AN INTRODUCTION TO DIFFERENTIAL GEOMETRY

EUGENE LERMAN

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typeset August 19, 2011.

1. Introduction: why manifolds?

There are many different ways to formulate mathematically the notion of a 'space' that occurs in different branches of science and engineering. For instance one can talk about the space of configurations of a physical system. This, of course, requires a decision as to the level of details one is trying to model. For example, we can regard the configuration space of a system consisting of a sun and a planet as $\mathbb{R}^3 \times \mathbb{R}^3$. We use three real numbers to describe the position of the center of mass of the sun and three real numbers to describe the position of the center of mass of the planet. In this model we assume that the sun and the planet are simply two points in space. We also allow collisions. If we exclude collisions (but still allow the sun and the planet to come arbitrarily close to each other), the configuration space is then

$$Q = \{(x,y) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid x \neq y\}.$$

Here is another idealized example: the configuration space of a penny tumbling through the air. Fix a frame of reference. We will need a triple of real numbers to describe the position of the penny's center of gravity and three orthonormal vectors to describe the orientation of the penny. Thus the configuration space in question is

$$Q = \mathbb{R}^3 \times O(3),$$

where $O(3)$ denotes the set of 3×3 orthogonal matrices (recall that an $n \times n$ matrix is orthogonal if (and only if) its columns form an orthonormal basis of $\mathbb{R}^n$).[1]

Exercise 1.1. What is the configuration space of a penny rolling on a plane?

Manifolds constitute a particular way to formalize the notion of a configuration space. These are the spaces that "locally look like $\mathbb{R}^n$." The reason we will limit ourselves to manifolds is that they are particularly suitable for generalizing the ideas of calculus — differentiation and integration. We will see that the two examples of configuration spaces given above: $Q = \{(x,y) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid x \neq y\}$ and $Q = \mathbb{R}^3 \times O(3)$ are, indeed, manifolds.

Remark 1.1. There are, of course, many other notions of a "space." In linear algebra one studies vector spaces and maps between them. In algebraic geometry one studies spaces of solutions of polynomial equations which give rise to the notion of an algebraic variety. In metric topology/geometry one studies metric spaces, spaces with a notion of a distance. In point set topology and in algebraic topology one talks about topological spaces. In analysis one may study the space of solutions of a partial differential equation. In geometry and topology one may be forced to study spaces that have singularities such as orbifolds and stratified spaces. Before we can discuss orbifolds and more complicated spaces we should first come to terms with manifolds which are smooth.

2. Smooth manifolds

2.1. Digression: smooth maps from open subsets of $\mathbb{R}^n$ to $\mathbb{R}^m$.

We start out by recalling the definition of a differentiable map.

Definition 2.1. Let $U \subset \mathbb{R}^n$ be an open subset. A map $f : U \to \mathbb{R}^m$ is *differentiable* at a point $x \in U$ if there is a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ so that

$$\lim_{h \to 0} \frac{1}{||h||}(f(x+h) - f(x) - Lh) = 0.$$

It is not hard to show that if such a map L exists, it is unique. The linear map L is variously called the derivative of f at x, the differential of f at x, ... and is denoted by df_x or by Df_x or by $Df(x)$ or by a similar notation. Moreover, the matrix corresponding to L with respect to the standard basis of $\mathbb{R}^n$ and $\mathbb{R}^m$ is the so called Jacobian matrix. That is, if $f = (f_1, \dots f_m)$ then

$$Df_x = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \dots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \dots & \frac{\partial f_m}{\partial x_n}(x) \end{pmatrix}$$

[1] Strictly speaking the configuration space is $\mathbb{R}^3 \times SO(3)$, where $SO(3)$ denotes the set of orthogonal matrices with *positive* determinant. Why?

Definition 2.2. Let $U \subset \mathbb{R}^n$ be an open subset. A map $f : U \to \mathbb{R}^m$ is *smooth* (or C^∞) on the set U if all partial derivatives of f to all orders exist at all points of U.

Here is a more "sophisticated" version of the the definition above. Suppose $f : U \to \mathbb{R}^m$ is differentiable at all points of U. Then we have a map $g(x) := Df_x : U \to \mathbb{R}^{nm}$. We can require that g is differentiable as a map from U to $\mathbb{R}^{nm}$. The derivative of g is a map from U to a bigger vector space $\mathbb{R}^N$ for an appropriate N. We can require that *this* map is differentiable and so on...In other words, if all derivatives of $f : U \to \mathbb{R}^n$ exist and are differentiable we say that f is smooth.

2.2. **Definitions and examples of manifolds.** A smooth manifold is a generalization of a smooth surface in $\mathbb{R}^3$. A smooth surface in $S \subset \mathbb{R}^3$ has local parameterizations: for every point $p \in S$ there is an open set $V \subset \mathbb{R}^3$ with $p \in V$ and a map $x : U \to S \cap V$ (where $U \subset \mathbb{R}^2$ is an *open* set) such that

(1) x is C^∞. That is $x(u_1, u_2) = (x_1(u_1, u_2), x_2(u_1, u_2), x_3(u_1, u_2))$ and each $x_i(u_1, u_2)$, $1 \leq i \leq 3$ is an infinitely differentiable function of $u = (u_1, u_2) \in U$;

(2) x is 1-1 (injective) and onto.

The map x is a *local parameterization* of S.

Example 2.3. The two sphere

$$S^2 = \{x \in \mathbb{R}^3 \mid ||x|| = 1\}$$

is a smooth surface: if $p = (p_1, p_2, p_3) \in S^2$ and $p_3 > 0$ take $V = \{x \in \mathbb{R}^3 \mid x_3 > 0\}$, $U = \{(u_1, u_2) \mid ||u|| < 1\}$ and a local parameterization $x : U \to S^2 \cap V$ to be $x(u_1, u_2) = (u_1, u_2, \sqrt{1 - u_1^2 - u_2^2})$. It's easy to check that this x is 1-1, onto and C^∞. If $p_3 < 0$ take the local parameterization $x(u) = (u_1, u_2, -\sqrt{1 - u_1^2 - u_2^2})$. If $p_3 = 0$ then either p_1 or p_2 is non-zero (or both) and there are formulas for local parameterizations similar to the ones above.

Note that if S is a smooth surface and $x_\alpha : \mathbb{R}^2 \supset U_\alpha \to S$ and $x_\beta : \mathbb{R}^2 \supset U_\beta \to S$ are two local parameterizations with

$$W_{\alpha\beta} := x_\alpha(U_\alpha) \cap x_\beta(U_\beta) \neq \emptyset$$

then

$$x_\beta^{-1} \circ x_\alpha : \mathbb{R}^2 \supset x_\alpha^{-1}(W_{\alpha\beta}) \to x_\beta^{-1}(W_{\alpha\beta}) \subset \mathbb{R}^2$$

is C^∞.

This motivates:

Definition 2.4. [of a C^∞ manifold, first approximation, not quite right] A C^∞ manifold of dimension m is a set M and a family of injective maps $\{x_\alpha : U_\alpha \to M\}$ where $U_\alpha \subset \mathbb{R}^m$ are open sets, such that

(1) $\bigcup x_\alpha(U_\alpha) = M$;

(2) if for some pair of indices α and β, the set $W_{\alpha\beta} := x_\alpha(U_\alpha) \cap x_\beta(U_\beta) \neq \emptyset$ then $x_\alpha^{-1}(W_{\alpha\beta})$, $x_\beta^{-1}(W_{\alpha\beta})$ are open in $\mathbb{R}^m$ and

$$x_\beta^{-1} \circ x_\alpha : x_\alpha^{-1}(W_{\alpha\beta}) \to x_\beta^{-1}(W_{\alpha\beta})$$

are C^∞.

One thing that is wrong with this definition is that there is no *topology* specified on M. The other is that instead of parameterizations one usually works with charts that go the other way. Namely

Definition 2.5 (Chart). Let X be a topological space. An $\mathbb{R}^n$ *(coordinate) chart* on X is a homeomorphism $\phi : X \supset U \to U' \subset \mathbb{R}^n$.

Notation. We will often write $\phi : U \to \mathbb{R}^n$ or even (U, ϕ) for a coordinate chart $\phi : X \supset U \to U' \subset \mathbb{R}^n$. Note that since ϕ takes values in $\mathbb{R}^n$, it is an n-tuple of functions $\phi = (x_1, \ldots, x_n)$ for some functions $x_i : U \to \mathbb{R}$, the *coordinate functions on* U associated to the coordinate chart $\phi : U \to \mathbb{R}^n$.

Notation. When dealing with charts it will be convenient to to adopt the notation where the standard coordinate functions on $\mathbb{R}^n$ are denote by r_i, $1 \leq i \leq n$. That is, r_i assigns to a point $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ the number a_i. If $\phi : U \to \mathbb{R}^n$ is a chart then

$$x_i = r_i \circ \phi : U \to \mathbb{R}$$

are the coordinate functions on U,

Definition 2.6 (**Atlas**). A C^∞ *atlas* on a topological space X is a collection of charts $\{\phi_\alpha : U_\alpha \to U'_\alpha\}$ (with all U''s being open subsets of one fixed $\mathbb{R}^n$ such that

(1) $\{U_\alpha\}$ is an open cover of X,[2] and
(2) If $U_\alpha \cap U_\beta \neq \emptyset$, then $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is C^∞ as a map from an open subset of $\mathbb{R}^n$ to $\mathbb{R}^n$. That is, changes of coordinates are smooth.

Example 2.7. The identity map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x$ is the *s*tandard chart on $\mathbb{R}$. The set $\{(f, \mathbb{R})\}$ consisting of one chart is an atlas on $\mathbb{R}$. The map $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^3$ is also a chart on $\mathbb{R}$; it defines a different atlas on $\mathbb{R}$.

Here is a third atlas on $\mathbb{R}$. For each integer $n \in \mathbb{Z}$, $\phi_n : (n, n+2) \to \mathbb{R}$, $\phi_n(x) = x$ is a chart. The set $\{(\phi_n, (n, n+2)\}$ is an atlas on $\mathbb{R}$.

Definition 2.8. We say that *two atlases are equivalent* if their union is also an atlas.

The definition above amounts to: an atlas $\{x_\alpha : U_\alpha \to U'_\alpha\}$ is equivalent to an atlas $\{y_\beta : V_\beta \to V'_\beta\}$ if for any indices α, β with $U_\alpha \cap V_\beta \neq \emptyset$ the map $x_\alpha \circ y_\beta^{-1} : y_\beta(U_\alpha \cap V_\beta) \to x_\alpha(U_\alpha \cap V_\beta)$ is smooth. One can easily verify that this is indeed an equivalence relation.

Exercise 2.1. Convince yourself that the first and the third atlases in Example 2.7 are equivalent. Show that the first and the second example of atlases are not equivalent.

Definition 2.9 (**Manifold**). An *n-dimensional (C^∞) manifold* a topological space M together with an equivalence class of C^∞ atlases.

Notation. We will denote the manifold and the underlying topological space by the same letter, with the equivalence class of atlases usually understood.

Example 2.10. Let $M = \mathbb{R}^n$. We cover M by one open set and take the identity map as our chart. This is the *standard* manifold structure on $\mathbb{R}^n$.

Example 2.11. Let $M = \mathbb{C}^n$. Again we cover $\mathbb{C}^n$ by one open set $U = \mathbb{C}^n$, and take as our coordinate chart the map $\phi : \mathbb{C}^n \to \mathbb{R}^{2n}$ which is given by

$$\phi(z_1, \ldots, z_n) = (Rez_1, Imz_1, \ldots).$$

Example 2.12. If M is a manifold, and $V \subset M$ is an open subset, then V is naturally a manifold. Check this!

Example 2.13. The set $\mathrm{M_n}(\mathbb{R})$ of $n \times n$ matrices with real coefficients is a manifold, since it is $\mathbb{R}^{n^2}$.

The subset $\mathrm{GL}(n, \mathbb{R}) \subset \mathrm{M_n}(\mathbb{R})$ of invertible matrices is an open subset: a matrix A is invertible if and only if its determinant is non-zero and determinant $\det : \mathrm{M_n}(\mathbb{R}) \to \mathbb{R}$ is a polynomial map, hence continuous. Hence the subset $\{A \in \mathrm{M_n}(\mathbb{R}) \mid \det A \neq 0\}$ is open. So by the previous example, $\mathrm{GL}(n, \mathbb{R})$ is a manifold.

Example 2.14. The two-sphere $S^2 := \{x \in \mathbb{R}^3 \mid ||x||^2 = 1\}$ is a manifold. To see this, we give S^2 the subspace topology that it inherits as a subset of $\mathbb{R}^3$. Next we define charts. To do this, let

$$U_i^+ = \{x = (x_1, x_2, x_3) \in S^2 : x_i > 0\}$$

and

$$U_i^- = \{x = (x_1, x_2, x_3) \in S^2 : x_i < 0\},$$

$i = 1, 2, 3$ (6 charts altogether) which gives us an open cover of S^2. Define $\phi_1^\pm(x) = (x_2, x_3)$, $\phi_2^\pm(x) = (x_1, x_3)$, and $\phi_3^\pm(x) = (x_1, x_2)$.

We need to verify that changes of coordinates are smooth. Consider, for example, $\phi_2^+ \circ (\phi_1^+)^{-1}(u_1, u_2) = (\sqrt{1 - u_1^2 - u_2^2}, u_2)$, which is smooth in its region of definition. The other compositions yield similar results. It follows that S^2 is indeed a manifold.

[2] That is, each $U_\alpha \subset X$ is open and $\cup_\alpha U_\alpha = X$

Example 2.15. Now we consider a slightly more interesting example of a manifold, the real projective space $\mathbb{R}P^{n-1}$ which is, by definition, the space of lines through the origin in $\mathbb{R}^n$. To give $\mathbb{R}P^{n-1}$ a topology, we think of it as the set of equivalence classes of nonzero vectors in $\mathbb{R}^n$. That is,

$$\mathbb{R}P^{n-1} = (\mathbb{R}^n \smallsetminus \{0\})/\sim,$$

where two non-zero vectors v and v' are equivalent if and only if there is a constant $\lambda \neq 0$ such that $v = \lambda v'$. Note that this is an equivalence relation. We then have a surjective map

$$\pi : \mathbb{R}^n - \{0\} \to \mathbb{R}P^{n-1}, \quad \pi(v) = [v],$$

where $[v]$ denotes the equivalence class of v ($[v]$ is the line through v).

We put on $\mathbb{R}P^{n-1}$ the *quotient topology*: $U \subset \mathbb{R}P^{n-1}$ is open if and only if $\pi^{-1}(U)$ is open in $\mathbb{R}^n - \{0\}$. I leave it to the reader to check that this topology is Hausdorff.

Charts here are given as follows: for each $1 \leq i \leq n$, let

$$U_i = \{[x_1, ..., x_n] \in \mathbb{R}P^{n-1} : x_i \neq 0\}$$

and define

$$\phi_i : U_i \to \mathbb{R}^{n-1}$$

by

$$[x_1, ..., x_n] \mapsto \left(\frac{x_1}{x_i}, \cdots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \cdots \frac{x_n}{x_i}\right).$$

Note that the inverse ϕ_i^{-1} is given by

$$\phi_i^{-1} : (x_1, \cdots, x_{n-1}) \mapsto [x_1, \cdots, x_{i-1}, 1, \cdots, x_n].$$

We must check that the change of coordinates maps are smooth. If $j < i$, then on the interesection $U_i \cap U_j$

$$\phi_j \circ \phi_i^{-1}(u_1, \cdots, u_{n-1}) = \phi_j(u_1, \cdots, u_{i-1}, 1, \cdots, u_n) = \left(\frac{u_1}{u_j}, \cdots, \frac{u_{i-1}}{u_j}, \frac{1}{u_j}, \cdots, \frac{u_n}{u_j}\right),$$

which is smooth. Other computations are similar (and are left to the reader).

Exercise 2.2. Define the complex projective space $\mathbb{C}P^{n-1}$ to be the set of complex lines through the origin in $\mathbb{C}^n$ and prove that it is a manifold.

Exercise 2.3. If M and N are manifolds, show that $M \times N$ is also naturally a manifold.

Exercise 2.4. Let V be a finite-dimensional vector space over $\mathbb{R}$. Then V is a manifold: a choice of basis $v_1, \ldots, v_n$ ($n = \dim V$) of V defines a linear bijection $\sigma : \mathbb{R}^n \to V$, $\sigma(r_1, \ldots, r_n) = \sum r_i v_i$. Define a topology on V by requiring that σ is a homeomorphism (that is, $U \subset V$ is open $\Leftrightarrow \sigma^{-1}(U) \subset \mathbb{R}^n$ is open). Check that this is indeed a Hausdorff second countable topology. Define $\sigma^{-1} : V \to \mathbb{R}^n$ to be a chart and $\{\sigma^{-1} : V \to \mathbb{R}^n\}$ to be an atlas (one chart!). Prove that a different choice of basis of V defines the same topology and an equivalent atlas.

Exercise 2.5. Let M be a manifold. Show that for each point $x \in M$ there is a coordinate chart $\phi : U \to \mathbb{R}^n$ with $x \in U$ such that $\phi(x) = 0$ and $\phi(U)$ is $B_1(0)$, the ball of radius 1 centered at 0.

Remark 2.16. In Definition 2.9 we have made no assumption on the topology of our manifolds. It is standard to assume that the manifolds are Hausdorff. Otherwise all sorts of pathologies turn up. Another set of standard assumptions guarantees the existence of partitions of unity (see subsection 2.4 below). For this the simplest assumption to make is that the manifold in question is *second countable.* However, this assumption is too stringent and paracompactness is much more reasonable. All of this will be discussed later on.

2.3. **Maps of manifolds.** In the Bourbakist view every area of mathematics has its collection of objects and its collection of maps between objects (or, more generally, morphisms). While it is enjoyable to make fun of Bourbaki and Bourbakists, there is some merit to this point of view. A map $f : M \to N$ between two manifolds is smooth if it is continuous and is smooth in coordinates. More precisely we have:

Definition 2.17 (smooth map)**.** Let M and N be two smooth manifolds with atlases $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$, respectively. A continuous map $f : M \to N$ is a *smooth map* (or a *morphism* of C^∞ manifolds) if for all α and β with

$$f^{-1}(V_\beta) \cap U_\alpha \neq \emptyset,$$

the composition

$$\psi_\beta \circ f \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap f^{-1}(V_\beta)) \to \psi_\beta(V_\beta)$$

is C^∞.

We will write $C^\infty(M, N)$ to denote the set of all smooth maps from M to N. Note that this definition does not depend on which atlases on M and N we choose [check this].

Also note a special case of this definition is that of a *s*mooth function on a manifold, which is a map from M to $\mathbb{R}$. To wit

Definition 2.18. A function $f : M \to \mathbb{R}$ is smooth if f is continuous and if for all coordinate charts $\{(U_\alpha, \phi_\alpha)\}$, $f \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha) \to \mathbb{R}$ is C^∞. It's consistent with the previous definition: we think of the real line $\mathbb{R}$ as a manifold with the standard coordinate chart $id : \mathbb{R} \to \mathbb{R}$. We denote the collection of all smooth functions on a manifold M by $C^\infty(M) = C^\infty(M, \mathbb{R})$.

Exercise 2.6. Let M be a manifold. Check that $C^\infty(M)$ is a vector space over the reals under the standard addition of functions and multiplication by scalars. Is it finite dimensional?

Exercise 2.7. Let M be a manifold. Check that a constant function on a manifold M is smooth.

Here are some examples of smooth maps.

Example 2.19. Take $M = \mathbb{R}^n \setminus \{0\}$, and let $N = \mathbb{R}P^{n-1}$. Let $\pi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}P^{n-1}$ be the projection $\pi(v) = [v]$. I claim that π is a smooth map. Let's check it.

