \). The normal bundle of the embedding \( \mathbb{S}^2 \subset \mathbb{R}^3 \) is

\[
E = \{(q, v) \in \mathbb{S}^2 \times \mathbb{R}^3 \mid v = \lambda q \text{ for some } \lambda \in \mathbb{R}\}
\]