Last Time. We constructed for every finite dimensional vector space $V$ a non-degenerate bilinear pairing

$$\Lambda^k(V^*) \times \Lambda^l(V) \to \mathbb{R} \quad \text{with} (l_1 \wedge \cdots \wedge l_k, v_1 \wedge \cdots \wedge v_k) = \det(l_i(v_j))$$

for any $l_1, \ldots, l_k \in V^*$, any $v_1, \ldots, v_k \in V$. As a consequence we can identify any product $l_1 \wedge \cdots \wedge l_k \in \Lambda^k(V^*)$ with an alternating $k$-linear map that whose value on the $k$-tuple $v_1, \ldots, v_k$ is $\det(l_i(v_j))$. From now on we identify

$$\Lambda^k(V^*) \xrightarrow{\sim} \text{Alt}^k(V; \mathbb{R}),$$

where $\text{Alt}^k(V; \mathbb{R})$ denotes the space of alternating $k$-linear maps. As a result, since $\Lambda^*(V^*)$ is a graded algebra, we can now multiply $k$-linear and $\ell$-linear alternating maps and get $k + \ell$-linear alternating maps:

$$\Lambda^k(V^*) \times \Lambda^\ell(V^*) \longrightarrow \Lambda^{k+\ell}(V^*)$$

$$(\alpha, \beta) \mapsto \alpha \wedge \beta.$$

Recall. For a manifold $M$, the tangent bundle $TM = \bigsqcup_{q \in M} T_qM$ can be given the structure of a manifold. In particular if $\varphi = (x_1, \ldots, x_m) : U \to \mathbb{R}^m$ is a chart on $M$, then

$$\tilde{\varphi} = (x_1, \ldots, x_m, dx_1, \ldots, dx_m) :TU \to \mathbb{R}^m \times \mathbb{R}^m$$

is a chart on $TM$. Recall also that Vector fields on $M$ are sections of $TM \xrightarrow{\pi} M$, that is, maps $X : M \to TM$ with $\pi \circ X = \text{id}_M$.

Similarly one can define the cotangent bundle $T^*M = \bigsqcup_{q \in M} T^*_qM$ and give it the structure of a smooth manifold in more or less the same way we made the tangent bundle into a manifold. That is, we manufacture new coordinate charts on $T^*M$ out of coordinate charts on $M$ and check that the transition maps between the new coordinate charts are smooth. If $\varphi$ is a chart then

$$\tilde{\varphi} = \left(x_1, \ldots, x_m, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}\right) : T^*U \to \mathbb{R}^m \times \mathbb{R}^m$$

$$(q, \eta) \mapsto \left(x_1(q), \ldots, x_m(q), \left.\eta_i \frac{\partial}{\partial x_i}\right|_{q}, \ldots, \left.\eta_i \frac{\partial}{\partial x_m}\right|_{q}\right)$$

is a chart on $T^*M$.

Here is an excerpt from the old notes checking the smoothness of the transition maps. Let $\psi = (y_1, \ldots, y_n) : V \to \mathbb{R}^n$ be a coordinate chart on $M$ with $V \cap U \neq \emptyset$. Then

$$\tilde{\psi} \circ \tilde{\varphi}^{-1}(r_1, \ldots, r_n, w_1, \ldots, w_n) = \tilde{\psi}(\sum_{i=1}^n w_i(dx_i)_{\varphi^{-1}(r)})$$

$$= ((\psi \circ \varphi^{-1})(r), \left.\frac{\partial}{\partial y_1}(\sum_{i=1}^n w_i dx_i), \ldots, \frac{\partial}{\partial y_n}(\sum_{i=1}^n w_i dx_i)\right)$$

$$= ((\psi \circ \varphi^{-1})(r), \sum_{i=1}^n \frac{\partial x_i}{\partial y_1} w_i, \ldots, \sum_{i=1}^n \frac{\partial x_i}{\partial y_n} w_i).$$

We conclude that

$$\tilde{\psi} \circ \tilde{\varphi}^{-1}(r_1, \ldots, r_n, w_1, \ldots, w_n) = (\psi \circ \varphi^{-1}(r), \left.\frac{\partial x_i}{\partial y_1}(r)\right| \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}, \ldots, \left.\frac{\partial x_i}{\partial y_n}(r)\right| \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}),$$

which is smooth. The rest of the argument proceeds as in the case of the tangent bundle.