The atlas on M is given by one chart — the inclusion ϕ of M into $\mathbb{R}^n$. The charts on $\mathbb{R}P^{n-1}$ are the same as last time. Note that $\pi^{-1}(U_i) = \{v \in \mathbb{R}^n \setminus \{0\} : v_i \neq 0\}$. To see that π is smooth, we need to check that $\phi_i \circ \pi \circ \phi^{-1} : \pi^{-1}(U_i) \to \mathbb{R}^{n-1}$ is C^∞. But note that

$$(\phi_i \circ \pi \circ \phi^{-1})(v) = \phi_i(\pi(v)) = \phi_i([v]) = \left(\frac{v_1}{v_i}, \cdots, \frac{v_n}{v_i}\right).$$

□

Example 2.20. Let $M = \mathbb{R}$ with the coordinate chart $\phi(x) = x^3$. Let $N = \mathbb{R}$ with the coordinate chart $\psi(x) = x$. Let $f : M \to N$ be the map $x \mapsto x^3$. Is f a C^∞ map?

$$(\psi \circ f \circ \phi^{-1})(x) = \psi \circ f(x^{1/3}) = \psi(x) = x,$$

which is smooth. So f is smooth.

Now let us see if the map $h : M \to N$, $h(x) = x$ is smooth. We have $\psi \circ h \circ \phi^{-1}(x) = x^{1/3}$, which is not differentiable at 0. So h is not smooth.

Finally note that $f^{-1} : N \to M$ *is* smooth:

$$\phi \circ f^{-1} \circ \psi^{-1}(x) = (x^{1/3})^3 = x.$$

Example 2.21. Constant functions are smooth maps of manifolds □

The appropriate notion of "isomorphism" in differential geometry is the following one:

Definition 2.22 (Diffeomorphism)**.** A C^∞ map $f : M \to N$ between two smooth manifolds is a *diffeomorphism* if f is a homeomorphism and both f and f^{-1} are C^∞ maps.

Example 2.23. The map $f : M \to N$ of Example 2.20 is a diffeomorphism.

Exercise 2.8. If M and N are manifolds, prove that $M \times N$ is diffeomorphic to $N \times M$.

Exercise 2.9. Show that the composition of smooth maps is smooth.

Exercise 2.10. Let $L_A : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ be left multiplication by $A \in \mathrm{GL}(n, \mathbb{R})$. Prove that L_A is a diffeomorphism. [Recall that $\mathrm{GL}(n, \mathbb{R}) \subset \mathbb{R}^{n^2}$ is the set of all invertible $n \times n$ matrices and that it is open in $\mathbb{R}^{n^2}$.]

2.4. **Partitions of unity.** In this subsection we define partitions of unity (that is, writing the constant function 1 as a sum of bump functions with certain properties) and prove the existence of a partition of unity subordinate to a cover on a second countable manifold. The existence of such partitions of unity is very useful. The proof of the existence of the partition of unity is not terribly useful and should be skipped on the first (and second) reading. The reason for this advice is that the proof is technical and the techniques will never be used again in this course. We start with a string of definitions.

Definition 2.24 (second countable)**.** A topological space X is *second countable* if there is a *countable* collection of open subsets $\{U_i\}$ of X such that any open set in X is the union of some collection of U_i's. In other words, the topology of X has a countable basis.

Example 2.25. The real line $\mathbb{R}$ with the standard topology is second countable: the collection $\{U_i\}$ is consists of open intervals (a, b) where a and b are rational numbers.

Similarly $\mathbb{R}^n$ is second countable: the collection $\{U_i\}$ consists of open balls $B_r(x)$ of rational radius r centered at points x with rational coordinates.

Remark 2.26. Any (topological) subspace of a second countable space is second countable [prove it]. Hence any manifold that can be realized as a subspace of some $\mathbb{R}^n$ has to be second countable.

The condition of second countability is much more than necessary for the existence of the partition of unity. One can get away with assuming only *paracompactness.* Here, for the record, is its definition. It takes a paragraph to state because we have to define a few more things first.

Definition 2.27. Let M be a topological space. A collection $\{U_\alpha\}$ of subsets of M is a *cover* of a subset $W \subset M$ if $W \subset \bigcup U_\alpha$. It is an *open cover* if each $\{U_\alpha\}$ is open. A *refinement* $\{V_\beta\}$ of a cover $\{U_\alpha\}$ is a cover such that for each index β there is an index $\alpha = \alpha(\beta)$ with $V_\beta \subset U_\alpha$.

A collection of subsets $\{U_\alpha\}$ of M is *locally finite* if for every point $m \in M$ there is a neighborhood W of M with $W \cap U_\alpha \neq 0$ for only finitely many α.

Example 2.28. The cover $\{(n, n+2\}_{n \in \mathbb{Z}}$ is a locally finite cover of $\mathbb{R}$. The cover $\{[-\frac{1}{n}, \frac{1}{n}]\}$ is a cover of $(-1, 1)$ which is not locally finite — there is a problem at 0.

Definition 2.29 (paracompact)**.** A topological space is *paracompact* if every open cover has a locally finite refinement.

Example 2.30. Any compact space is paracompact. We will see shortly that second countable Hausdorff manifolds are paracompact.

Definition 2.31 (support)**.** The *support* $\operatorname{supp} f$ of a continuous function $f : X \to \mathbb{R}$ is the closure of the set of points where f is non-zero:

$$\operatorname{supp} f = \overline{\{x \in X : f(x) \neq 0\}}.$$

Definition 2.32 (Partition of Unity)**.** Let $\{U_\alpha\}$ be an open cover of a manifold M. A *partition of unity subordinate to the cover* $\{U_\alpha\}$ is a collection of smooth functions $\{\rho_\beta : M \to [0, 1]\}$ such that :

(1) For each index β there is an index α with $\operatorname{supp}(\rho_\beta) \subset U_\alpha$.
(2) For each point $m \in M$, there is a neighborhood W of m such that $\rho_\beta|_W \neq 0$ for only finitely many β. That is, the collection of supports $\{\operatorname{supp} \rho_\beta\}$ is locally finite.
(3) $\sum_\beta \rho_\beta = 1$.

Remark 2.33. Note that we need condition (2) to make sense of the sum in (3): by (2), for each point $m \in M$ the sum $\sum \rho_\beta(m)$ is actually a *finite* sum. So there are no problems with convergence.

Theorem 2.34. *Let M be a second countable Hausdorff manifold. Then every open cover of M has a partition of unity subordinate to it.*

Proof. (You should not read this proof the first time around)
Step 1. We first construct a collection $\{X_k\}_{k=1}^\infty$ of open subsets of M such that their closures $\overline{X}_k$ are compact, $\overline{X}_k \subset X_{k+1}$ and $M = \bigcup_{k=1}^\infty X_k$. Since M is second countable, there is a countable basis of the topology of M. Out of this collection of open sets choose those that have compact closure and denote them by W_1, W_2, ... We claim that that they cover M: $M = \bigcup W_i$. Indeed, a point $x \in M$ has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$ ($n = \dim M$, of course). For any point y in an open set $U \subset \mathbb{R}^n$ there is a closed ball $\bar{B}_r(y)$ centered at y with $\bar{B}_r(y) \subset U$. Closed balls in $\mathbb{R}^n$ are compact. Hence every point $x \in M$ has a neighborhood $U(x)$ whose closure $\overline{U(x)}$ is compact. Now $U(x)$ is a union of a certain number of elements of the countable basis of the topology of M. The closure of each of these elements is compact. Therefore $x \in W_i$ for some index i. This proves that $M = \bigcup W_i$.

Let $X_1 = W_1$. The whole collection $\{W_i\}_{i=1}^\infty$ covers $\overline{X}_1$. Since $\overline{X}_1$ is compact, $\overline{X}_1 = W_{i_1} \cup W_{i_2} \cup \ldots \cup W_{i_p}$ for some $i_1 < i_2 < \cdots < i_p$. Let $X_2 = W_{i_1} \cup W_{i_2} \cup \ldots \cup W_{i_p}$. Then $\overline{X}_2$ is compact ... Continuing in this manner we get the desired collection $\{X_k\}_{k=1}^\infty$.

Step 2. We construct three open countable covers $\{V_{\beta,1}\}$, $\{V_{\beta,2}\}$, $\{V_{\beta,3}\}$ with $\{V_{\beta,1}\} \subset \{V_{\beta,2}\} \subset \{V_{\beta,3}\}$, $\bigcup_\beta \{V_{\beta,1}\} = M$ and $\{V_{\beta,3}\}$ is locally finite and subordinate to $\{U_\alpha\}$, the cover we started out with. Note that this will prove that any Hausdorff second countable manifold is paracompact, as promised.

Fix an index k. For each point $z \in \overline{X}_k \smallsetminus X_{k-1}$ choose an open set $V_{z,3}$ such that $V_{z,3} \subset U_\alpha$ for some α, $V_{z,3} \subset X_{k+1}$ and $V_{z,3} \cap \overline{X}_{k-1} = \emptyset$. Additionally we require that there is a coordinate chart ψ_z mapping $V_{z,3}$ homeomorphically onto

$$B_3(0) := \{x \in \mathbb{R}^n \mid ||x|| < 3\}.$$

Let $V_{z,i} = \psi_z^{-1} B_i(0)$ for $i = 1, 2$. The open sets $V_{z,1}$ cover the compact set $\overline{X}_k \smallsetminus X_{k-1}$ (and are contained in $X_{k+1} \smallsetminus \overline{X}_{k-2}$). Therefore, for each k, there is a *finite* collection of $V_{z,1}$'s covering $\overline{X}_k \smallsetminus X_{k-1}$. Take all of these finite collections. We get a cover $\{V_{\beta,1}\}$ of M. Similarly we get two more covers: $\{V_{\beta,2}\}$ and $\{V_{\beta,3}\}$. Note that by construction they are locally finite and are subordinate to $\{U_\alpha\}$: for each β there is $\alpha(\beta)$ with $V_{\beta,i} \subset U_{\alpha(\beta)}$.

Step 3. Now we construct a partition of unity. The function

$$f(t) = \begin{cases} e^{-\frac{1}{t}}, & \text{if } t > 0 \\ 0, & \text{if } t \leq 0 \end{cases}$$

is smooth on all of $\mathbb{R}$ [this fact is not entirely trivial]. Hence

$$\tilde{f}(t) = \begin{cases} e^{-\frac{1}{1-t}}, & \text{if } t < 1 \\ 0, & \text{if } t \geq 1 \end{cases}$$

is smooth on all of $\mathbb{R}$. Therefore $h : \mathbb{R}^n \to [0, \infty)$ given by

$$h(x) = \tilde{f}(||x||^2/4)$$

is also smooth. Note that $h(x) > 0$ for all $x \in B_2(0)$ and $h(x) = 0$ for all $x \notin B_2(0)$. Therefore, for each index β,

$$g_\beta(x) = \begin{cases} h(\psi_\beta(x)) & \text{if } x \in V_{\beta,3} \\ 0, & \text{if } x \notin V_{\beta,3}, \end{cases}$$

where $\psi_\beta : V_{\beta,3} \to B_3(0)$ is the corresponding coordinate chart, is a smooth function on M. Moreover, $g_\beta(x) > 0$ for $x \in V_{\beta,1}$. Since the cover $\{V_{\beta,3}\}$ is locally finite, the sum

$$G(x) = \sum_\beta g_\beta(x)$$

makes sense [converges for each x] and defines a smooth function on M. Since $\{V_{\beta,1}\}$ covers M, $G(x) > 0$ for all $x \in M$. Let

$$\rho_\beta(x) = g_\beta(x)/G(x).$$

Then $1 \geq \rho_\beta(x) \geq 0$, $\sum \rho_\beta = 1$ and $\operatorname{supp} \rho_\beta \subset V_{\beta,3} \subset U_{\alpha(\beta)}$. Thus the collection $\{\rho_\beta\}$ is the desired partition of 1. □

Corollary 2.34.1. *Let M be a second countable Hausdorff manifold and $\{U_i\}_{i=1}^{\infty}$ a* countable *open cover. Then there is a partition of unity $\{\rho_i\}$ with* $\operatorname{supp}\rho_i \subset U_i$.

Proof. By Theorem 2.34 there is a partition of unity $\{\tau_\beta\}$ with $\operatorname{supp}\tau_\beta \subset U_i$ for some $i = i(\beta)$. Let

$$I(i) = \{\beta \mid \operatorname{supp}\tau_\beta \subset U_i \text{ and } \operatorname{supp}\tau_\beta \not\subset U_j \text{ for } j < i\}.$$

Define

$$\rho_i = \sum_{\beta \in I(i)} \tau_\beta.$$

The collection $\{\rho_i\}$ is the desired partition of 1. □

Proposition 2.35. *Suppose that M is a second countable Hausdorff manifold, $K \subset M$ a closed subset and $U \subset M$ an open set with $K \subset U$. Then there is a smooth function $f : M \to [0,1]$ such that*

(1) $f|_K \equiv 1$ *and*
(2) $\operatorname{supp}(f) \subset U$.

Proof. Let $U_1 = U$ and $U_2 = M \smallsetminus K$. By Corollary 2.34.1 there exists smooth functions $\rho_1, \rho_2 : M \to [0,1]$ with $\operatorname{supp}\rho_i \subset U_i$ and $\rho_1 + \rho_2 = 1$. Since $\operatorname{supp}\rho_2 \subset M \smallsetminus K$, $\rho_2|_K \equiv 0$. Hence $\rho_1|_K \equiv 1$. Now let $f = \rho_1$. □

Corollary 2.35.1. *Let M be a (second countable Hausdorff) manifold. For any point $x \in M$ and any neighborhood U of x in M there is a smooth function $f : M \to \mathbb{R}$ so that*

(1) $f \equiv 1$ *on a neighborhood V of x contained in U and*
(2) $\operatorname{supp}(f) \subset U$.

Proof. Exercise. You can use the proposition above. Alternatively prove it directly first in the case where $M = \mathbb{R}^n$ and then use a coordinate chart around x to prove it for arbitrary M. Is the condition that M is second countable really necessary? □

3. Tangent vectors and tangent spaces

3.1. **Tangent vectors and tangent spaces.** We learn in physics that a vector is an arrow sticking out of a point in space and that a vector field assigns an arrow to each point in space. When we learn linear algebra, we are told to forget this point of view: all vectors are sticking out of one point — the origin. For the purposes of differential geometry the physics point of view is correct after all: all our vectors are anchored at various points in space.

There is another issue we need to deal with. If $S \subset \mathbb{R}^3$ is a smooth convex surface, one can imagine that for every point $p \in S$ there is a two-plane T_pS touching S at that point, a plane *tangent* to S at p. (It is not entirely clear that such a plane is unique, but that's another story.) A vector tangent to S at p would be an arrow anchored at p and lying in T_pS. This raises a problem: our manifolds are defined abstractly and not as subsets of some $\mathbb{R}^n$. So what would a tangent plane be in this case? and what vector space would it lie in?

The solution is to think of vectors as directional derivatives. A directional derivative of a function on $\mathbb{R}^n$ depends on two things: a direction and the point at which the function is being differentiated. For a smooth function $f \in C^\infty(\mathbb{R}^n)$, we write

$$D_v f(p) = \frac{d}{dt}|_0 f(p + tv)$$

for the directional derivative of f at a point $p \in \mathbb{R}^n$ in the direction $v \in \mathbb{R}^n$. Observe that

(1) the directional derivatives are linear: for any $f, g \in C^\infty(\mathbb{R}^n)$ and any $\lambda, \mu \in \mathbb{R}$

$$D_v(\lambda f + \mu g)(p) = \lambda D_v f(p) + \mu D_v g(p);$$

(2) the directional derivatives have a *derivation* property:

$$D_v(fg)(p) = f(p)\, D_v g(p) + D_v f(p)\, g(p).$$

This motivates the following definition:

Definition 3.1 (Tangent vector)**.** Let M be a manifold and $a \in M$ a point. A *tangent vector* to M at a is an $\mathbb{R}$-linear map $v : C^\infty(M) \to \mathbb{R}$ such that

$$v(fg) = f(a)v(g) + g(a)v(f) \tag{3.1}$$

for all functions $f, g \in C^\infty(M)$.

Linear maps $C^\infty(M) \to \mathbb{R}$ satisfying (3.1) are also said to have a *derivation* property and are called *derivations* (into $\mathbb{R}$).

Definition 3.2 (Tangent space)**.** The *tangent space* T_aM to a manifold M at a point a is the collection of all tangent vectors to M at a.

Exercise 3.1. The tangent space T_aM is a *vector space over the reals.* [That's why the elements of the tangent space are called "vectors"!] That is, if $v, w \in T_aM$ and $\lambda, \mu \in \mathbb{R}$ then the linear map $\lambda v + \mu w : C^\infty(M) \to \mathbb{R}$ is a derivation.

Note that by our definition every direction derivative at a point $p \in \mathbb{R}^n$ is a tangent vector at p to $\mathbb{R}^n$ This begs a question: are there tangent vectors that are not directional derivatives? The answer is no, tangent vectors to points of $\mathbb{R}^n$ are directional derivatives and that's all there is to it:

Proposition 3.3. *Let $w \in T_a\mathbb{R}^n$ be a tangent vector. That is, suppose $w : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is a linear map satisfying (3.1). Then*

$$w(f) = D_v f\,(a)$$

for some $v \in \mathbb{R}^n$. The same result holds with $\mathbb{R}^n$ replaced by some open ball $B_r(a)$.

To prove the proposition we first "recall" a version of Taylor's theorem.

Lemma 3.4. *Let f be a smooth function on $\mathbb{R}^n$. Fix a point $a \in \mathbb{R}^n$ Then for any $x \in \mathbb{R}^n$*

$$f(x) = f(a) + \sum (x_i - a_i)h_i(x) \tag{3.2}$$

where $h_i(x)$ are smooth functions with

$$h_i(a) = \frac{\partial f}{\partial x_i}(a).$$

Proof. Suppose first that $a = 0$. Then, by the fundamental theorem of calculus and chain rule,

$$f(x) - f(0) = \int_0^1 \frac{d}{dt} f(tx)\,dt = \int_0^1 (\sum x_i \frac{\partial f}{\partial x_i}(tx))\,dt = \sum x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx)\,dt.$$

Let $h_i(x) = \int_0^1 \frac{\partial f}{\partial x_i}(tx)\,dt$. These are the desired functions. If $a \neq 0$ apply the previous argument to $\bar{f}(x) = f(x-a)$. □

Remark 3.5. If f is a smooth function on an open ball $B_r(a)$ then (3.2) still holds at all $x \in B_r(a)$, except now $h_i \in C^\infty(B_r(a))$. The proof is exactly the same.

Before proving the proposition we need one more simple lemma.

Lemma 3.6. *Let M be a manifold and $w \in T_aM$ a tangent vector. Then for any constant function c we have $w(c) = 0$.*

Proof. Apply the tangent vector w to the constant function 1:

$$w(1) = w(1 \cdot 1) = 1w(1) + w(1)1 = 2w(1). \Rightarrow w(1) = 0.$$

Since w is linear, for any constant function $c = c \cdot 1$

$$w(c) = w(c \cdot 1) = c\,w(1) = 0.$$

□

Proof of Proposition 3.3. By Lemma 3.4, $f(x) = f(a) + \sum(x_i - a_i)h_i(x)$. Hence

$$w(f) = w(f(a)) + \sum(w(x_i - a_i)h_i(a) + (a_i - a_i)w(h_i)) = 0 + \sum w(x_i)h_i(a) + 0 = \sum w(x_i)\frac{\partial f}{\partial x_i}(a).$$

Therefore $w = D_v f\,(a)$, were $v = (w(x_1), \dots, w(x_n))$.

