1. Let $M$ be an open subset of $\mathbb{R}^3$. In $\mathbb{R}^3$, the standard inner product $(\cdot, \cdot)$ defines an isomorphism $\mathbb{R}^3 \rightarrow (\mathbb{R}^3)^*$, $v \mapsto (v, \cdot)$, which in turn induces an isomorphism of spaces of sections

$$A : \Gamma(TM) \rightarrow \Omega^1(M), \quad A(X) = (X, \cdot).$$

The standard volume form $\mu = dx_1 \wedge dx_2 \wedge dx_3$ defines an isomorphism $\mathbb{R}^3 \rightarrow \Lambda^2((\mathbb{R}^3)^*)$ by $v \mapsto \iota(v)\mu$, which also induces an isomorphism

$$B : \Gamma(TM) \rightarrow \Omega^2(M) \quad B(X) = \iota(X)\mu.$$

Finally, the map

$$C : C^\infty(M) \rightarrow \Omega^3(M) \quad C(f) = f\mu$$

is also an isomorphism. (Check these facts!)

Show that the standard vector calculus notions of div, grad, and curl can be defined as

1. grad($f$) = $A^{-1}(df)$ for any smooth function $f$ on $M$.
2. curl($X$) = $B^{-1}(d(A(X)))$ for any vector field $X$ on $M$.
3. div($X$) = $C^{-1}(d(B(X)))$ for any vector field $X$ on $M$.

In other words prove that the diagram

\[
\begin{array}{ccc}
C^\infty(M) & \xrightarrow{\text{grad}} & \Gamma(TM) & \xrightarrow{\text{curl}} & \Gamma(TM) & \xrightarrow{\text{div}} & C^\infty(M) \\
\downarrow & & \downarrow A & & \downarrow B & & \downarrow C \\
C^\infty(M) & \xrightarrow{d} & \Omega^1(M) & \xrightarrow{d} & \Omega^2(M) & \xrightarrow{d} & \Omega^3(M)
\end{array}
\]

commutes.

2. Prove that for any manifold $M$, a 1-form $\alpha$ on $M$ and any vector fields $X, Y$ on $M$,

$$L_X(\iota(Y)\alpha) = \iota(L_XY)\alpha + \iota(Y)L_X\alpha.$$

3. Let $V$ be a real vector space of dimension $n$, $v_1, \ldots, v_k \in V$ a finite collection of vectors. Prove:

$$v_1 \wedge v_2 \wedge \cdots \wedge v_k \neq 0$$

if and only if the set $\{v_1, \ldots, v_k\}$ is linearly independent.

4. Prove that for any $k$-form $\alpha$ and any $\ell$-form $\beta$ on a manifold $M$, we have

$$\alpha \wedge \beta = (-1)^{|\alpha||\beta|}\beta \wedge \alpha.$$