We will see in a week that one can construct the exterior powers $\Lambda^k(T^*M) = \bigsqcup_{q \in M} \Lambda^k(T^*_qM)$ and get manifolds over $M$. The smooth sections of $\Lambda^k(T^*M) \to M$ are called differential $k$-forms.

Notation. $\Omega^k(M) \triangleq \Gamma(\Lambda^k(T^*M))$ such that $\pi \circ \omega = \text{id}_M$. What then is $\Omega^0(M)$? By convention $\Lambda^0(T^*M) = M \times \mathbb{R}$. Consequently

$$\Omega^0(M) = \left\{M \xrightarrow{\tau} M \times \mathbb{R} \mid \tau(q) = (q, f(q)), \ f : M \to \mathbb{R}\right\} = C^\infty(M)$$
Remark 23.1. Differential forms on $M$ can be multiplied point-wise: $\forall \alpha \in \Omega^k(M) \forall \beta \in \Omega^l(M)$

$$\langle \alpha \wedge \beta \rangle_q \equiv \alpha_q \wedge \beta_q$$

for all points $q \in M$.

Example 23.2. Let $M = \mathbb{R}^m$. Then $TM = \mathbb{R}^m \times \mathbb{R}^m$ and $T^*M = \mathbb{R}^m \times (\mathbb{R}^m)^*$. At every $q \in \mathbb{R}^m$ we have a basis of $T_q^*\mathbb{R}^m: (dx_1)_q, \ldots, (dx_m)_q$. So $\Lambda^k(T^*M) = \mathbb{R}^m \times \Lambda^k((\mathbb{R}^m)^*)$ and

$$\alpha \in \Omega^k(\mathbb{R}^m) \iff \alpha = \sum_{|I|=k} a_I \, dx_I$$

where $I = \{i_1, \ldots, i_m\} \subseteq \{1, \ldots, m\}$, $a_I \in C^\infty(M)$, and $dx_I \equiv dx_{i_1} \wedge \cdots \wedge dx_{i_m}$

Example 23.3. 1-forms on $\mathbb{R}^2$ look like

$$M(x, y) \, dx + N(x, y) \, dy$$

where $M(x, y), N(x, y)$ are smooth functions.

Example 23.4. 2-forms on $\mathbb{R}^3$ look like

$$P(x, y, z) \, dx \wedge dy + Q(x, y, z) \, dy \wedge dz + R(x, y, z) \, dz \wedge dx,$$

where, again, $P, Q$ and $R$ are smooth functions.

Example 23.5.

$$\alpha = \cos v \, du - u \sin v \, dv \in \Omega^1(\mathbb{R}^2)$$

$$\beta = \sin v \, du + u \cos v \, dv \in \Omega^1(\mathbb{R}^2)$$

$$\alpha \wedge \beta = (\cos v \, du - u \sin v \, dv) \wedge (\sin v \, du + u \cos v \, dv)$$

$$= \cos v \sin v \, du \wedge du + u \cos^2 v \, du \wedge dv$$

$$- u \sin^2 v \, dv \wedge du + u^2 \sin v \cos v \, dv \wedge dv$$

$$= u \cos^2 v \, du \wedge dv - u \sin^2 v \, dv \wedge du$$

$$= u \, du \wedge dv$$

Remark 23.6. For any function $f \in C^\infty(M)$, $df$ is a 1-form. In particular given a coordinate chart $(x_1, \ldots, x_m)$ we have

$$df = \sum \left< df, \frac{\partial}{\partial x_i} \right> dx_i = \sum \frac{\partial f}{\partial x_i} \, dx_i$$

Example 23.7.

$$f(u, v) = u \cos v \quad df = \frac{\partial}{\partial x_i} \, dx_i = \frac{\partial f}{\partial x_i} \, dx_i$$

Hence in Example 23.5 we have $df \wedge dg = df(u \cos v) \wedge du (u \sin v) = du \wedge dv$.