We leave the ball version of the proof as an exercise. □

Remark 3.7. The proof above actually shows that the derivations $\{\frac{\partial}{\partial x_i}|_a\}$ form a basis of $T_a\mathbb{R}^n$.

For arbitrary manifolds a choice of coordinates near a point also defines a basis of the tangent space at the point. To express this precisely it will be convenient to slightly change our notation. To this end, denote the points of $\mathbb{R}^n$ by $r = (r_1, \dots, r_n)$. We also think of r_i as a function that assign to a point its i-th coordinate. If $\phi : U \to \mathbb{R}^n$ is a coordinate chart on a manifold M, then $\phi = (r_1 \circ \phi, \dots, r_n \circ \phi)$. We then think of $x_i = r_i \circ \phi$ as coordinate functions on U.

The coordinates define tangent vectors at points of U: for any $a \in U$ and any $f \in C^\infty(M)$ we define $\frac{\partial}{\partial x_i}|_a$ by

$$\frac{\partial}{\partial x_i}|_a(f) := \frac{\partial}{\partial r_i}|_{\phi(a)}(f \circ \phi^{-1}).$$

It is easy to see that these are, indeed, tangent vectors. It should come as no surprise that they form a basis of the tangent space T_aM. After all, manifolds locally look like $\mathbb{R}^n$ and in $\mathbb{R}^n$ the partial derivatives do form bases of tangent spaces. Now let's prove this. We first observe that tangent vectors are local.

Lemma 3.8. *Let M be a manifold and $v \in T_aM$ a tangent vector. Then for any two functions $f, g \in C^\infty(M)$ with $f = g$ in a neighborhood U of a, we have*

$$v(f) = v(g).$$

In particular, if h is constant on a neighborhood U of a, then $v(h) = 0$ (cf. Lemma 3.6).

Proof. As $v : C^\infty(M) \to \mathbb{R}$ is $\mathbb{R}$-linear, it is enough to show that $v(f - g) = 0$. Chose a smooth bump function $\rho : M \to [0, 1]$ with $\operatorname{supp} \rho \subset U$ which is identically 1 on a neighborhood V of a. We then have that $\rho \cdot (f - g) = 0$ on all of M by construction. Furthermore, because v is linear, $v(0) = 0$, hence

$$0 = v(\rho\,(f - g)) = v(\rho)\,(f - g)(a) + \rho(a)\,v(f - g) = v(f - g).$$

□

What's the point of the lemma, aside from its esthetic appeal? If $\phi = (x_1, \dots, x_n) : U \to \mathbb{R}^n$ is a coordinate chart on a manifold M and $v \in T_aM$ is a tangent vector at some point $a \in U$, then we cannot apply v to a coordinate function x_i. The function x_i is only defined on U; it is not a smooth function on all of M. However, there is a way around this problem. Pick a smooth bump function $\rho : M \to [0, 1]$ with $\operatorname{supp} \rho \subset U$ which is identically 1 on some neighborhood of a. Then $x_i\rho$ *is* a smooth function on M and so $v(x_i\rho)$ does make sense. Moreover, this number *does not depend on the choice of the bump function*: if $\tau : M \to [0, 1]$ is another choice of a bump function with the same properties, then $x_i\rho = x_i\tau$ on some (perhaps smaller) neighborhood of a. Therefore, by the preceding lemma, $v(x_i\rho) = v(x_i\tau)$. We therefore *define*

$$v(x_i) := v(x_i\rho)$$

for some choice of the bump function ρ. Similarly, if $h \in C^\infty(U)$ we *define*

$$v(h) := v(h\rho)$$

for some (any) choice of the appropriate bump function ρ.

Lemma 3.9. *If $\phi = (x_1, \dots, x_n) : U \to \mathbb{R}^n$ is a coordinate chart on a manifold M and $v \in T_aM$ is a tangent vector at some point $a \in U$. Then*

$$v = \sum_i v(x_i) \frac{\partial}{\partial x_i}|_a. \tag{3.3}$$

Moreover, the vectors $\{\frac{\partial}{\partial x_i}|_a\}$ form a basis of T_aM.

Proof. We evaluate both sides of (3.3) on a function $f \in C^\infty(M)$. It is no loss of generality to assume that $\phi(U)$ is a ball and that $\phi(a) = 0$. By Lemma 3.4,

$$(f \circ \phi^{-1})(r) = (f \circ \phi^{-1})(0) + \sum r_i h_i(r)$$

where $h_i(0) = \frac{\partial}{\partial r_i}(f \circ \phi^{-1})|_0$. Thus,

$$f(x) = f(a) + \sum x_i \cdot f_i(x),$$

where

$$f_i(a) = \frac{\partial}{\partial r_i}(f \circ \phi^{-1})(0) = \frac{\partial}{\partial x_i}|_a(f),$$

for all $x \in U$. Hence, for any $v \in T_aM$, we have

$$\begin{aligned} v(f) &= v(f(a) + \sum x_i f_i) \\ &= \sum x_i(a)v(f_i) + \sum v(x_i) f_i(a) \\ &= \sum v(x_i) f_i(a) \\ &= \sum v(x_i) \frac{\partial}{\partial x_i}|_a(f). \end{aligned}$$

This shows that $\{\frac{\partial}{\partial x_i}|_a\}$ span T_aM. To check linear independence observe that

$$\frac{\partial}{\partial x_i}|_a(x_j) = \delta_{ij},$$

where δ_{ij} denotes the Kronecker delta function: it's 1 if $i = j$ and zero otherwise. □

Remark 3.10. We have seen in the preceding discussion that for any $p \in \mathbb{R}^n$ the tangent space $T_p\mathbb{R}^n$ is isomorphic to $\mathbb{R}^n$. Explicitly the isomorphism is give by taking a vector $v \in \mathbb{R}^n$ to the directional derivative at p in the direction of v:

$$\mathbb{R}^n \xrightarrow{\simeq} T_p\mathbb{R}^n \quad v \mapsto D_v(\cdot)(p).$$

In particular

$$\mathbb{R} \xrightarrow{\simeq} T_a\mathbb{R} \quad s \mapsto s\frac{d}{dr}|_a.$$

3.2. **Digression: vector spaces and their duals.** Given two (finite dimensional) vector spaces V and W we denote the set of all linear maps from V to W by $\mathrm{Hom}(V, W)$. It is a vector space: any linear combination of two linear maps is again a linear map. Of special interest is the vector space $V^* := \mathrm{Hom}(V, \mathbb{R})$ of linear maps from a vector space V to $\mathbb{R}$, the so called *dual* vector space. If $\{v_i\}_{i=1}^n$ is a basis of V, the *dual basis* is a basis $\{v_i^*\}$ of V^* defined by

$$v_i^*(v_j) = \delta_{ij}$$

for all $1 \leq i, j \leq n$. This is indeed a basis. If $\ell \in V^*$ is an arbitrary functional, then

$$\ell = \sum \ell(v_i)v_i^*$$

because both sides of the formula above agree on the basis vectors v_j (I am tacitly using the fact that if two linear maps $\mu, \nu : V \to \mathbb{R}$ agree on basis vectors, then they agree). It follows that $\dim V^* = \dim V$. Finally observe that for any vector $u \in V$,

$$u = \sum v_i^*(u)v_i.$$

Why is the formula above true? Apply v_j^* to both sides.

Exercise 3.2. Show that a choice of basis of vector spaces V and W identifies $\mathrm{Hom}(V, W)$ with a space of matrices. Conclude that $\dim \mathrm{Hom}(V, W) = \dim V \cdot \dim W$.

3.3. **Differentials.**

Definition 3.11. Let $f : M \to N$ be a smooth map of manifolds and $a \in M$ a point. The *differential* of f at a is the linear map

$$df_a : T_aM \to T_{f(a)}N$$

defined by

$$(df_a(v))(h) = v(h \circ f)$$

for all $v \in T_aM$ and all $h \in C^\infty(N)$.

Exercise 3.3. Check that the definition above makes sense. That is, given $v \in T_aM$, check that the map

$$C^\infty(N) \to \mathbb{R}, \quad h \mapsto v(h \circ f)$$

is a linear map satisfying (3.1).

We will check shortly that in the case of a smooth map $f : \mathbb{R}^n \to \mathbb{R}^m$, $df_a = Df_a$ under the natural identification $T_a\mathbb{R}^n \simeq \mathbb{R}^n$.

We next sort out what the definition of a differential amounts to in the case where $f : M \to \mathbb{R}$ is a smooth *function* (in other words the target manifold $N = \mathbb{R}$). By definition 3.11, df_a is a map from T_aM to $T_{f(a)}\mathbb{R} \simeq \mathbb{R}$. That is, if we compose df_a with the isomorphism $T_{f(a)}\mathbb{R} \xrightarrow{\simeq} \mathbb{R}$ (see Remark 3.10, we get a linear map

$$\underline{df_a} : T_aM \to \mathbb{R}$$

By definition, $\underline{df_a}$ an element of the dual vector space $T_a^*M := \mathrm{Hom}(T_aM, \mathbb{R})$. I claim that the linear map $\underline{df_a}$ is given by

(3.4) $$\underline{df_a}(v) = v(f).$$

for any tangent vector $v \in T_aM$.

Proof. Let $r : \mathbb{R} \to \mathbb{R}$ denote the identity map. We think of it as the standard coordinates on $\mathbb{R}$. Then for every point $x \in \mathbb{R}$ the vector $\frac{d}{dr}|_x$ is a basis vector of $T_x\mathbb{R}$, which gives us an isomorphism

$$T_x\mathbb{R} \to \mathbb{R}, \quad t\frac{d}{dr}|_x \mapsto t.$$

The map above has a "coordinate free" description as well. It is:

$$T_x\mathbb{R} \ni v \mapsto v(r).$$

Therefore

$$\underline{df_a}(v) = (df_a(v))(r) = v(r \circ f) = v(f).$$

□

Remark 3.12. It is customary not to distinguish between df_a and $\underline{df_a}$. Thus, in the case of $f \in C^\infty(M)$, the differential df_a denotes *both* the linear map $df_a : T_aM \to T_{f(a)}\mathbb{R}$ and the linear functional $df_a : T_aM \to \mathbb{R}$. In other words, from now on we drop the notation $\underline{df_a}$ and write (3.4) as

(3.5) $$df_a(v) = v(f).$$

for all $f \in C^\infty(M)$, $a \in M$, $v \in T_aM$.

Definition 3.13. The vector space

$$T_a^*M := \mathrm{Hom}(T_aM, \mathbb{R})$$

is called the *cotangent* space of M at a.

The new concept of the differential allows us to re-interpret the formula (3.3). Recall that a choice of coordinates $\phi = (x_1, \ldots, x_n) : U \to \mathbb{R}^n$ on a manifold M gives rise to a basis $\{\frac{\partial}{\partial x_i}|_a\}$ of T_aM for any point $a \in U$. We claim that $\{(dx_i)_a\}$ form the dual basis of the cotangent space T_a^*M. Indeed, by (3.5),

$$(dx_j)_a\left(\frac{\partial}{\partial x_i}|_a\right) = \frac{\partial}{\partial x_i}|_a(x_j) = \delta_{ij}.$$

Since for $v \in T_aM$ we have $v(x_i) = (dx_i)_a(v)$, (3.3) becomes

(3.6) $$v = \sum (dx_i)_a(v)\, \frac{\partial}{\partial x_i}\bigg|_a.$$

Let $f = (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map. We are now in the position to compare $df_a : T_a\mathbb{R}^n \to T_{f(a)}\mathbb{R}^m$ with $Df_a : \mathbb{R}^n \to \mathbb{R}^m$. Let $r_1, \ldots r_n$ denote the standard coordinates on $\mathbb{R}^n$ and $s_1, \ldots, s_m$ the standard coordinates on $\mathbb{R}^m$. Using (3.6) we compute:

$$\begin{aligned}(ds_i)_{f(a)}(df_a(\frac{\partial}{\partial r_j}\bigg|_a)) = (df_a(\frac{\partial}{\partial r_j}\bigg|_a))(s_i) &= \frac{\partial}{\partial r_j}\bigg|_a(s_i \circ f)\\ &= \frac{\partial}{\partial r_j}\bigg|_a(f_i)\\ &= \frac{\partial f_i}{\partial r_j}(a)\end{aligned}$$

Thus the matrix of the linear map $df_a : T_a\mathbb{R}^n \to T_{f(a)}\mathbb{R}^m$ with respect to the basis $\{\frac{\partial}{\partial r_j}\big|_a\}$ and $\{\frac{\partial}{\partial s_i}\big|_{f(a)}\}$ is the Jacobian matrix of Df_a. $\square$.

It is worth singling out another special case of the definition of a differential of a map: $M = \mathbb{R}$. In this case $f : \mathbb{R} \to N$ is a smooth curve. We define the tangent vector to f at $t \in \mathbb{R}$ to be

$$f'(t) := df_t\left(\frac{d}{dr}|_t\right).$$

Note that by definition $f'(t)$ is a tangent vector in $T_{f(t)}N$, the tangent space to N at $f(t)$.

Exercise 3.4. Let M be a manifold, $p \in M$ a point and $v \in T_pM$ a tangent vector at the point p. Show that there is a curve $\gamma : I \to M$ (where I is an open interval containing 0) with $\gamma(0) = p$ and $\gamma'(0) = v$.

We next observe that the chain rule holds for the differentials of smooth maps.

Theorem 3.14 (Chain Rule)**.** *If $F : X \to Y$ and $H : Y \to Z$ are smooth maps of manifolds, then*

$$d(H \circ F)_a = dH_{F(a)} \circ dF_a$$

for any point $a \in X$.

Proof. Fix $a \in X$, $v \in T_aX$, and $f \in C^\infty(Z)$. Then

$$\begin{aligned} (d(H \circ F)_a(v))(f) &= v(f \circ (H \circ F)) \\ &= v((f \circ H) \circ F) \\ &= (dF_a(v))(f \circ H) \\ &= (dH_{F(a)}(dF_a(v)))(f). \end{aligned}$$

$\square$

Remark 3.15. Theorem 3.14 and Exercise 3.4 give us a useful way of computing differentials $df_a : T_aM \to T_{f(a)}N$. By the exercise, for any $v \in T_aM$ we can find a curve $\gamma : I \to M$ with $\gamma(0) = a$ and $\gamma'(0) = v$. Then, by the chain rule,

$$df_a(v) = df_a(\gamma'(0)) = df_a(d\gamma(\frac{d}{dr}|_0)) = d(f \circ \gamma)_0(\frac{d}{dr}|_0) = (f \circ \gamma)'(0).$$

Exercise 3.5. Prove that if $F : M \to N$ is a diffeomorphism then the differential $dF_a : T_aM \to T_{F(a)}N$ is an isomorphism.

Exercise 3.6. Let M and N be manifolds. Prove that for any $(a, b) \in M \times N$ the tangent space $T_{(a,b)}(M \times N)$ is isomorphic to $T_aM \times T_bN$.

Exercise 3.7. Suppose that $\gamma : \mathbb{R} \to \mathbb{R}^n$, $\gamma(t) = (\gamma_1(t), \ldots \gamma_n(t))$ is a smooth curve. Show that

$$d\gamma\,(\frac{d}{dt}) = \sum_i {\gamma_i}'(t)\frac{\partial}{\partial r_i},$$

where ${\gamma_i}'(t)$ are ordinary derivatives.

3.4. **The tangent bundle.**

Definition 3.16 (provisional)**.** The *tangent bundle* TM of a manifold M is (as a set)

$$TM = \bigsqcup_{a \in M} T_aM.$$

Note that there is a natural projection (the tangent bundle projection)

$$\pi : TM \to M$$

which sends a tangent vector $v \in T_aM$ to the corresponding point a of M.

We want to show that the tangent bundle TM itself is a manifold in a natural way and the projection map $\pi : TM \to M$ is smooth. Strictly speaking, we first should specify a topology on TM. However, our strategy will be different. We will first find candidates for coordinate charts on the tangent bundle TM. They will be constructed out of coordinate charts on M. We will check that the change of these candidate coordinates on TM is smooth. We will then use these candidate coordinates to manufacture a topology on TM.

Let $\phi = (x_1, \cdots, x_n) : U \to \mathbb{R}^n$ be a coordinate chart on M. Out of it we construct a chart on TU. The first n functions come for free: we take the functions $x_1 \circ \pi, \ldots, x_n \circ \pi$. Another set of n functions come for free also: by (3.6), given a vector $v \in T_aU$,

$$v = \sum (dx_i)_a(v) \frac{\partial}{\partial x_i}|_a.$$

Hence, abusing the notation a bit, we get maps

$$dx_i : TU \to \mathbb{R}, \quad TU \ni v \mapsto (dx_i)_a(v), \text{ where } a = \pi(v).$$

Thus we define a candidate coordinate chart

$$\tilde{\phi} := (x_1 \circ \pi, \cdots, x_n \circ \pi, dx_1, \cdots, dx_n) : TU \to \mathbb{R}^n \times \mathbb{R}^n$$

by

$$\tilde{\phi}(v) = (x_1(\pi(v)), \ldots, x_n(\pi(v)), (dx_1)_{\pi(v)}(v), \ldots, (dx_n)_{\pi(v)}(v)).$$

If $\{U_\alpha, \phi_\alpha)\}$ is an atlas on M, we get a candidate atlas $\{(TU_\alpha, \tilde{\phi}_\alpha)\}$ on TM. To see why this could possibly be an atlas, we need to check that the change of coordinates in this new purported atlas is smooth. To this end pick two coordinate charts $(U, \phi = (x_1, \cdots, x_n))$ and $(V, \psi = (y_1, \cdots, y_n))$ on M with $U \cap V \neq \emptyset$. Then $T(U \cap V) = TU \cap TV \neq \emptyset$. Let

$$\tilde{\phi} = (x_1, \cdots, x_n, dx_1, \cdots, dx_n) : TU \to \mathbb{R}^n \times \mathbb{R}^n$$

and

$$\tilde{\psi} = (y_1, \cdots, y_n, dy_1, \cdots, dy_n) : TV \to \mathbb{R}^n \times \mathbb{R}^n$$

be the corresponding candidates charts on TM. Now let us compute the change of coordinates $\tilde{\psi} \circ \tilde{\phi}^{-1}$.

First, note that

$$\tilde{\phi}^{-1}(r_1, \cdots, r_n, u_1, \cdots, u_n) = \sum_i u_i \frac{\partial}{\partial x_i}\Big|_{\phi^{-1}(r_1, \cdots, r_n)} \in T_{\phi^{-1}(r_1, \cdots, r_n)}M.$$

So

$$\tilde{\psi}(\sum u_i \frac{\partial}{\partial x_i}\Big|_{\phi^{-1}(r_1, \cdots, r_n)}) = (\psi(\phi^{-1}(r_1, \cdots, r_n)), dy_1(\sum_i u_i \frac{\partial}{\partial x_i}), \cdots, dy_n(\sum_i u_i \frac{\partial}{\partial x_i})).$$

But

$$dy_j(\sum_i u_i \frac{\partial}{\partial x_i}) = \sum_i u_i(\frac{\partial}{\partial x_i}(y_j)) = \sum_i \frac{\partial y_j}{\partial x_i} u_i = \sum_i \frac{\partial}{\partial r_i}(r_j(\psi \circ \phi^{-1}))u_i.$$

Thus the change of the candidate coordinates is given by

$$\begin{aligned} \tilde{\psi} \circ \tilde{\phi}^{-1}(r_1, \cdots, r_n, u_1, \cdots, u_n) =& (\psi \circ \phi^{-1}(r), (\sum_i \frac{\partial y_1}{\partial x_i}(r)u_i, \ldots, \sum_i \frac{\partial y_n}{\partial x_i}(r)u_i)) \\ =& (\psi \circ \phi^{-1}(r), \left(\frac{\partial y_j}{\partial x_i}(r)\right) \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}), \end{aligned} \tag{3.7}$$

where $r = (r_1, \ldots r_n)$. Clearly $\tilde{\psi} \circ \tilde{\phi}^{-1}$ is smooth wherever it is defined. It remains to define a topology on TM so that the charts $\tilde{\phi} : TU \to \phi(U) \times \mathbb{R}^n$ are homeomorphisms. We declare a subset $O \subset TM$ to be open if for any coordinate chart $\phi : U \to \mathbb{R}^n$ on M, the set $\tilde{\phi}(O \cap TU) \subset \mathbb{R}^n \times \mathbb{R}^n$ is open.

