Recall that an inner product on a (finite dimensional) real vector space $V$ is a bilinear map $g: V \times V \to \mathbb{R}$ (i.e., $g \in V^* \otimes V^*$) which is

1. symmetric: $g(v, w) = g(w, v)$ for all $v, w \in V$; and
2. positive definite: $g(v, v) > 0$ for all nonzero vectors $v \in V$.

1. Prove that an inner product $g$ on a vector space $V$ defines an isomorphism $g^\#: V \to V^*$ with
   
   $$(g^\#(v))(w) := g(v, w)$$

   for all $v, w \in V$. That is, $g^\#(v) = g(v, -) \in V^*$.

2. A Riemannian metric $g$ on a manifold $M$ is a smooth section of the bundle $T^*M \otimes T^*M \to M$ so that for all $q \in M$, the bilinear map $g_q: T_qM \times T_qM \to \mathbb{R}$ is an inner product. If $g$ is a Riemannian metric on a manifold $M$, the pair $(M, g)$ is called a Riemannian manifold. Prove that every manifold $M$ has a Riemannian metric. Hint: Construct the metric locally in coordinates first.

3. Prove that if $(M, g)$ is a Riemannian manifold then the metric $g$ defines an isomorphism of vector bundles $g^\#: TM \to T^*M$ by

   $$T_qM \ni v \mapsto g_q(v, -) \in T_q^*M$$

   for all $(q, v) \in T_qM$. That is

   $$g^\#(q, v) = g_q(v, -).$$

4. Prove that given a smooth function $f$ on a Riemannian manifold $(M, g)$ there is a unique vector field $\nabla f$ with

   $$g_q(\nabla f_q, v) = df_q(v)$$

   for all $q \in M$, $v \in T_qM$. This vector field $\nabla f$ is called the gradient vector field of $f$.

5. Let $f$ be a smooth function on a Riemannian manifold $(M, g)$ and $\gamma(t)$ an integral curve of the gradient vector field $\nabla f$ of $f$. Prove that

   $$\frac{d}{dt} f(\gamma(t)) \geq 0$$

   for all $t$ that $\gamma$ is defined. When is $\frac{d}{dt} f(\gamma(t)) = 0$?