Remark 23.8. Once we define pullback of differential forms we’ll see that $u \, du \wedge dv$ is the pullback of $dx \wedge dy$ by $f(u, v) = (u \cos v, u \sin v)$.

We now proceed to define pullbacks of differential forms by smooth maps.

Recall. Given a linear map $A : V \to W$ between two vector spaces we get $\Lambda^k A : \Lambda^k V \to \Lambda^k W$. We also have $A^* : W^* \to V^*$ where

$$(A \ast l)(v) = l(Av) = (l \circ A)(v)$$

Hence given a linear map $A : V \to W$ we get $\Lambda^k(A^*) : \Lambda^k W^* \to \Lambda^k V^*$ with $(l_1 \wedge \cdots \wedge l_k) \mapsto (A^* l_1) \wedge \cdots \wedge (A^* l_k)$.
What does this map $\Lambda^k(A^*)$ amount to when we identify exterior powers of the dual vector spaces with alternating multilinear maps? We compute:

\[
((\Lambda^k(A^*))l_1 \wedge \cdots \wedge l_k)(v_1, \ldots, v_k) = (l_1 \circ A) \wedge \cdots \wedge (l_k \circ A)(v_1, \ldots, v_k)
\]

\[
= \det(l_i(Av_j))
\]

\[
= (l_1 \wedge \cdots \wedge l_k)(Av_1, \ldots, Av_k)
\]

**Remark 23.9.** Note that for all $\alpha \in \Lambda^k(W^*)$ and all $\beta \in \Lambda^n(W^*)$ we have

\[
\Lambda^k(A^*) \alpha \wedge \Lambda^n(A^*) \beta = \Lambda^{k+n}(A^*)(\alpha \wedge \beta)
\]

since $\Lambda^*(A^*)$ is a map of algebras!

With these preliminaries out of the way we are now set to define pullbacks of differential forms. If $F : M \to N$ is a map of manifolds then for all $q \in M$ we have $dF_q : T_qM \to T_{F(q)}N$ which gives us the following map of algebras

\[
\Lambda^*((dF_q)^*) : \Lambda^*(T^*_F(q)N) \to \Lambda^*(T^*_qM)
\]

So for $\alpha \in \Omega^k(N)$ we get $F^* \alpha \in \Omega^k(M)$ defined by

\[
(23.2) \quad (F^* \alpha)_q = \Lambda^*((dF_q)^*) \alpha_{F(q)}.
\]

Strictly speaking we should check that if $\alpha$ is a smooth differential form on $N$ then its pullback $F^* \alpha$ is also smooth. But let’s not worry about this for the time being, certainly not until after we define $\Lambda^*(T^*M)$.

Equation (23.2) translates into:

\[
(23.3) \quad (F^* \alpha)_q(v_1, \ldots, v_k) = \alpha_{F(q)}((dF_q)v_1, \ldots, (dF_q)v_k)
\]

Why did we define pullback of differential forms by (23.2) and not by (23.3)? Because (23.2) automatically implies that

\[
F^*(\alpha \wedge \beta) = F^* \alpha \wedge F^* \beta
\]

for all differential forms $\alpha, \beta$ on $N$.

**Next Time.** $F^*(df) = df \circ F$. Hence if $x = r \cos \theta$ and $y = r \sin \theta$ then $dx \wedge dy = r \, dr \wedge d\theta$. 

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