Proposition 3.17. *The collection of open sets on TM defined above does indeed form a topology. Moreover, if M is Hausdorff and second countable, so is TM.*

Proof. An exercise for the reader. ☐

We conclude that if M is an n-dimensional Hausdorff second countable manifold then its tangent bundle TM is a $2n$-dimensional Hausdorff second countable manifold. Moreover, each coordinate chart $(x_1, \dots x_n) : U \to \mathbb{R}^n$ on M gives rise to a coordinate chart $(x_1 \circ \pi, \dots x_n \circ \pi, dx_1, \dots, dx_n) : TU \to \mathbb{R}^{2n}$.

Remark 3.18. The following notation is suggestive: we write $(m, v) \in TM$ for $v \in T_m(M)$. Strictly speaking, it is redundant since $m = \pi(v)$.

Remark 3.19. It is customary to simply write $x_i : TU \to \mathbb{R}$ for $x_i \circ \pi : TU \to \mathbb{R}$.

Exercise 3.8. Prove that the map $\pi : TM \to M$ is smooth and that the differential $d\pi_v : T_v(TM) \to T_{\pi(v)}M$ is surjective for all tangent vectors $v \in TM$. Hint: do it in (convenient) coordinates.

3.5. **The cotangent bundle.** As a set, the cotangent bundle T^*M is the disjoint union of cotangent spaces:

$$T^*M = \bigsqcup_{a \in M} T_a^*M.$$

Note that there is a natural projection (the cotangent bundle projection)

$$\pi : T^*M \to M$$

which sends a cotangent vector (a covector for short) $\eta \in T_a^*M$ to the corresponding point a of M. We make the cotangent bundle T^*M into a manifold in more or less the same way we made the tangent bundle into a manifold. That is, we manufacture new coordinate charts on T^*M out of coordinate charts on M and check that the transition maps between the new coordinate charts are smooth.

So let $\phi = (x_1, \dots, x_n) : U \to \mathbb{R}^n$ be a coordinate chart on M. Then for each point $a \in U$ the covectors $\{(dx_i)_a\}$ form a basis of T_a^*M. The partials $\{\frac{\partial}{\partial x_i}|_a\}$ form the dual basis. Hence for any $\eta \in T_a^*M$,

$$\eta = \sum \eta(\frac{\partial}{\partial x_i}|_a)\, (dx_i)_a.$$

Therefore the partials $\{\frac{\partial}{\partial x_i}\}$ give us coordinate functions on T^*U:

$$\frac{\partial}{\partial x_i} : T^*U \to \mathbb{R}^n, \quad T^*U \ni \eta \mapsto \eta(\frac{\partial}{\partial x_i}|_a),$$

where $a = \pi(\eta)$. We now define the candidate coordinates

$$\bar{\phi} : T^*U \to \mathbb{R}^n \times \mathbb{R}^n$$

by

$$\bar{\phi} = (x_1 \circ \pi, \dots, x_n \circ \pi, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}).$$

Note that

$$\bar{\phi}^{-1}(r_1, \dots, r_n, w_1, \dots, w_n) = \sum_{i=1}^n w_i (dx_i)_{\phi^{-1}(r)} \in T^*_{\phi^{-1}(r)}M,$$

where again we have abbreviated $(r_1, \dots, r_n)$ as r. We now check the transition maps. Let $\psi = (y_1, \dots, y_n) : V \to \mathbb{R}^n$ be a coordinate chart on M with $V \cap U \neq \emptyset$. Then

$$\begin{aligned}
\bar{\psi} \circ \bar{\phi}^{-1}(r_1, \dots, r_n, w_1, \dots, w_n) =& \bar{\psi}(\sum_{i=1}^n w_i (dx_i)_{\phi^{-1}(r)}) \\
=& ((\psi \circ \phi^{-1})(r), \frac{\partial}{\partial y_1}(\sum_{i=1}^n w_i\, dx_i), \dots, \frac{\partial}{\partial y_n}(\sum_{i=1}^n w_i\, dx_i)) \\
=& ((\psi \circ \phi^{-1})(r), \sum_i w_i \frac{\partial x_i}{\partial y_1}, \dots, \sum_i w_i \frac{\partial x_i}{\partial y_n}).
\end{aligned}$$

We conclude that

$$\bar{\psi} \circ \bar{\phi}^{-1}(r_1, \cdots, r_n, w_1, \cdots, w_n) = (\psi \circ \phi^{-1}(r), \left(\frac{\partial x_i}{\partial y_j}(r)\right) \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}), \tag{3.8}$$

which is smooth. The rest of the argument proceeds as in the case of the tangent bundle.

Remark 3.20. Later on, when we look at the general vector bundles, it will be instructive to compare the formulas for the change of coordinates in the tangent and the cotangent bundles. In particular note that the matrices $\left(\frac{\partial y_j}{\partial x_i}(r)\right)$ and $\left(\frac{\partial x_i}{\partial y_j}(r)\right)$ are inverse transposes of each other.

3.6. **Vector fields.** A vector field X on a manifold M smoothly assigns to a point $a \in M$ a tangent vector $X(a) \in T_aM$.[3] What does "smoothly" mean? If X is a vector field in $\mathbb{R}^n$ then

$$X(a) = \sum f_i(a) \frac{\partial}{\partial r_i}|_a$$

for certain functions $f_i(a) \in \mathbb{R}$ of the point $a \in \mathbb{R}^n$. So whatever we mean by "smooth" should amount to the functions f_i being smooth. This suggests one definition of a smooth vector field:

Definition 3.21. A vector field X on a manifold M is *smooth* if for any coordinate chart $\phi = (x_1, \ldots, x_n) : U \to \mathbb{R}^n$ we have, for any point $a \in U$,

$$X(a) = \sum f_i(a) \frac{\partial}{\partial x_i}|_a \tag{3.9}$$

for some smooth functions $f_i : U \to \mathbb{R}$.

There is something a bit unsatisfying about this definition: is it possible that the functions f_i in (3.9) are smooth for one choice of coordinates and not smooth for another choice? So we will use it as as starting point for a better one. Note that the functions f_i in (3.9) are given by:

$$f_i(a) = (dx_i)_a(X(a)),$$

for any $a \in U$. Thus Definition 3.21 simply says that the composite $(x_1, \ldots, x_n, dx_1, \ldots, dx_n) \circ X : U \to \mathbb{R}^n \times \mathbb{R}^n$ is smooth. But this is the same thing as saying that the map $X : M \to TM$ is smooth. Not every map $Z : M \to TM$ is a vector field: we need to make sure that $Z(a) \in T_aM$. The condition is equivalent to

$$\pi(Z(a)) = a$$

for all $a \in M$. Here, as before, $\pi : TM \to M$ is the natural projection. This gives us a slightly more "sophisticated" definition of a vector field:

Definition 3.22. A (smooth) *vector field* X on a manifold M is a smooth map $X : M \to TM$ such that $\pi \circ X = id$.

There is yet another definition of a vector field, which is quite useful from some points of view:

Definition 3.23. A smooth *vector field* X on a manifold M is a linear map $X : C^\infty(M) \to C^\infty(M)$ such that

$$X(fg) = fX(g) + gX(f) \quad \text{for all } f, g \in C^\infty(M). \tag{3.10}$$

Proposition 3.24. *Definitions 3.22 and 3.23 are equivalent.*

Proof. Exercise. Here are a few hints. Given a vector field $X : M \to TM$ define a map $\tilde{X}$ from $C^\infty(M)$ to functions on M by

$$(\tilde{X}(f))(a) = X_a(f)$$

for all $f \in C^\infty(M)$ and all $a \in M$. Check that $\tilde{X}(f)$ is a smooth function and that the map $\tilde{X}$ so defined is a derivation. That is, show that (3.10) holds with X replaced by $\tilde{X}$.

Conversely, given a map $\tilde{X} : C^\infty(M) \to C^\infty(M)$ with the derivation property as above, define $X : M \to TM$ by

$$X_a(f) = (\tilde{X}(f))(a)$$

for all $f \in C^\infty(M)$ and all $a \in M$. Check that X_a is indeed a tangent vector in T_aM and that the map $X : M \to TM$, $a \mapsto X_a$ is smooth in a. ∎

[3]Sometimes this is also written X_a.

Remark 3.25. From now on we will not distinguish between the two definitions and will think of vector fields as either smooth maps $M \to TM$ satisfying certain conditions or as $\mathbb{R}$-linear maps $C^\infty(M) \to C^\infty(M)$ satisfying the appropriate conditions. We will make no notation distinction between the two ways of looking at vector fields. Thus $X(a)$ will stand for the value of a vector field at a point a if a is a point. On the other hand, if f is a smooth function, $X(f)$ will stand for a new smooth function, the "derivative" of f with respect to the vector field X.

Notation. There are several standard ways to denote the space of all smooth vector fields on a given manifold M. The two most common ones are $\Gamma(TM)$ [vector fields are sections of the tangent bundle, see below] and $\mathcal{X}(M)$.

Remark 3.26. 1. The space of vector fields $\Gamma(TM)$ is a vector space over $\mathbb{R}$: if $X, Y \in \Gamma(TM)$ are (smooth) vector fields and $\lambda, \mu \in \mathbb{R}$ are scalars, then their linear combination $\lambda X + \mu Y$ is defined by

$$(\lambda X + \mu Y)(a) := \lambda X(a) + \mu Y(a)$$

for any $a \in M$. It is again a smooth vector field.

2. We can also multiply vector fields on M by smooth functions: if $X \in \Gamma(TM)$ and $f \in C^\infty(M)$ then fX is defined by

$$(fX)(a) := f(a)X(a)$$

for all $a \in M$.

A fancy way of describing 2. is to say that $\Gamma(TM)$ is a *module* over the ring of smooth functions $C^\infty(M)$. See if you can impress your date.

If $X, Y \in \Gamma(TM)$ are two vector fields on a manifold M then it is *not true* that the $\mathbb{R}$-linear map

$$C^\infty(M) \to C^\infty(M), \quad f \mapsto X(Y(f)).$$

is a vector field — it does not have the correct derivation property. For example, if $M = \mathbb{R}$ and $X = Y = \frac{d}{dt}$, then $X(Y(f)) = f''$ and $(fg)'' = (f'g + fg')' = f''g + 2f'g' + fg'' \neq f''g + fg''$. However,

Lemma 3.27. *Let $X, Y \in \Gamma(TM)$ be two smooth vector fields on a manifold M. Then the map*

$$[X, Y] : C^\infty(M) \to C^\infty(M), \quad f \mapsto X(Y(f)) - Y(X(f)) \tag{3.11}$$

is a vector field.

Proof. Clearly the map $[X, Y]$ is $\mathbb{R}$-linear. We need to check that it has the correct derivation property. This is a mindless computation. Pick two functions $f, g \in C^\infty(M)$. Then

$$\begin{aligned}
[X, Y](fg) =& X(Y(fg)) - Y(X(fg)) \\
=& X(Y(f)g + fY(g)) - Y(X(f)g + fX(g)) \\
=& X(Y(f))g + Y(f)X(g) + X(f)Y(g) + fX(Y(g)) - Y(X(f))g - X(f)Y(g) - Y(f)X(g) - fY(X(g)) \\
=& X(Y(f))g - Y(X(f))g + fX(Y(g)) - fY(X(g)) \\
=& ([X, Y](f))g + f([X, Y](g)).
\end{aligned}$$

□

Definition 3.28. The *Lie bracket* of two vector fields X and Y on a manifold M is the vector field $[X, Y]$ defined by (3.11).

We now quickly recall the definitions of bilinear and skew-symmetric bilinear maps, the point being that Lie bracket will turn out to be a skew-symmetric bilinear map.

Definition 3.29. Let V, U and W be three vector spaces over the reals. A map

$$b : V \times U \to W$$

is *bilinear* if it is ($\mathbb{R}$-) linear in each argument: for all $u_1, u_2 \in U$, $c_1, c_2 \in \mathbb{R}$ and all $v \in V$,

$$b(v, c_1u_1 + c_2u_2) = c_1b(v, u_1) + c_2b(v, u_2);$$

and for all $v_1, v_2 \in V$, $c_1, c_2 \in \mathbb{R}$ and all $u \in U$,

$$b(c_1v_1 + c_2v_2, u) = c_1b(v_1, u) + c_2b(v_2, u).$$

Definition 3.30. A bilinear map $b : U \times U \to V$ is *skew-symmetric* if

$$b(u_1, u_2) = -b(u_2, u_1)$$

for all $u_1, u_2 \in U$.

It is easy to see that the Lie bracket on a manifold M is $\mathbb{R}$-bilinear and skew-symmetric. Note that it is *not* $C^\infty(M)$-bilinear:

$$[X, hY] = X(h)Y + h[X, Y]$$

for any $X, Y \in \Gamma(TM)$, $h \in C^\infty(M)$ (prove this).

Somewhat surprisingly the Lie bracket has a kind of derivation property:

Lemma 3.31 (Jacobi identity)**.** *For any three vector fields $X, Y, Z \in \Gamma(TM)$ on a manifold M*

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]. \tag{3.12}$$

Here is how one sees this as a derivation property: for a vector field $X \in \Gamma(TM)$ define

$$L_X : \Gamma(TM) \to \Gamma(TM)$$

by

$$L_X(Y) = [X, Y].$$

With this definition (3.12) becomes:

$$L_X([Y, Z]) = [L_X(Y), Z] + [Y, L_X(Z)]$$

Proof of Lemma 3.31. This is another computation that's easier to do yourself than watch someone else doing it. To keep the notation from getting out of hand, we will drop parentheses. Thus $XYZf$ stands for $X(Y(Z(f))))$ etc. We pick a function $f \in C^\infty(M)$ and compute:

$$\begin{aligned}([[X, Y], Z] + [Y, [X, Z]])f =& [X, Y]Zf - Z[X, Y]f + Y[X, Z]f - [X, Z]Yf \\ &= XYZf - YXZf - ZXYf + ZYXf + YXZf - YZXf - XZYf + ZXYf \\ &= XYZf + ZYXf - YZXf - XZYf \\ &= X(YZf - ZYf) + (ZY - YZ)Xf = [X, [Y, Z]]f.\end{aligned}$$

This proves the Jacobi identity. □

Equation (3.12) is called the Jacobi identity and is often written as

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

(it is equivalent to (3.12) by skew-symmetry of $[\cdot, \cdot]$.

Definition 3.32. A (real) *Lie algebra* is a vector space V over $\mathbb{R}$ (perhaps infinite dimensional) together with a map $[\cdot, \cdot] : V \times V \to V$, a Lie bracket, such that

(1) $[\cdot, \cdot]$ is bilinear,
(2) $[\cdot, \cdot]$ is skew-symmetric, and
(3) $[\cdot, \cdot]$ satisfies the Jacobi identity: for all $v, u, w \in V$

$$[u, [v, w]] = [[u, v], w] + [v, [u, w]].$$

Example 3.33. We have proved that the space of vector fields $\Gamma(TM)$ on a manifold M forms a Lie algebra.

Example 3.34. $\mathbb{R}^3$ with the cross (vector) product is a Lie algebra.

Remark 3.35. The bracket on a Lie algebra can be thought of as a multiplication. Note that it is *not* associative in general because of the Jacobi identity.

The geometric meaning of the Lie brackets of vector fields will be discussed later.

4. Submanifolds and the implicit function theorem

Given a smooth function $F : \mathbb{R}^m \to \mathbb{R}^n$ and a point $c \in \mathbb{R}^n$ the level set

$$F^{-1}(c) := \{x \in \mathbb{R}^m \mid F(x) = c\}$$

may or may not be a smooth manifold. For example, take $f(x, y) = x^2 - y^2$, a smooth function on $\mathbb{R}^2$. Then $f^{-1}(0)$ is the union of two lines: $y = \pm x$. It is not a manifold. However, for $c \neq 0$, $f^{-1}(c)$ is a union of two smooth curves, hence a 1 dimensional manifold. The goal of this section is to describe a sufficient condition for the level sets $F^{-1}(c)$ to be manifolds. We then generalize this to level sets of smooth maps between manifolds. The key technical result that makes it all possible is the inverse function theorem.

4.1. The inverse function theorem and a few of its consequence.

Theorem 4.1 (Inverse function theorem)**.** *Let $U, U' \subset \mathbb{R}^n$, be open sets and $F : U \to U'$ a smooth map. Suppose for some point $a \in U$ the differential*

$$dF_a : \mathbb{R}^n \to \mathbb{R}^n$$

is invertible. Then there are open neighborhoods U_0 of a in U and U_0' of $F(a)$ in U' so that

$$F : U_0 \to U_0'$$

is a diffeomorphism.

We will assume this result without proof. It is not essential that U and U' are open subsets of $\mathbb{R}^n$ — any finite dimensional vector space will do. It is even true with $\mathbb{R}^n$ replaced by a Banach space. We now discuss various consequences of the inverse function theorem. The most famous one is the implicit function theorem. But first we prove the manifold version.

Proposition 4.2. *Let $f : N \to M$ be a smooth map of manifolds with $f(p) = q$ ($p \in N$, $q \in M$). Suppose*

$$df_p : T_pN \to T_qM$$

is an isomorphism (invertible linear map). There there are neighborhoods U of $p \in N$, V of q in M so that

$$f|_U : U \to V$$

is a diffeomorphism (invertible map with a smooth inverse).

Proof. Note first that if $\phi : U' \to \mathbb{R}^n$ is a coordinate chart on N then for any $z \in U'$ the map $d\phi_z : T_zN \to T_{\phi(z)}\mathbb{R}^n$ is an isomorphism (for instance if $\phi = (x_1, \dots, x_n)$, $d\phi_x(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial r_i}$).

So let $p \in U' \xrightarrow{\phi} \mathbb{R}^n$ and $q \in V' \xrightarrow{\psi} \mathbb{R}^m$ be two coordinate charts on M and N respectively. Then the diagram

$$\begin{array}{ccc} U' & \xrightarrow{f} & V' \\ \phi\downarrow & & \downarrow\psi \\ \phi(U') & \xrightarrow[(\psi\circ f\circ\phi^{-1})]{} & \psi(V') \end{array} \tag{4.1}$$

commutes: $\psi \circ f = (\psi \circ f \circ \phi^{-1}) \circ \phi$. Hence the diagram of differentials

$$\begin{array}{ccc} T_pN & \xrightarrow{df_p} & T_qM \\ d\phi_p\downarrow & & \downarrow d\psi_q \\ T_{\phi(p)}\phi(U') & \xrightarrow[d(\psi\circ f\circ\phi^{-1})_{\phi(p)}]{} & T_{\psi(q)}\psi(V') \end{array} \tag{4.2}$$

commutes as well. By the inverse function theorem, there are neighborhoods $\bar{U}$ of $\phi(p)$ and $\bar{V}$ of $\psi(q)$ so that

$$(\psi \circ f \circ \phi^{-1})|_{\bar{U}} : \bar{U} \to \bar{V}$$

is a diffeomorphism. Consequently,

$$f : \phi^{-1}(\bar{U}) \to \psi^{-1}(\bar{V})$$

is a diffeomorphism. ☐

Next we turn to the implicit function theorem, the vector space version.

Theorem 4.3 (Implicit function theorem)**.** *Let $F : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^k$ be a smooth map, $(a,b) \in \mathbb{R}^n \times \mathbb{R}^k$ a point and $c = F(a,b)$. Suppose that the restriction of the differential*

$$dF_{(a,b)}|_{\{0\}\times\mathbb{R}^k} : \{0\} \times \mathbb{R}^k \to \mathbb{R}^k$$

is onto. Then there are neighborhoods U of $a \in \mathbb{R}^n$, W of (a,b) in $\mathbb{R}^n \times \mathbb{R}^k$ and a smooth map $g : U \to \mathbb{R}^k$ with $g(a) = b$ such that the

$$F^{-1}(c) \cap W = \text{ graph } \{g : U \to \mathbb{R}^k\}.$$

That is, for $(x,y) \in W$

$$F(x,y) = c \Leftrightarrow y = g(x).$$

In other words the function g is implicitly defined by the equation $F(x, g(x)) = c$.

Proof. We write suggestively $\frac{\partial F}{\partial x}(a,b)$ for the restriction $dF_{(a,b)}|_{\mathbb{R}^n\times\{0\}}$ and $\frac{\partial F}{\partial y}(a,b)$ for $dF_{(a,b)}|_{\{0\}\times\mathbb{R}^k}$. Consider the smooth map $H : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n \times \mathbb{R}^k$ defined by

$$H(x,y) = (x, F(x,y))$$

for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^k$. Then the differential of H at (a,b) is of the form

$$dH_{(a,b)} = \left(\begin{array}{c|c} I & 0 \\ \hline \frac{\partial F}{\partial x}(a,b) & \frac{\partial F}{\partial y}(a,b) \end{array} \right),$$

where $I : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map. By assumption $\frac{\partial F}{\partial y}(a,b)$ is invertible. Hence $dH_{(a,b)}$ is invertible. By the inverse function theorem the function H is invertible on a neighborhood of (a,b). Let $G(u,v) = (G_1(u,v), G_2(u,v))$ denote its inverse, which is defined on a neighborhood of $H(a,b) = (a, F(a,b)) = (a,c)$. We may take this neighborhood to be of the form $U \times V$, with $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^k$ being open. Let $W = G(U \times V)$. Then

$$(u,v) = H(G(u,v)) = (G_1(u,v), F(G_1(u,v), G_2(u,v))$$

for all $(u,v) \in U \times V$. Hence $G_1(u,v) = u$. Therefore

$$F(u, G_2(u,v)) = v$$

for all $(u,v) \in U \times V$. Conversely, if for any $(x,y) \in W$ we have $F(x,y) = v$ then

$$(x,y) = G(H(x,y)) = G(x, F(x,y)) = G(x,v) = (G_1(x,v), G_2(x,v))$$

and therefore $y = G_2(x,v)$.

Define the function $g : U \to \mathbb{R}^k$ by

$$g(x) = G_2(x,c).$$

It is a smooth function and, by the above discussion,

$$F(x,y) = c \;\Leftrightarrow\; y = g(x)$$

for any $(x,y) \in W$. ☐

Remark 4.4. Here is a slightly different and ultimately more useful way to look at what we have proved. The argument above shows that there is a diffeomorphism

$$H : W \to U \times V$$

mapping bijectively the set

$$\{F = c\} \cap W := \{(x,y) \in W \mid F(x,y) = c\}$$

onto the set

$$H(W) \cap (\mathbb{R}^n \times \{c\})$$

This motivates the following definition.

Definition 4.5 (Submanifold)**.** Let M be an m-dimensional manifold. A subset $N \subset M$ is an n-dimensional *embedded submanifold* if for every point $q \in N$, there is a coordinate chart $\phi = (x_1, \cdots, x_m) : U \to \mathbb{R}^m$ with $q \in U$ such that

$$\phi(U \cap N) = \phi(U) \cap (\mathbb{R}^n \times \{0\}).$$

That is, for all $a \in N \cap U$,

$$\phi(a) = (x_1(a), \cdots, x_n(a), 0, \cdots, 0).$$

Such charts are said to be *adapted to N.*

Example 4.6. The sphere S^2 is an embedded submanifold of $\mathbb{R}^3$. For example if $(x_1, x_2, x_3) \in S^2$ and $x_3 > 0$ then

$$\phi(x_1, x_2, x_3) = (x_1, x_2, x_3 - \sqrt{1 - x_1^2 - x_2^2})$$

is a chart adapted to S^2 (and there are 5 more charts like this).

Thus the implicit function theorem says that, under certain conditions, portions of a level set of a map $F : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^k$ are embedded submanifolds. Naturally the embedded submanifolds *are* manifolds in their own right.

Lemma 4.7. *If $N \subset M$ is an n-dimensional embedded submanifold of an m-dimensional manifold M then it is naturally an n-dimensional manifold in its own right, and the inclusion map $\iota : N \hookrightarrow M$, $\iota(a) = a$ is smooth.*

Proof. We make N into a topological space by giving it the subspace topology. If $\phi : U \to \mathbb{R}^m$ is a chart on M adapted to N, then $p \circ \phi|_N : N \cap U \to \phi(U) \cap \mathbb{R}^n$ is a homeomorphism. Here $p : \mathbb{R}^m \to \mathbb{R}^n$ is the projection $p(x_1, \ldots, x_n, \ldots, x_m) = (x_1, \ldots, x_n)$. If $\psi : V \to \mathbb{R}^m$ is another chart adapted to N, then $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ maps $\phi(U \cap V) \cap (\mathbb{R}^n \times \{0\})$ diffeomorphically to $\psi(U \cap V) \cap (\mathbb{R}^n \times \{0\})$. Hence if $\{\phi_\alpha : U_\alpha \to \mathbb{R}^m\}$ is a collection of charts on M adapted to N with $M = \bigcup U_\alpha$ then $\{p \circ \phi_\alpha|_{U_\alpha \cap N} : U_\alpha \cap N \to \mathbb{R}^n\}$ is an atlas on N. Checking that the inclusion map ι is smooth is easy: in coordinates it's the inclusion $\mathbb{R}^n \to \mathbb{R}^m$, $(r_1, \ldots, r_n) \mapsto (r_1, \ldots, r_n, 0, \ldots, 0)$ ☐

We now generalize the implicit function theorem.

Proposition 4.8. *Let $F : \mathbb{R}^m \to \mathbb{R}^k$ be a smooth map and $c \in F(\mathbb{R}^m) \subset \mathbb{R}^k$ a point. Suppose that for all points $q \in F^{-1}(c)$ the differential*

$$dF_q : \mathbb{R}^m \to \mathbb{R}^k$$

is onto. Then the level set $F^{-1}(c)$ is a submanifold of $\mathbb{R}^m$ and (if $F^{-1}(c)$ is nonempty)

$$\dim F^{-1}(c) = \dim \mathbb{R}^m - \dim \mathbb{R}^k.$$

Proof. Fix a point $q \in F^{-1}(c)$. Let $Z = \ker dF_p$. Let $X \subset \mathbb{R}^m$ be the vector space complement to Z so that

$$\mathbb{R}^m = Z \oplus X \simeq Z \times X.$$

We can thus think of a point $p \in \mathbb{R}^m$ as a pair $(z, x) \in Z \times X$. By assumption on dF_q and by construction of X, the restriction

$$dF_q|_X : X \to \mathbb{R}^k$$

is an isomorphism of vector spaces. We now proceed as in the proof of the implicit function theorem. Consider

$$H : Z \times X \to Z \times \mathbb{R}^k, \quad H(z, x) = (z, F(z, x)).$$

Write $\frac{\partial F}{\partial z}$ for $dF|_Z$ and $\frac{\partial F}{\partial x}$ for $dF|_X$. Then

Then the differential of H is of the form

$$dH_{(z,x)} = \left(\begin{array}{c|c} I & 0 \\ \hline \frac{\partial F}{\partial z} & \frac{\partial F}{\partial x} \end{array} \right).$$

By construction $\frac{\partial F}{\partial x}(q) : X \to \mathbb{R}^k$ is a bijection. Hence dH_q is a bijection. By the inverse function theorem there exist neighborhoods W of q in $\mathbb{R}^m$ and $U \times V$ of $H(q)$ in $Z \times \mathbb{R}^k$ so that $H : W \to U \times V$ is a diffeomorphism. Moreover, as in the proof of the implicit function theorem H maps bijectively $\{F = c\} \cap W$ to $(U \times V) \cap (Z \times \{c\})$. Therefore $F^{-1}(c) = \{F = c\}$ is a submanifold of $\mathbb{R}^m$ of dimension

$$\dim Z = \dim \mathbb{R}^m - \dim \mathbb{R}^k.$$

□

Example 4.9. Consider $F : \mathbb{R}^n \to \mathbb{R}$, $F(x) = \sum x_i^2$. Then $dF_x = (2x_1, \dots, 2x_n)$. Hence dF_x is surjective for all nonzero x. In particular $F^{-1}(1) = \{x \in \mathbb{R}^n \mid \sum x_i^2 = 1\}$ is a submanifold of $\mathbb{R}^n$ of dimension $n-1$. This is, of course, the standard sphere of radius 1.

Definition 4.10 (Regular value)**.** Suppose $f : M \to N$ is a smooth map of manifolds. A point $c \in N$ is a *regular value* of f if for all $x \in f^{-1}(c)$ the differential

$$df_x : T_xM \to T_cN$$

is surjective.

The previous proposition then simply states that non-empty preimages of a regular values of a map $F : \mathbb{R}^m \to \mathbb{R}^k$ are submanifolds of $\mathbb{R}^m$.

Remark 4.11. Note that if $f^{-1}(c) = \emptyset$, then c is a regular value of f. It seems silly to construct a definition this way. The reason for the peculiar phrasing is that it makes easier to state Sard's theorem.

Theorem 4.12 (Sard's Theorem)**.** *Let $f : M \to N$ be a smooth map. Then the set of regular values of f is dense in M (and in fact its compliment has measure* 0*).*

Note that if $F : M \to N$ maps everything to one point $\{c\}$ then c is not a regular value (the differential of F is 0 everywhere), but $N \smallsetminus \{c\}$ does consist of regular values. So Sard's theorem does hold for constant maps, except for the preimage of every regular value of a constant map is empty. It will take us too far afield to prove Sard's theorem, so we won't do it. On the other hand Proposition 4.8 nicely generalizes to manifolds:

Theorem 4.13. *If c is a regular value of a smooth map of manifolds $f : M \to N$ and if $f^{-1}(c) \neq \emptyset$ then the level set $f^{-1}(c)$ is an embedded submanifold of M of dimension*

$$\dim f^{-1}(c) = \dim(M) - \dim(N).$$

Before we proceed with the proof of Theorem 4.13, we make a two observations.

1. Let $\{\phi_\alpha : U_\alpha \to \mathbb{R}^m\}$ be an atlas on a manifold M. Suppose for some index β there is a diffeomorphism $\sigma : \phi_\beta(U_\beta) \to W \subset \mathbb{R}^m$ (W is some open set). Then
   (i) $\sigma \circ \phi_\beta : U_\beta \to \mathbb{R}^m$ is a chart on M,
   (ii) this chart is compatible with the atlas $\{\phi_\alpha : U_\alpha \to \mathbb{R}^m\}$ we started out with.

The implies that

3. If Z is a submanifold of a manifold M and $H : M \to M'$ is a diffeomorphism, then $H(Z)$ is a submanifold of M'.

Proof of Theorem 4.13. It is enough to show that for every point a of $f^{-1}(c)$ there is a neighborhood U of a such that $U \cap f^{-1}(c)$ is a submanifold of U of dimension $m-n$.

Let $a \in f^{-1}(c)$ be a point. Let $\phi : U \to \mathbb{R}^m$ be a chart of M with $a \in U$ and $\psi : V \to \mathbb{R}^n$ be a chart on N with $c \in V$. Then

$$\psi \circ f \circ \phi^{-1} : U' \to V$$

is a smooth map. Moreover, by the chain rule,

$$d(\psi \circ f \circ \phi^{-1})_0 = d\psi_c \circ df_a \circ d(\phi^{-1})_0.$$

Since $d\psi_c$ and $d\phi_a$ are isomorphisms and df_a is onto for any $a \in f^{-1}(c)$ by assumption, $d(\psi \circ f \circ \phi^{-1})_{\phi(a)} : T_{\phi(a)}\mathbb{R}^m \to T_{\psi(c)}\mathbb{R}^n$ is onto for any $a \in f^{-1}(c) \cap U$. By Proposition 4.8 $(\psi \circ f \circ \phi^{-1})^{-1}(\psi(c)) = \phi(U \cap f^{-1}(c))$ is a submanifold of $\phi(U)$ of dimension $m-n$. Therefore $U \cap f^{-1}(c)$ is a submanifold of $U \subset M$ of dimension $m-n$. Since a is arbitrary, $f^{-1}(c)$ is a submanifold of all of M of the desired dimension. □

The next statement describes the tangent bundle of a regular level set $f^{-1}(c)$.

Corollary 4.13.1. *Suppose that c is a regular value of $f : M \to N$ and $f^{-1}(c) \neq \emptyset$. Then for all $a \in f^{-1}(c)$,*

$$T_a f^{-1}(c) = \ker(df_a).$$

Proof. Since $\dim T_a f^{-1}(c) = \dim f^{-1}(c) = \dim M - \dim N = \dim \ker df_a$, it is enough to prove that $T_a f^{-1}(a) \subset \ker df_a$. Let $v \in T_a f^{-1}(c)$ be a vector.

By exercise 3.4 there is a curve $\gamma : I \to f^{-1}(c)$ (where I is an interval containing 0) such that $\gamma(0) = a$ and $d\gamma(\frac{d}{dt}) = v$. Since $f \circ \gamma$ is a constant map, $d(f \circ \gamma)_0 = 0$. By the chain rule, $d(f \circ \gamma)_0(\frac{d}{dt}) = df_{\gamma(0)}(d\gamma_0(\frac{d}{dt})) = df_a(v)$. Therefore $T_a f^{-1}(c) \subset \ker df_a$ and we are done. □

Example 4.14. Let $f : \mathbb{R}^n \to \mathbb{R}$ be given by $f(x) = \sum x_i^2$. Then, as we have seen before, 1 is a regular value of f and $df_x = (2x_1, \ldots, 2x_n)$ for all $x \in \mathbb{R}^n$. Therefore, for any $x \in f^{-1}(1) = S^{n-1}$ the tangent space $T_x S^{n-1}$ is naturally isomorphic to $\ker\{v \mapsto \sum 2x_i v_i\}$, which is the $(n-1)$ dimensional hyperplane in $\mathbb{R}^n \simeq T_x\mathbb{R}^n$ orthogonal to the vector x. □

Exercise 4.1. Show that $\mathrm{O}(n)$, the set of all $n \times n$ orthogonal matrices, is a submanifold of $\mathrm{GL}(n, \mathbb{R})$. Hint: Consider the map $f : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{Sym}(n, \mathbb{R})$ given by $A \mapsto AA^T$. Show that the identity matrix I is a regular value of f.

4.2. **Transversality.** We now have enough tools to do a bit of differential topology.

Definition 4.15 (Transversality)**.** A smooth map $F : M \to N$ of manifolds is *transverse* to a submanifold Z of N if for every $z \in Z$ and any $m \in F^{-1}(z)$, we have

$$T_z Z + dF_m(T_m M) = T_z N$$

(not necessarily as a direct sum!).

Notation. We write $F \pitchfork Z$ if a map F is transverse to a submanifold Z.

Example 4.16.

Let $N = \mathbb{R}^2$, $M = \mathbb{R}^3$, $Z = S^2 \subset M$ and $f : N \to M$ is given by $f(x_1, x_2) = (x_1, x_2, 0)$. Then $f \pitchfork S^2$.

Remark 4.17. A map $F : M \to N$ is transverse to submanifold Z consisting of one point c if and only if c is a regular value of F.

Example 4.18. Take $M = N = \mathbb{R}^2$. Consider $F : M \to N$ given by $F(x, y) = (x, x^2)$. Then F is transverse to $\{0\} \times \mathbb{R}$, but it is not transverse to $\mathbb{R} \times \{0\}$. □

Theorem 4.19. *If a smooth map $F : M \to N$ of manifolds is transverse to a submanifold Z of N, then $F^{-1}(Z)$ is a submanifold of M. Moreover,*

$$T_a(F^{-1}(Z)) = (dF_a)^{-1}(T_{F(a)}Z),$$

for all $a \in F^{-1}(Z)$, and

$$\dim(M) - \dim(F^{-1}(Z)) = \dim(N) - \dim(Z).$$

Proof. We first consider a special case: assume that $N = \mathbb{R}^n$, $Z = \mathbb{R}^k \times \{0\} \subset \mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^n$. Let $\pi : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ denote the canonical projection map. Then

$$\pi^{-1}(0) = \mathbb{R}^k \times \{0\} = Z,$$

hence

$$(\pi \circ F)^{-1}(0) = F^{-1}(Z).$$

Additionally, for all $a \in F^{-1}(Z)$

$$d(\pi \circ F)_a(T_a M) = d\pi_{F(a)}(dF_a(T_a M)) = d\pi_{F(a)}(dF_a(T_a M) + T_{F(a)}Z) = d\pi_{F(a)}(\mathbb{R}^n) = \mathbb{R}^{n-k},$$

where for the second equality we used the fact that $d\pi_{F(a)}(T_{F(a)}Z) = 0$. Therefore 0 is a regular value of $\pi \circ F$ and consequently $(\pi \circ F)^{-1}(0) = F^{-1}(Z)$ is a submanifold of M. Moreover,

$$T_a F^{-1}(Z) = T_a(\pi \circ F)^{-1}(0) = \ker d(\pi \circ F)_a = \ker\{d\pi_{F(a)} \circ dF_a\} = (dF_a)^{-1}(\ker d\pi_{F(a)}) = (dF_a)^{-1}(T_{F(a)}Z).$$

Finally, since $(d\pi \circ F)_m$ is surjective,

$$\dim F^{-1}(Z) = \dim(\ker(d\pi \circ F)_a) = \dim M - \dim \mathbb{R}^{n-k}.$$

Therefore

$$\dim M - \dim F^{-1}(Z) = \dim M - (\dim M - \dim \mathbb{R}^{n-k}) = \dim N - \dim Z.$$

What about the general case? Since Z is an embedded submanifold for all $z \in Z$, there is a coordinate chart $\psi = (x_1, \ldots, x_n) : N \to \mathbb{R}^n$ adapted to Z with $z \in V$. Hence $\psi(Z) = \psi(V) \cap (\mathbb{R}^k \times \{0\})$. Now apply the previous argument to $\psi \circ F : F^{-1}(V) \to \mathbb{R}^n$ and $\psi(V) \cap (\mathbb{R}^k \times \{0\})$. □

Example 4.20. Consider two surfaces S_1 and S_2 in $\mathbb{R}^3$ such that $T_xS_1 \neq T_xS_2$ for every $x \in S_1 \cap S_2$. Then $T_xS_1 + T_xS_2 = \mathbb{R}^3$ for all $x \in S_1 \cap S_2$.

Let $F : S_1 \hookrightarrow \mathbb{R}^3$ be the inclusion map. Then $dF_x(T_xS_1) = T_xS_1$. Thus, F is transverse to S_2. By the theorem above $F^{-1}(S_2) = S_1 \cap S_2$ is a submanifold of S_1 of dimension 1. In other words, if two surfaces are nowhere tangent then they intersect in a collection of curves.

4.3. **Embeddings, Immersions, and Rank.**

Definition 4.21 (Immersion)**.** A smooth map of manifold $f : Z \to M$ is an *immersion* if its differential is injective at every point of Z.

Immersions need not be injective: consider the map $f : S^1 \to S^1$, $f(e^{i\theta}) = e^{2i\theta}$. It's a 2-1 map but its differential everywhere is a bijection.

Example 4.22. The inclusion map of an submanifold is a 1-1 immersion.

Definition 4.23 (Submersion)**.** A map $f : M \to N$ is called a *submersion* if its differential at every point is surjective.

Exercise 4.2. Show that for any manifold M the canonical projection $\pi : TM \to M$ is a submersion — compute in the appropriate coordinates.

Exercise 4.3. Show that if $Z \subset M$ is an embedded submanifold, then $\pi^{-1}(Z) \subset TM$ is an embedded submanifold of the tangent bundle TM of M. Here again $\pi : TM \to M$ is the projection. Note that $\pi^{-1}(Z) = \cup_{a \in Z} T_aM$. It is often denoted by $TM\Big|_Z$.

Definition 4.24 (Embedding)**.** A smooth map of manifold $f : Z \to M$ is an *embedding* if $f(Z) \subset M$ is an embedded submanifold and $f : Z \to f(Z)$ is a diffeomorphism.

This says, in particular, that every embedding is a 1-1 immersion. The converse is not true.

Example 4.25. Let Z be an interval and consider a map f that sends it to figure 8 as in the picture. Then $f : Z \to \mathbb{R}^2$ is a 1-1 immersion which is not an embedding: the topology on $f(Z)$ as a subspace of $\mathbb{R}^2$ is coarser than the topology on $f(Z)$ that makes $f : Z \to f(Z)$ a homeomorphism. Or, if you prefer $f^{-1} : f(Z) \to Z$ is not continuous if $f(Z)$ is given the subspace topology.

Example 4.26. Consider the map $f : \mathbb{R} \to S^1 \times S^1$ given by

$$f(t) = (e^{2\pi i t}, e^{2\pi\sqrt{2}t}).$$

The image of f is dense in $S^1 \times S^1$. Hence f is a 1-1 immersion which is not an embedding. □

Definition 4.27 (Rank)**.** The rank of a smooth map $f : M \to N$ of manifold at a point $a \in M$ is the rank of the linear map $df_a : T_aM \to T_{f(a)}N$.

Proposition 4.28. *If $f : M \to N$ is a smooth and* $\operatorname{rank}(f) = k$ *at some point $a \in M$, then for all a' sufficiently close to a, $(rank f_s) \geq k$.*

Proof. The rank of f at a is the rank of the matrix $((\frac{\partial y_i \circ f}{\partial x_j}(a)))$, where $(x_1, \ldots, x_m)$ are coordinates on M near a and $(y_1, \ldots, y_n)$ are coordinates on N near $f(a)$. By a suitable permutation of coordinates, we may assume that $\det((\frac{\partial f_i}{\partial x_j}(a)))_{i,j \leq k} \neq 0$. Since the determinant is a continuous mapping, this determinant is also non-zero for points sufficiently close to a. □

The following theorem, which is a generalization of the Implicit Function Theorem, applies in particular to immersions, but we state the more general version.

Theorem 4.29 (Rank Theorem)**.** *Suppose that a smooth $f : M \to N$ has rank k at all points $a \in M$. Then for any point $a \in M$ there are coordinate chart $\phi : U \to \mathbb{R}^m$ on M about a and a chart $\psi : V \to \mathbb{R}^n$ on N about $f(a)$ such that*

$$(\psi \circ f \circ \phi^{-1})(r_1, \cdots, r_m) = (r_1, \cdots, r_k, 0, \cdots, 0).$$

We will not prove this theorem since we don't have the time

Exercise 4.4. Define $f : R^3 \to \mathbb{R}^6$ by $f(x, y, z) = (x^2, y^2, z^2, yz, zx, xy)$. Is f an immersion? Show that the restriction of f to S^2 is an immersion of S^2 into $\mathbb{R}^6$.

Exercise 4.5. Show that there is no immersion $f : S^2 \to \mathbb{R}^2$.

Exercise 4.6. (a) Let N be a manifold. Prove that the diagonal $\Delta_N = \{(n, n) \in N \times N : n \in N\}$ is an embedded submanifold of $N \times N$.
(b) Let $F : M \to N$ and $g : L \to N$ be smooth maps such that, for all $m \in M$ and $l \in L$ with $f(m) = g(l)$ we have

$$df_m(T_m M) + dg_l(T_l L) = T_r N, \; r = f(m) = g(l).$$

Show that

$$Z = \{(m, l) \in M \times L : f(m) = g(l)\}$$

is a submanifold of $M \times L$.

Exercise 4.7. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map such that for every x with $||x|| \geq 2$, we have $||f(x)|| < 1/||x||$. Show that (a) $||f||$ attains its maximum value at a point of $\mathbb{R}^n$.
(b) f is not an immersion.

Exercise 4.8. Let N be a closed embedded submanifold of M. Show that every vector field X on N can be extended to a vector field Y on M.

Hint: First extend the vector field in adapted coordinates. Next, use a partition of unity to combine each of the locally defined extensions into a global vector field.

Exercise 4.9. Consider $f(x, y) = y^2 + \frac{1}{6}x^6 - \frac{1}{2}x^2$ on $\mathbb{R}^2$. For each $c \in \mathbb{R}$, determine whether or not $f^{-1}(c)$ is a submanifold of $\mathbb{R}^2$. Justify your answer.

5. Vector fields and flows

5.1. **Definitions, examples, correspondence between vector fields and flows.** We start with a few words about notation. In this section I and J will stand for an open connected subset of the reals containing the origin, such as an open interval (a, b) or half-infinite intervals $(-\infty, b)$ and $(a, +\infty)$ or the whole of $\mathbb{R}$ (of course $a < 0 < b$).

Recall next that given a curve $\gamma : I \to M$ in a manifold M, the *tangent vector* $\dot{\gamma}(t)$ to the curve at $\gamma(t)$ is

$$\dot{\gamma}(t) := d\gamma_t(\frac{d}{dt}).$$

As you have proved in the homework if $\gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t))$ is a curve in $\mathbb{R}^n$, then $\dot{\gamma}(t)$ is the vector $(\gamma_1'(t), \ldots, \gamma_n'(t))$ where $'$ denotes the ordinary derivative. Next, an important definition:

Definition 5.1. A curve $\gamma : I \to M$ is an *integral curve* of a vector field X on a manifold M through the point q if

$$\begin{cases} \dot{\gamma}(t) = X_{\gamma(t)} & \text{for all } t \in I \\ \gamma(0) = q. \end{cases}$$

In other words the tangent vector to the curve γ at t is the value of the vector field X at $\gamma(t)$.

We are now in position to summarize the goals of this subsection. We will see that vector fields are the geometric version of ordinary differential equations (ODEs) and integral curves are the geometric version of the solutions of ODEs. Using this connection with ODEs we will show that integral curves of vector fields exist and that on Hausdorff manifolds integrals curves are unique. We will then assume that all our manifolds are Hausdorff. With this assumption we will show all integral curves of a given vector field can be put together to form a flow. Moreover, there is a bijective correspondence between vector fields and flows. We will then use flows to give the Lie bracket a geometric meaning.

We first interpret the problem of existence of integral curves in coordinates. Let $\phi = (x_1, \dots, x_m) : U \to \mathbb{R}^m$ be a coordinate chart on a manifold M. Suppose $\gamma : I \to U$ is an integral curve of a vector field X. Since X is a smooth vector field, there are smooth functions $f_i : U \to \mathbb{R}$, $1 \le i \le m$ so that

$$X_a = \sum f_i(a) \frac{\partial}{\partial x_i}|_a$$

for all $a \in U$ (of course $f_i = dx_i(X)$). Similarly,

$$\begin{aligned}\dot{\gamma}(t) &= d\gamma_t(\frac{d}{dt}) \\ &= \sum dx_i(d\gamma_t(\frac{d}{dt}))\frac{\partial}{\partial x_i}|_{\gamma(t)} \\ &= \sum \frac{d}{dt}|_t(x_i \circ \gamma)\ \frac{\partial}{\partial x_i}|_{\gamma(t)} \\ &= \sum (x_i \circ \gamma)'(t)\frac{\partial}{\partial x_i}|_{\gamma(t)}\end{aligned}$$

Therefore, the equation $\dot{\gamma}(t) = X_{\gamma(t)}$ is equivalent to

$$\sum (x_i \circ \gamma)'(t)\frac{\partial}{\partial x_i}|_{\gamma(t)} = \sum f_i(\gamma(t))\frac{\partial}{\partial x_i}|_{\gamma(t)}$$

for all $t \in I$. Thus γ is an integral curve of X in U if and only if

$$(x_i \circ \gamma)'(t) = f_i(\gamma(t)), \quad t \in I, \quad 1 \le i \le m.$$

This is a system of ordinary differential equations. Conversely, any solution to the above system defines an integral curve of the vector field X inside the open set U. We now quote without proof the appropriated theorem from the theory of ODEs.

Theorem 5.2. *Let $V \subset \mathbb{R}^m$ be an open set and $F = (F_1, \dots, F_m) : V \to \mathbb{R}^m$ a smooth map. For any point $q_0 \in V$ there is an open neighborhood V_0 of q_0, $\epsilon > 0$ and a smooth map*

$$\Phi : (-\epsilon, \epsilon) \times V_0 \to V$$

so that for each $q \in V_0$ the curve $\gamma_q(t) := \Phi(t, q)$ is the unique solution of the ODE

$$\gamma_i'(t) = F_i(\gamma(t)), \quad t \in (-\epsilon, \epsilon), \quad 1 \le i \le m$$

subject to the initial condition

$$\gamma_q(0) = q.$$

The proof uses a contraction mapping principle and is similar to the proof of the inverse function theorem. We will have no time for it.

Corollary 5.2.1. *Suppose X is a vector field on a manifold M. For every point $q_0 \in M$ there is a neighborhood U of q_0, $\epsilon > 0$ and a smooth map*

$$\Phi : (-\epsilon, \epsilon) \times U \to M$$

so that for any $q \in U$,

$$\gamma_q(t) := \Phi(t, q)$$

is the unique integral curve of X through q. In particular, if $\sigma : I \to U$ is another integral curve of X with $\sigma(0) = q$ then $\sigma(t) = \gamma_q(t)$ for all $t \in I \cap (-\epsilon, \epsilon)$.

It is important to note that uniqueness of integral curves does depend on the fact that we keep track of the initial conditions.

Lemma 5.3. *If $\gamma : I \to M$ is an integral curve of a vector field X passing through p then for any $s \in \mathbb{R}$, the curve*

$$\sigma(t) = \gamma(t+s)$$

is also an integral curve of X. However, at time 0 it passes through $q = \gamma(s)$[4] *(The curve σ is defined on $I' = \{t \in \mathbb{R} \mid t+s \in I\}$.)*

Proof. This is an easy application of the chain rule. Here are the gory details. Define the translation $\tau_s : I' \to I$ by $\tau_s(t) = t+s$. Then $\sigma = \gamma \circ \tau_s$. Note that $d(\tau_s)_t : \mathbb{R} \to \mathbb{R} = id$. Hence

$$\begin{aligned}\dot{\sigma}(t) &= d\sigma_t(\frac{d}{dt}) = d(\gamma \circ \tau_s)_t(\frac{d}{dt}) = (d\gamma_{s+t} \circ d(\tau_s)_t)\,(\frac{d}{dt}) \\ &= d\gamma_{s+t}(\frac{d}{dt}) = \dot{\gamma}(t+s) = X_{\gamma(t+s)} = X_{\sigma(t)}.\end{aligned}$$

□

The open set U in Corollary 5.2.1 above lies inside some coordinate chart on M. Therefore the general uniqueness of integral curves of X doesn't quite follow from the corollary. Here is an example where the uniqueness fails.

Example 5.4. Consider first the real line $\mathbb{R}$ with the constant vector field $\frac{d}{dt}$. The corresponding differential equation is

$$\gamma'(t) = 1.$$

The solutions are curves of the form $\gamma(t) = p + t$.

Now consider the non-Hausdorff manifold M obtained by gluing two copies of $\mathbb{R}$ along $\mathbb{R} \smallsetminus \{0\}$. More precisely, let $\tilde{M} = \mathbb{R} \times \{0,1\}$. Define an equivalence relation $\sim$ by $(x,0) \sim (x,1)$ for all $x \neq 0$. Let $M = \tilde{M}/\sim$. We write $[x,0]$ and $[x,1]$ for the equivalence classes of $(x,0)$ and $(x,1)$ respectively. Note that by design $[0,0] \neq [0,1]$. These are the "two origins" of the "line" M. For $x \neq 0$ we have $[x,0] = [x,1]$.

Note that M comes with two natural coordinate charts: $\phi([x,0]) = x$ and $\psi([x,1]) = x$ for all $x \in \mathbb{R}$. The change of coordinates $\phi \circ \psi^{-1}$ is defined on all of $\mathbb{R} \smallsetminus \{0\}$ and is the identity map. It follows that the constant vector field $\frac{d}{dt}$ defines a vector field X on M. Moreover, $\gamma(t) = \phi^{-1}(t+1)$ and $\sigma(t) = \psi^{-1}(t+1)$ are integral curves of X with $\gamma(0) = [1,0] = [1,1] = \sigma(0)$. Additionally $\gamma(t) = \sigma(t)$ **except** for $t = -1$. □

Why do problems like these not occur on Hausdorff manifolds? The key point is: a manifold M is Hausdorff if and only if the diagonal

$$\Delta_M := \{(m,m) \in M \times M \mid m \in M\}$$

is **closed** in $M \times M$ [prove it]. Consequently, if $\gamma : I \to M$ and $\sigma : J \to M$ are two curves, then the set

$$K := \{t \in I \cap J \mid \gamma(t) = \sigma(t)\},$$

where the two curves agree, is closed in $I \cap J$. Indeed, K is the preimage of Δ_M under the map $I \cap J \ni t \mapsto (\gamma(t), \sigma(t)) \in M \times M$.

Now suppose additionally that γ and σ are two integral curves of a vector field $X \in \Gamma(TM)$. Then, by Corollary 5.2.1 and Lemma 5.3, the set of points K is also **open** in $I \cap J$. Since $I \cap J$ is an interval and K is open and closed, it follows that the set K has to be all of $I \cap J$. This gives us uniqueness: two integral curves of a given vector field passing through a given point at $t = 0$ agree for all t in the intersection of their domains of definition.

Furthermore it makes sense to take the union of the integral curves γ and σ:

$$(\gamma \cup \sigma)(t) := \begin{cases} \gamma(t) & t \in I \\ \sigma(t) & t \in J \end{cases}$$

Taking the union of all integral curves of a vector field X passing through a given point p we get a *maximal* integral curve $\gamma_p : I_p \to M$ of X passing through p. It is maximal in the following sense: if $\gamma : I \to M$ is

[4] $\gamma(s)$ is not $\gamma(0)$ unless $\gamma(t) = \gamma(0)$ for all t, in which case $\gamma(t) = \sigma(t)$.

any other integral curve of X passing through p then $I \subset I_p$ and $\gamma(t) = \gamma_p(t)$ for all $t \in I$. We have proved the following lemma.

Lemma 5.5. *Let M be a Hausdorff manifold and $X \in \Gamma(TM)$ a vector field. For any two integral curves $\gamma : I \to M$ and $\sigma : J \to M$ of X*

$$\{t \in I \cap J \mid \gamma(t) = \sigma(t)\} = I \cap J.$$

Consequently, for any point $p \in M$ there is a unique maximal integral curve γ_p of X passing through p.

From now on, unless noted otherwise, all manifolds are assumed to be Hausdorff.

Example 5.6. An integral curve of a vector field need not be defined for all time. Here is a simple example. Let $M = (-\infty, 0)$ and $X = \frac{d}{dt}$. Then the maximal integral curve γ_p of X passing through $p \in (-\infty, 0)$ is given by $\gamma_p(t) = p + t$, hence is defined only when $p + t < 0$, i.e., $t < -p$.

Corollary 5.2.1 has another important consequence: the maximal integral curve γ_p of the vector field X depends smoothly on the point p. We can therefore put the maximal integral curves together and obtain a map

$$\Phi(t, p) = \gamma_p(t) \quad \text{for all } t \in I_p \text{ and all } p \in M. \tag{5.1}$$

We have to be a bit careful about the set where the map Φ is defined. It is defined on a *subset* A of $\mathbb{R} \times M$ containing $\{0\} \times M$. Moreover, by Corollary 5.2.1, the subset A is open.

Definition 5.7. We use the notation above: γ_p denotes the maximal integral curve through the point p of a vector field X on a manifold M. The map

$$\Phi : \mathbb{R} \times M \supset A \to M$$

defined by (5.1) is called the *(local) flow of the vector field X.*

The word "local" refers to the fact that the flow Φ need not be defined for all time t but only for t in some neighborhood of 0, the neighborhood that depends on the point p. If the set A in the definition above is all of $\mathbb{R} \times M$, we say that X has a *global* flow.

Lemma 5.8. *Let X be a vector field on a Hausdorff manifold M and let $\Phi : \mathbb{R} \times M \supset A \to M$ denote its local flow. Then*

$$\Phi(t, \Phi(s, p)) = \Phi(s + t, p)$$

for all $p \in M$ and all $s, t \in \mathbb{R}$ for which both sides of the equation make sense.

Proof. Fix $p \in M$ and $s \in \mathbb{R}$. Let $\gamma(t) = \Phi(t, \Phi(s, p))$ and let $\sigma(t) = \Phi(s + t, p)$. Then γ is the maximal integral curve of X passing through $\Phi(s, p)$. By Lemma 5.3 $\sigma(t)$ is the maximal integral curve of X passing through $\sigma(0) = \Phi(s + 0, p)$. Therefore, since maximal integral curves are unique on Hausdorff manifolds, $\gamma(t) = \sigma(t)$ for all t. ☐

This motivates the following definition.

Definition 5.9 (abstract local flow)**.** A *local flow* on a manifold M is a map $\Psi : A \to M$, where A is open subset of $\mathbb{R} \times M$ containing $\{0\} \times M$, having the following two properties

(1) $\Psi(0, p) = p$ for all $p \in M$
(2) $\Psi(t, \Psi(s, p)) = \Psi(s + t, p)$ whenever both sides make sense.

Example 5.10. Let $M = \mathbb{R}^n$. The map $\Psi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $\Psi(t, p) = e^t p$ is a flow.

Example 5.11. Let $M = \mathbb{R}^2$. The map

$$\Psi(t, (x, y)) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

is a flow. The example can be described more succinctly in complex coordinates: let $M = \mathbb{C}$ and write $\Psi(t, z) = e^{it} z$.

It may be a bit hard to see what the meaning of the two conditions of Definition 5.9 really is. It's easier to understand what's going on in the case where Ψ is a *global flow*, that is, when the domain of the definition A of Ψ is all of $\mathbb{R} \times M$.

Given a global flow $\Psi : \mathbb{R} \times M \to M$, we have, for each $t \in \mathbb{R}$ a map

$$\Psi_t : M \to M, \quad \Psi_t(q) := \Psi(t, q).$$

Condition (1) in the definition of local flow then simply says that Ψ_0 is the identity map id_M on M. Condition (2) becomes

$$\Psi_t(\Psi_s(q)) = \Psi_{t+s}(q) \tag{5.2}$$

for all $t, s \in \mathbb{R}$ and $q \in M$. Hence $\Psi_t \circ \Psi_{-t} = id_M = \Psi_{-t} \circ \Psi_t$. Consequently $\Psi_t : M \to M$ is a *diffeomorphism* for each $t \in \mathbb{R}$. Moreover, we can interpret (5.2) as saying that we have a *homomorphism of groups*

$$\mathbb{R} \ni t \mapsto \Psi_t \in \mathrm{Diff}(M),$$

where $\mathrm{Diff}(M)$ denotes the group of diffeomorphisms of M (it's a group under composition). This is why global flows are also referred to as 1-parameter groups of diffeomorphisms.

Now let's return to vector fields. The point of much of the preceding discussion is that for a vector field X on (Hausdorff) manifold M the collection of integral curves taken together forms a local flow. The converse is true as well.

Lemma 5.12. *Let $\Psi : \mathbb{R} \times M \supset A \to M$ be a local flow. Then the map $X : C^\infty(M) \to C^\infty(M)$ defined by*

$$Xf(p) = \frac{d}{dt}\Big|_0 f(\Psi(t, p))$$

for all $f \in C^\infty(M)$ is a vector field. Moreover, Ψ is the local flow of X.

Proof. Since $\Psi(t, p)$ is a smooth function of t and p, $f(\Psi(t, p))$ is also a smooth function of t and p and its derivative $\frac{d}{dt}|_0 f(\Psi(t, p))$ is a smooth function of p. For any $f, g \in C^\infty(M)$, we have

$$(fg)(\Psi(t, p)) = f(\Psi(t, p))\, g(\Psi(t, p)).$$

Hence $X(fg) = X(f)g + fX(g)$, i.e., X is a vector field.

It remains to check that Ψ is the local flow of X. We need to show that for each $p \in M$,

$$\gamma_p'(t) = X_{\gamma_p(t)}$$

where $\gamma_p(t) := \Psi(t, p)$. Let $f \in C^\infty(M)$ be a function. Then

$$\begin{aligned}
(\gamma_p'(t))\, f &= \frac{d}{ds}|_{s=t} f(\gamma_p(s)) \\
&= \frac{d}{ds}|_{s=t} f(\Psi(s, p)) \\
&= \frac{d}{ds}|_{s=0} f(\Psi(s + t, p)) \\
&= \frac{d}{ds}|_{s=0} f(\Psi(s, \Psi(t, p))) \\
&= (Xf)(\Psi(t, p)) = X_{\Psi(t,p)} f = X_{\gamma_p(t)} f.
\end{aligned}$$

□

Definition 5.13. A vector field is *complete* if its local flow is a global flow. That is, each integral curve is defined for all $t \in \mathbb{R}$.

Here are two examples of vector fields that are *not* complete.

Example 5.14. The vector field $\frac{d}{dt}$ on $(-\infty, 0)$ is not complete.

Example 5.15. The vector field $x^2 \frac{d}{dx}$ on $\mathbb{R}$ is not complete: $\Phi(t, x) = \frac{x}{1 - xt}$ is its local flow [check it]. The flow is defined for $t \in (-\infty, 1/x)$ if $x > 0$ and for $t \in (1/x, +\infty)$ if $x < 0$.

It is nice to know when a vector field has a global flow. For this purpose we define:

Definition 5.16. The *support* of a vector field X on a manifold M is

$$\operatorname{supp}(X) = \overline{\{p \in M \mid X_p \neq 0\}},$$

the closure of the set of points where X is non-zero.

Theorem 5.17. *A vector field with compact support is complete. In particular any vector field on a compact manifold defines a global flow.*

Recall that we are tacitly assuming throughout that all our manifolds are Hausdorff. Also, recall that any closed subset of a compact space is compact. Hence any vector field on a compact manifold has compact support. There are more than one way to prove the theorem above. For our proof we will need the following lemma.

Lemma 5.18. *Let $X \in \Gamma(TM)$ be a vector field with the flow $\Phi : \mathbb{R} \times M \supset A \to M$. Suppose that $\{\tau\} \times M \subset A$, that is, the flow of X is defined for time τ at all points of M.Then*

$$d(\Phi_\tau)_m(X_m) = X_{\Phi_\tau(m)}$$

for all $m \in M$. Here, as before, $\Phi_\tau(m) := \Phi(\tau, m)$.

Proof. Let $\gamma_m(t)$ be the maximal integral curve of X through m: $\gamma_m(t) = \Phi(t, m)$. Then $X_m = \gamma_m'(0)$. Also,

$$\begin{aligned}\Phi_\tau(\gamma_m(t)) &= \Phi(\tau, \Phi(t, m)) = \Phi(\tau + t, m) \\ &= \Phi(t, \Phi(\tau, m)) = \gamma_{\Phi_\tau(m)}(t).\end{aligned}$$

Hence

$$\begin{aligned}d(\Phi_\tau)_m(X_m) &= d(\Phi_\tau)_m(\gamma_m'(0)) = d(\Phi_\tau)_m \circ d(\gamma_m)_0\, (\frac{d}{dt}) \\ &= d(\Phi_\tau \circ \gamma_m)_0\, (\frac{d}{dt}) = (\Phi_\tau(\gamma_m))'(0) = (\gamma_{\Phi_\tau(m)})'(0) = X_{\gamma_{\Phi_\tau(m)}(0)} = X_{\Phi_\tau(m)}.\end{aligned}$$

□

Proof of Theorem 5.17. We want to show that the domain of definition A of the local flow $\Phi(t, p)$ of X is all of $\mathbb{R} \times M$. If $X_m = 0$ then the constant curve $\gamma_m(t) = m$ is the integral curve of X through m. It is defined for all t. Therefore on $M \smallsetminus \operatorname{supp} X$ the flow is defined for all t: $\mathbb{R} \times (M \smallsetminus \operatorname{supp} X) \subset A$. Also $\{0\} \times M \subset A$ by definition of the flow. In particular $\{0\} \times \operatorname{supp} X \subset A$. Since $\operatorname{supp} X$ is compact and A is open, there is $\epsilon > 0$ so that $[-\epsilon, \epsilon] \times \operatorname{supp} X \subset A$. Hence $[-\epsilon, \epsilon] \times M \subset A$. We now define $\tilde{\Phi} : [0, 2\epsilon] \times M \to M$ by

$$\tilde{\Phi}(t, p) = \Phi(\epsilon, \Phi(t - \epsilon, p)) = \Phi_\epsilon(\Phi(t - \epsilon, p)).$$

Here, as before, $\Phi_\epsilon(q) = \Phi(\epsilon, q)$. We claim that for any $p \in M$ the curve

$$\tilde{\gamma}_p(t) = \begin{cases} \Phi(t, p) & t \in [-\epsilon, \epsilon] \\ \tilde{\Phi}(t, p) & t \in [0, 2\epsilon] \end{cases}$$

is an integral curve of X. Indeed, for $t \in [0, 2\epsilon]$, by definition of $\tilde{\gamma}_p$ and $\tilde{\Phi}$,

$$\begin{aligned}\tilde{\gamma}_p'(t) &= d\Phi_\epsilon(\tilde{\gamma}_p'(t - \epsilon)) \\ &= d\Phi_\epsilon(X_{\Phi(p, t-\epsilon)}) \\ &= X_{\Phi_\epsilon(\Phi(p, t-\epsilon))} = X_{\tilde{\Phi}(t,p)} = X_{\tilde{\gamma}_p(t)},\end{aligned}$$

where the third equality holds by Lemma 5.18. It follows that the maximal integral curve γ_p of X through p is defined for $t \in [-\epsilon, 2\epsilon]$. Hence $[-\epsilon, 2\epsilon] \times M \subset A$. Arguing inductively we get $[-k\epsilon, n\epsilon] \times M \subset A$ for all positive integers k and n. Therefore $A = \mathbb{R} \times M$ and X is complete. □

5.2. **The geometry of the Lie bracket.** As before, we continue to assume that all manifolds are Hausdorff. Additionally, in this subsection we will pretend that all flows are global, equivalently, that all vector fields are complete. Assuming completeness is not necessary. On the other hand carrying out the argument in full generality obscures the main simple ideas.

Definition 5.19. Let X and Y be two vector fields with Φ denoting the flow of X. The *Lie derivative* $L_X Y$ of Y with respect to X is a vector field defined by

$$(L_X Y)_p := \lim_{t\to 0} \frac{1}{t}(d(\Phi_{-t})_p(Y_{\Phi_t(p)} - Y_p) = \frac{d}{dt}\Big|_{t=0} d(\Phi_{-t})_p(Y_{\Phi_t(p)})$$

for all $p \in M$.

Several remarks are in order. It is not entirely clear that the Lie derivative, as defined above, is a smooth vector field. We will prove this shortly. Second, $t \to d(\Phi_{-t})_p(Y_{\Phi_t(p)})$ is a curve in a finite dimensional vector space T_pM. We have seen that for $\mathbb{R}^n$, we can always canonically identify $T_p\mathbb{R}^n$ with $\mathbb{R}^n$. A moment of reflection should convince you that the same identification works for any finite dimensional vector space. Hence it does make sense to think of the Lie derivative $(L_X Y)_p$ as a vector in the tangent space T_pM. Finally note that if $\gamma : I \to T_pM$ is any smooth curve, then

$$\left(\frac{d}{dt}\Big|_{t=0}\gamma(t)\right) f = \frac{d}{dt}\Big|_{t=0}(\gamma(t)f) \quad \text{for any } f \in C^\infty(M). \tag{5.3}$$

We will need the equation in the proof of Theorem 5.20 below. There are many ways to prove (5.3). For example, pick a basis of T_pM and compute both sides of (5.3) in coordinates defined by the basis. What makes the proof work is the fact that partials commute. With these preliminaries out of the way we are ready to state the main result of the subsection.

Theorem 5.20. *Lie derivative is a Lie bracket. That is, for any two vector fields X and Y on a manifold M*

$$(L_X Y)_p = ([X, Y])_p$$

for all points $p \in M$.

Proof. Denote the flow of Y by Ψ. We evaluate $(L_X Y)_p$ on an arbitrary smooth function $f \in C^\infty(M)$:

$$\begin{aligned}
(L_X Y)_p f &= \left(\frac{d}{dt}\Big|_{t=0} d(\Phi_{-t})_p(Y_{\Phi_t(p)})\right) f \\
&= \frac{d}{dt}\Big|_{t=0}\left(d(\Phi_{-t})_p(Y_{\Phi_t(p)})f\right) \\
&= \frac{d}{dt}\Big|_{t=0} Y_{\Phi_t(p)}(f\circ\Phi_{-t}) \\
&= \frac{d}{dt}\Big|_{t=0}\left(\frac{\partial}{\partial s}\Big|_{s=0}(f\circ\Phi_{-t})(\Psi_s(\Phi_t(p)))\right) \\
&= \frac{\partial^2}{\partial t\partial s}\Big|_{(0,0)}(f\circ\Phi_{-t}\circ\Psi_s\circ\Phi_t)(p) \\
&= \frac{d}{ds}\Big|_{s=0}\left(\frac{\partial}{\partial t}\Big|_{t=0}(f\circ\Phi_{-t}\circ\Psi_s\circ\Phi_t)(p)\right) \\
&= \frac{d}{ds}\Big|_{s=0}\left(\frac{\partial}{\partial t}\Big|_{t=0}(f\circ\Phi_{-t})(\Psi_s(\Phi_0(p))) + \frac{\partial}{\partial t}\Big|_{t=0}(f\circ\Phi_{-0}\circ\Psi_s)(\Phi_t(p))\right) \\
&= \frac{d}{ds}\Big|_{s=0}\left(-X_{\Psi_s(p)}f + \frac{\partial}{\partial t}\Big|_{t=0}(f\circ\Psi_s\circ\Phi_t)(p)\right) \\
&= -\frac{d}{ds}\Big|_{s=0}(Xf)(\Psi_s(p)) + \frac{d}{dt}\Big|_{t=0}\frac{\partial}{\partial s}\Big|_{s=0}(f\circ\Psi_s)(\Phi_t(p)) \\
&= -Y(Xf)(p) + \frac{d}{dt}\Big|_{t=0}(Yf)(\Phi_t(p)) \\
&= -Y(Xf)(p) + X(Yf)(p) = ([X, Y]_p)f.
\end{aligned}$$

□

In particular this proves that the Lie derivative $L_X Y$ *is* a vector field. As a corollary to the above proof, we get:

Corollary 5.20.1. *Let Φ and Ψ denote the flows of vector fields X and Y respectively. Then for any smooth function f,*

$$([X,Y]f)(p) = \frac{\partial^2}{\partial t \partial s}\Big|_{(0,0)} (f \circ \Phi_{-t} \circ \Psi_s \circ \Phi_t)(p). \tag{5.4}$$

Note that if the flows Φ_t and Ψ_s commute, that is,

$$\Phi_t \circ \Psi_s = \Psi_s \circ \Phi_t \text{ for all } t \text{ and } s,$$

then $\Phi_{-t} \circ \Psi_s \circ \Phi_t = \Psi_s \circ \Phi_{-t} \circ \Phi_t = \Psi_s$. In particular, it's independent of t. Hence the right hand side of (5.4) is 0. Therefore $[X,Y] = 0$. The converse is true as well.

Lemma 5.21. *Let Φ and Ψ denote the flows on a manifold M of vector fields X and Y respectively. Then*

$$[X,Y] = 0 \textit{ if and only if } \Phi_t \circ \Psi_s = \Psi_s \circ \Phi_t.$$

Proof. We have just proved that if the flows commute the Lie bracket has to vanish. Now suppose $[X,Y] = 0$. Our proof will use the following observation. Let V and W be two finite dimensional vector spaces, $T : V \to W$ a linear map and $\gamma : I \to V$ a smooth curve. Then, since $dT = T$,

$$(T \circ \gamma)'(t) = T(\gamma'(t)).$$

Here, again we identify $\gamma'(t)$ with a vector in V and similarly for $(T \circ \gamma)'(t)$. With the preliminaries out of the way, we proceed with the actual proof. Since $[X,Y] = 0$,

$$0 = \frac{d}{dh}\Big|_{h=0} (d\Phi_{-h})(Y_{\Phi_h(p)})$$

for all points p. Hence, with $\bullet$ denoting the appropriate point,

$$\begin{aligned} \frac{d}{dt}\Big|_{t=s} d(\Phi_{-t})_\bullet(Y_{\Phi_t(p)}) &= \frac{d}{dh}\Big|_{h=0} (d\Phi_{-(s+h)})_\bullet(Y_{\Phi_{s+h}(p)}) \\ &= \frac{d}{dh}\Big|_{h=0} d(\Phi_{-s})_\bullet[(d\Phi_{-h})_\bullet(Y_{\Phi_h(\Phi_s(p))})] \\ &= d(\Phi_{-s})_\bullet[\frac{d}{dh}\Big|_{h=0} d(\Phi_{-h})_\bullet(Y_{\Phi_h(\Phi_s(p))})] = 0. \end{aligned}$$

Here, in the last equality we used the fact that $d(\Phi_{-s})_\bullet$ is a linear map between tangent spaces and $h \mapsto d(\Phi_{-h})_\bullet(Y_{\Phi_h(\Phi_s(p))})$ is a curve in the tangent space $T_{\Phi_s(p)}M$. Hence the curve $t \mapsto d(\Phi_{-t})_\bullet(Y_{\Phi_t(p)})$ is a constant curve. In particular,

$$d(\Phi_{-t})_\bullet(Y_{\Phi_t(p)}) = d(\Phi_{-0})_\bullet(Y_{\Phi_0(p)}) = Y_p$$

for all t. Consequently

$$Y_{\Phi_t(p)} = d(\Phi_t)_p Y_p \text{ for all } t. \tag{5.5}$$

We use the equation above to argue that $\sigma(s) = \Phi_t(\Psi_s(p))$ is an integral curve of Y passing through $\Phi_t(p)$:

$$\begin{aligned} \sigma'(s) &= \frac{d}{d\tau}\Big|_{\tau=s} [\Phi_t(\Psi_\tau(p))] \\ &= (d\Phi_t)(\frac{d}{d\tau}\Big|_{\tau=s} \Psi_\tau(p) \\ &= (d\Phi_t)(Y_{\Psi_\tau(p)}) = Y_{(\Phi_t \circ \Psi_s)(p)} \text{ by (5.5)} \\ &= Y(\sigma(s)). \end{aligned}$$

On the other hand $s \mapsto \Psi_s(\Phi_t(p))$ is also an integral curve of Y passing through $\Phi_t(p)$. Therefore the two curves are equal:

$$\Phi_t(\Psi_s(p)) = \Psi_s(\Phi_t(p)).$$

□

We end the section with a somewhat technical subsection. The point of this subsection will not be apparent for some time.

5.3. **Map-related vector fields.** Recall that given a smooth map between manifolds $f : M \to N$, for each point $p \in M$ we get a map of tangent spaces $df_p : T_pM \to T_{f(p)}N$. Therefore, given a vector field $X : M \to TM$ we get for each $p \in M$ a vector $df_p(X_p) \in T_{f(p)}N$. It is not a vector field on N. If additionally f is diffeomorphism we can make it into a vector field: define

$$X'(q) = df_{f^{-1}(q)} X_{f^{-1}(q)}$$

The new vector field X' is related to the old vector field X by

$$df \circ X = X' \circ f$$

where we think of X, X' and df as maps $X : M \to TM$, $X' : N \to TN$ and $df : TM \to TN$ respectively. In other words, the diagram

$$\begin{array}{ccc} TM & \xrightarrow{df} & TN \\ {\scriptstyle X}\uparrow & & \uparrow{\scriptstyle X'} \\ M & \xrightarrow{f} & N \end{array}$$

commutes.

Definition 5.22. Let $X : M \to TM$, $Y : N \to TN$ be two vector fields and $f : M \to N$ a smooth map. The two vector fields X and Y are *f-related* if

$$df \circ X = Y \circ f.$$

Example 5.23. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the projection onto the first factor: $f(x, y) = x$. Then any vector field of the form $X = \frac{\partial}{\partial x} + g(x, y)\frac{\partial}{\partial y}$, where $g : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function is f-related to $Y = \frac{d}{dx}$.

The main fact worth remembering about related vector fields is that Lie brackets go to Lie brackets. More precisely,

Lemma 5.24. *Let $X_1, X_2 : M \to TM$ and $Y_1, Y_2 : N \to TN$ be two pairs of vector fields related by a map $f : M \to N$:*

$$df \circ X_i = Y_i \circ f \quad i = 1, 2.$$

Then

$$df \circ [X_1, X_2] = [Y_1, Y_2] \circ f.$$

Proof. Note that two vector fields X and Y are f-related ($f : M \to N$) if and only if for any smooth function $h \in C^\infty(N)$,

$$Y(h)_{f(p)} = X_p(h \circ f)$$

for all $p \in M$. Or, more concisely,

$$(Yh) \circ f = X(h \circ f)$$

We now compute:

$$\begin{aligned} [X_1, X_2](h \circ f) &= X_1(X_2(h \circ f)) - X_2(X_1(h \circ f)) \\ &= X_1((Y_2h) \circ f)) - X_2((Y_1h) \circ f) \\ &= (Y_1(Y_2h)) \circ f - (Y_2(Y_1h)) \circ f = ([Y_1, Y_2]h) \circ f. \end{aligned}$$

□

Exercise 5.1. Find the flows of the following vector fields on $\mathbb{R}^2$:

$$X = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}$$

and

$$Y = x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}.$$

Exercise 5.2. Prove that if a vector field X on a manifold M vanishes at a point p, $X(p) = 0$, then there is an open set W containing p such that the flow of X on W exists for all $t \in [0, 1]$.

Exercise 5.3. Let M be a manifold. An **isotopy** on M is a collection of diffeomorphisms $\{f_t : M \to M\}_{t\in(-\epsilon,\epsilon)}$ such that

(1) f_0 is the identity, and
(2) the map $(-\epsilon, \epsilon) \times M \to M$ given by $(t, m) \mapsto f_t(m)$ is smooth.

A **time-dependent vector field** $\{X_t\}$ is a smooth map $(-\epsilon, \epsilon) \times M \to TM$ of the form $(t, m) \mapsto (X_t)_m =: X_t(m)$. An isotopy $\{f_t\}$ defines a time-dependent vector field $\{X_t\}$ by

$$X_s(f_s(m)) = \frac{d}{dt}\Big|_{t=s} f_t(m).$$

Prove that given a time-dependent vector field $\{X_t\}$, there is an isotopy $\{f_t\}$ such that the equation above holds.
Hint: Let $\overline{X}(t, m) = (\frac{d}{dt}, X_t(m))$; it is a vector field on $\mathbb{R} \times M$. The local flow $\Phi_s(t, m)$ of $\overline{X}$ is of the form $\Phi_s(t, m) = (\Phi^1_s(t, m), \Phi^2_s(t, m))$. Show that $\Phi^1_s(t, m) = s + t$.

Exercise 5.4. Consider a time-dependent vector field $X_t(m) = t\frac{d}{d\theta}$ on S^1. Compute the corresponding isotopy.

Exercise 5.5. Suppose that M and N are manifolds. If $X \in \Gamma(TM)$ is a vector field, show that $\bar{X} : M \times N \to T(M \times N) \simeq TM \times TN$ given by $\bar{X}(m, n) = (X_m, 0)$ is a well-defined vector field on $M \times N$. Similarly, given $Y \in \Gamma(TN)$ we get $\bar{Y} \in \Gamma(T(M \times N))$. Show that $[\bar{X}, \bar{Y}] = 0$.

Exercise 5.6. Suppose that X and Y are vector fields on M. Compute an expression for $[X, Y]$ in local coordinates.

6. (Multi)linear algebra

The goal of this section is to define tensors, tensor algebra and Grassmann (exterior) algebra. We will use these constructions to define tensors and differential forms on manifolds. In this section, unless noted otherwise, all vector spaces are over the real number and are finite dimensional. There are two ways to think about tensors:

(1) tensors are multi-linear maps;
(2) tensors are elements of a "tensor product" of two or more vector spaces.

The first way is more concrete. The second is more abstract but also more powerful.

6.1. Tensor products.
We start by reviewing multi-linear maps.

Definition 6.1. Let $V_1, \ldots, V_n$ and U be vector spaces. A map

$$f : \overbrace{V_1 \times \cdots \times V_n}^{n \text{ factors}} \to U, \quad (v_1, \ldots, v_n) \mapsto f(v_1, \ldots, v_n)$$

is *multi-linear* if for each fixed index i and a fixed $(n-1)$-tuple of vectors $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n$ the map

$$V_i \to U, \quad w \mapsto f(v_1, \ldots, v_{i-1}, w, v_{i+1}, \ldots, v_n)$$

is linear. When the number of factors is n, as above, we will also say that f is *n-linear.*

For example, if we identify $\mathbb{R}^{n^2} \simeq \overbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}^{n \text{ factors}}$ by thinking of an $n \times n$ matrix as an n-tuple of column vectors, then the determinant

$$\det : \overbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}^{n \text{ factors}} \to \mathbb{R}, \quad (v_1, \ldots, v_n) \mapsto \det(v_1|\ldots|v_n)$$

is an n-linear map. Here is an example of a bilinear map. Any inner product on a vector space V:

$$V \times V \ni (v, w) \mapsto v \cdot w \in \mathbb{R}$$

is bilinear. There is no standard notation for the space of n-linear maps from $V_1 \times \cdots \times V_n$ to U. We will denote it by

$$\mathrm{Mult}(V_1 \times \cdots \times V_n, U) = \mathrm{Mult}_n(V_1 \times \cdots \times V_n, U)$$

(n is to indicate that these are n-linear maps). This space, $\mathrm{Mult}(V_1 \times \cdots \times V_n, U)$, is a vector space: any linear combination of two n-linear maps is n-linear. We now take a closer look at the space of bilinear maps $\mathrm{Mult}_2(V \times W, U)$. This case is complicated enough to understand what happens with multi-linear maps in general, but simple enough not to bog down in notation.

Lemma 6.2. *Let $\{v_i\}$, $\{w_j\}$ and $\{u_k\}$ denote the bases of V, W and U respectively and $\{v_i^*\}$, $\{w_j^*\}$ and $\{u_k^*\}$ the corresponding dual bases. Then the maps*

$$\phi_{ij}^k : V \times W \to U, \quad \phi_{ij}^k(v, w) = v_i^*(v) w_j^*(w) u_k$$

are bilinear and form a basis of $\mathrm{Mult}_2(V \times W, U)$*. Hence*

$$\dim \mathrm{Mult}_2(V \times W, U) = \dim V \dim W \dim U.$$

Proof. It is easy to see that ϕ_{ij}^k are bilinear. Next, for any $b \in \mathrm{Mult}_2(V \times W, U)$, any $w \in W$ and any $v \in V$,

$$\begin{aligned} b(v, w) &= b(\sum v_i^*(v) v_i, \sum w_j^*(w) w_j) \\ &= \sum_{i,j} v_i^*(v) w_j^*(w) b(v_i, w_j) \\ &= \sum_{i,j,k} v_i^*(v) w_j^*(w) u_k^*(b(v_i, w_j)) u_k \\ &= \sum_{i,j,k} u_k^*(b(v_i, w_j)) \phi_{ij}^k(v, w). \end{aligned}$$

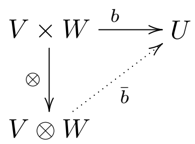
Hence the maps ϕ_{ij}^k span $\mathrm{Mult}_2(V \times W, U)$. Also, the collection of numbers $u_k^*(b(v_i, w_j))$ uniquely determine the bilinear form b. Hence ϕ_{ij}^k's are linearly independent. □

We now turn to the definition of the tensor product $V \otimes W$ [pronounced "V tensor W"] of two vector spaces V and W. Informally it consists of finite linear combinations of symbols $v \otimes w$, where $v \in V$ and $w \in W$. Additionally, these symbols are subject to the following identities:

$$\begin{aligned} &(v_1 + v_2) \otimes w - v_1 \otimes w - v_2 \otimes w = 0 \\ &v \otimes (w_1 + w_2) - v \otimes w_1 - v \otimes w_2 = 0 \\ &\alpha\,(v \otimes w) - (\alpha v) \otimes w = 0 \\ &\alpha\,(v \otimes w) - v \otimes (\alpha w) = 0, \end{aligned}$$

for all $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$ and $\alpha \in \mathbb{R}$. These identities simply say that the map $\otimes : V \times W \to V \otimes W$, $(v, w) \mapsto v \otimes w$, is a bilinear map. The fact that everything in $V \otimes W$ is a linear combination of symbols $v \otimes w$ means that the image of the map $\otimes : V \times W \to V \otimes W$ spans $V \otimes W$.[5] Here is the formal definition of the tensor product of two vector spaces.

Definition 6.3. A tensor product of two finite dimensional vector spaces V and W is a vector space $V \otimes W$ *together with a bilinear map* $\otimes : V \times W \to V \otimes W$, $(v, w) \mapsto v \otimes w$[6] such that for any bilinear map $b : V \times W \to U$ there is a *unique linear map* $\bar{b} : V \otimes W \to U$ with $\bar{b}(v \otimes w) = b(v, w)$. That is, the diagram



commutes. The existence of the map $\bar{b}$ satisfying the above conditions is called the *universal property* of the tensor product.

[5]But the image of $\otimes$ is not *all* of $V \otimes W$. The elements in the image are called decomposable tensors.

[6]The symbol $v \otimes w$ stands for the value of the map $\otimes$ on the pair (v, w)

This definition is quite abstract. It is not clear that such objects exist and, if they exist, that they are unique. Setting the question of existence and uniqueness of tensor products aside, let's us sort out the relationship between $V \otimes W$ and bilinear maps $\mathrm{Mult}(V \times W, U)$. Recall that $\mathrm{Hom}(X, Y)$ is the space of all linear maps from a vector space X to a vector space Y and is itself a vector space (see p. 13).

Lemma 6.4. *Assume that $V \otimes W$ exists. Then*

$$\mathrm{Hom}(V \otimes W, U) \xrightarrow{\simeq} \mathrm{Mult}(V \times W, U).$$

Proof. The isomorphism in question is built into the definition of the tensor product. Given a linear map $A : V \otimes W \to U$ the composition $A \circ \otimes : V \times W \to U$ is bilinear. And conversely, given a bilinear map $b \in \mathrm{Mult}(V \times W, U)$ there is a unique linear map $\bar{b} : V \otimes W \to U$ so that $(\bar{b} \circ \otimes)(v, w) = b(v, w)$ for all $(v, w) \in V \times W$.

In other words the maps $\mathrm{Hom}(V \otimes W, U) \ni A \mapsto A \circ \otimes \in \mathrm{Mult}(V \times W, U)$ and $\mathrm{Mult}(V \times W, U) \ni b \mapsto \bar{b} \in \mathrm{Hom}(V \otimes W, U)$ are inverses of each other. □

Next we observed that the uniqueness of the tensor product is also built into the definition of the tensor product.

Proposition 6.5. *If tensor products exist, they are unique up to isomorphism.*

Proof. The proof is quite formal and uses nothing but the universal property. Suppose there are two vector spaces $V \otimes_1 W$ and $V \otimes_2 W$ with corresponding bilinear maps $\otimes_1 : V \times W \to V \otimes_1 W$ and $\otimes_2 : V \times W \to V \otimes_2 W$ which satisfy the conditions of the Definition 6.3. We will argue that these vector spaces are isomorphic. By the universal property there exist a unique *linear* map $\overline{\otimes}_1 : V \otimes_2 W \to V \otimes_1 W$ so that the diagram

$$\begin{array}{ccc} V \times W & \xrightarrow{\otimes_1} & V \otimes_1 W \\ {\scriptstyle \otimes_2}\downarrow & \nearrow{\scriptstyle \overline{\otimes}_1} & \\ V \otimes_2 W & & \end{array}$$

commutes. By the same argument, switching the roles of $\otimes_1$ and $\otimes_2$, there is a unique linear map $\overline{\otimes}_2 : V \otimes_1 W \to V \otimes_2 W$ making the diagram

$$\begin{array}{ccc} V \times W & \xrightarrow{\otimes_2} & V \otimes_2 W \\ {\scriptstyle \otimes_1}\downarrow & \nearrow{\scriptstyle \overline{\otimes}_2} & \\ V \otimes_1 W & & \end{array}$$

commute. Define

$$\begin{aligned} T_1 &= \overline{\otimes}_1 \circ \overline{\otimes}_2 : V \otimes_1 W \to V \otimes_1 W \\ T_2 &= \overline{\otimes}_2 \circ \overline{\otimes}_1 : V \otimes_2 W \to V \otimes_2 W. \end{aligned}$$

These are linear maps making the diagrams

$$\begin{array}{ccc} V \times W & \xrightarrow{\otimes_1} & V \otimes_1 W \\ {\scriptstyle \otimes_1}\downarrow & \nearrow{\scriptstyle T_1} & \\ V \otimes_1 W & & \end{array} \quad \text{and} \quad \begin{array}{ccc} V \times W & \xrightarrow{\otimes_2} & V \otimes_2 W \\ {\scriptstyle \otimes_2}\downarrow & \nearrow{\scriptstyle T_2} & \\ V \otimes_2 W & & \end{array}$$

commute. But the identity maps $id_i : V \otimes_i W \to V \otimes_i W$, $i = 1, 2$, are linear and also make the respective diagrams commute. By uniqueness $T_i = id_i$. Hence $\overline{\otimes}_1$ and $\overline{\otimes}_2$ are inverses of each other and provide the desired isomorphisms. □

Now we construct the tensor product as a quotient of an infinite dimensional vector space by an infinite dimensional subspace thereby proving its existence.

Proposition 6.6. *Tensor products exist.*

Proof. Let V and W be two finite dimensional vector spaces. We want to construct a new vector space $V \otimes W$ and a bilinear map $\otimes : V \times W \to V \otimes W$ satisfying the conditions of Definition 6.3. We start with a vector space $F(V \times W)$ made of formal finite linear combinations of ordered pairs (v, w), $v \in V$, $w \in W$. Its basis is the set $\{(v, w) \mid v \in V,\ w \in W\} = V \times W$.

If you prefer you can think of $F(V \times W)$ as the set of functions

$$\{f : V \times W \to \mathbb{R} \mid f(v, w) \neq 0 \text{ for only finitely many pairs } (v, w)\}.$$

This set of functions is an infinite dimensional vector space. Its basis consists of functions that take value 1 on a given pair (v_0, w_0) and 0 on all other pairs. It's tempting to call this function (v_0, w_0). The vector space $F(V \times W)$ is called the *free vector space* generated by the set $V \times W$.

Note that we have an inclusion map $\iota : V \times W \to F(V \times W)$, $\iota(v, w) = (v, w)$. It is not bilinear since $(v_1 + v_2, w) \neq (v_1, w) + (v_2, w)$ in $F(V, W)$.

Consider the smallest subspace K of $F(V, W)$ containing the following collection of vectors:

$$S = \left\{ \begin{array}{c} (v_1 + v_2, w) - (v_1, w) - (v_2, w) \\ (v, w_1 + w_2) - (v, w_1) - (v, w_2) \\ c(v, w) - (cv, w) \\ c(v, w) - (v, cw), \end{array} \middle| \; v, v_1, v_2 \in V,\ w, w_1, w_2 \in W \text{ and } c \in \mathbb{R} \right\}$$

In other words, consider the subspace K of $F(V \times W)$ spanned by the set S. Define $V \otimes W$ to be the quotient of $F(V \times W)$ by K:

$$V \otimes W := F(V \times W)/K.$$

Define the map $\otimes : V \times W \to V \otimes W$ to be the composite of the inclusion $\iota : V \times W \hookrightarrow F(V \times W)$ and the quotient map $F(V \times W) \to F(V \times W)/K$. The definition of K is rigged precisely so that this composite is bilinear. We write $v \otimes w$ for the value of $\otimes$ on the pair (v, w). By construction the set $\{v \otimes w \mid (v, w) \in V \times W\}$ spans $V \otimes W$ [but it's much too big to be a basis].

We check that the map $\otimes : V \times W \to V \otimes W$ has the required universal property. Suppose $b : V \times W \to U$ is bilinear. Since $V \times W$ is a basis for $F(V \times W)$, b defines a unique linear map $\tilde{b} : F(V \times W) \to U$ given on the basis by $\tilde{b}((v, w)) = b(v, w)$. As b is bilinear, $\tilde{b}$ is 0 on K by the definition of K. Thus we obtain a linear map $\bar{b} : F(V \times W)/K = V \otimes W \to U$ with $\bar{b}(v \otimes w) = \tilde{b}((v, w)) = b(v, w)$. Since the vectors of the form $v \otimes w$ span $V \otimes W$, $\bar{b}$ is unique. This verifies the universal property and thereby proves the existence of the tensor product. □

Lemma 6.7. *For any vector spaces V and W*

$$\dim(V \otimes W) = \dim V \cdot \dim W.$$

Proof.

$$\begin{aligned} \dim V \otimes W = \dim(V \otimes W)^* &= \dim \operatorname{Hom}(V \otimes W, \mathbb{R}) \\ &= \dim \operatorname{Mult}(V \times W, \mathbb{R}) \quad \text{by Lemma 6.4} \\ &= \dim V \cdot \dim W \cdot \dim \mathbb{R}. \end{aligned}$$

□

We are now in position to quickly prove a number of results about tensor products.

Corollary 6.7.1. *If $\{v_i\}$ and $\{w_j\}$ are a basis of V and W respectively, then $\{v_i \otimes w_j\}$ is a basis of $V \otimes W$.*

Proof. Since the vectors of the form $v \otimes w$, $v \in V$, $w \in W$, span $V \otimes W$, the much smaller set $\{v_i \otimes w_j\}$ also spans $V \otimes W$[7]. But $\dim(V\otimes)W = \dim V \cdot \dim W$ is precisely the number of elements in the set $\{v_i \otimes w_j\}$. Hence the set $\{v_i \otimes w_j\}$ is a basis. □

Lemma 6.8. *$V \otimes W$ is isomorphic to $W \otimes V$.*

[7] We are using here the fact that for any $(v, w) \in V \times W$, the tensor $v \otimes w$ is a linear combination of $v_i \otimes w_j$'s.

Proof. Consider the map $b : W \times V \to V \otimes W$ defined by

$$b(w, v) = v \otimes w.$$

Since b is bilinear, there is a unique linear map $\bar{b} : W \otimes V \to V \otimes W$ with $\bar{b}(w \otimes v) = v \otimes w$. Since the set $\{v \otimes w \mid v \in V, w \in W\}$ generates $V \otimes W$, the map $\bar{b}$ is surjective. It is an isomorphism by dimension count. $\square$

Lemma 6.9. *$V^* \otimes W$ is isomorphic to* $\mathrm{Hom}(V, W)$*.*

Proof. Consider $b : V^* \times W \to \mathrm{Hom}(V, W)$ defined by

$$(b(v^*, w))(v) = v^*(v)w \quad \text{for all } v^* \in V^*, v \in V, w \in W.$$

Since b is bilinear, it induces a linear map $\bar{b} : V^* \otimes W \to \mathrm{Hom}(V, W)$ with

$$(\bar{b}(v^* \otimes w))(v) = v^*(v)w \quad \text{for all } v^* \in V^*, v \in V, w \in W.$$

Observe that linear maps of the form $v \mapsto v^*(v)w$ span $\mathrm{Hom}(V, W)$ (The proof of this fact is very similar to the proof of Lemma 6.2 and is left as an exercise). Hence $\bar{b}$ is an isomorphism by dimension count. $\square$

Exercise 6.1. Show that if $\{v_i\}$ is a basis of a vector space V, $\{v_i^*\}$ the dual basis and $\{w_j\}$ the basis of a vector space W, then $\{v_i^*(\cdot)w_j\}$ is a basis of $\mathrm{Hom}(V, W)$.

Lemma 6.10. *If $A : V \to W$ and $B : V' \to W'$ are two linear maps, then there is a unique linear map $A \otimes B : V \otimes V' \to W \otimes W'$ such that $(A \otimes B)(v \otimes w) = A(v) \otimes B(w)$ for all $(v, w) \in V \times W$.*

Proof. Consider $b : V \times W \to V' \otimes W'$ given by

$$b(v, w) = Av \otimes Bw.$$

The map b is bilinear, whence the universal property gives us a unique linear map $\bar{b} : V \otimes W \to V' \otimes W'$ with

$$\bar{b}(v \otimes w) = Av \otimes Bw$$

for all $(v, w) \in V \times W$. $\square$

Exercise 6.2. Show that if $A : V \to W$ is represented by a matrix (a_{ij}) with respect to some bases of V and W and $B : V' \to W'$ is represented by a matrix (b_{kl}) with respect to bases of V' and W', then $A \otimes B$ is represented by the matrix $(a_{ij}b_{kl})$ with respect to the appropriate bases.

Exercise 6.3. Show that there is a natural isomorphism $\phi : V^* \otimes W^* \xrightarrow{\simeq} \mathrm{Mult}(V \times W, \mathbb{R})$ with

$$\phi(v^* \otimes w^*)(v, w) = v^*(v)w^*(w)$$

for all v^*, w^*, v, w.

Show that there is a natural isomorphism $\psi : V^* \otimes W^* \to (V \otimes W)^*$ with

$$\psi(v^* \otimes w^*)\,(v \otimes w) = v^*(v)w^*(w)$$

for all v^*, w^*, v, w.

Exercise 6.4. Show that the map $\mathbb{R} \times V \to V$, $(a, v) \mapsto av$ gives rise to an isomorphism $R \otimes V \xrightarrow{\simeq} V$ which sends $a \otimes v$ to av for all $a \in \mathbb{R}$ and $v \in V$.

Exercise 6.5. Show that taking tensor product is associative:

$$V \otimes (U \otimes W) \simeq (V \otimes U) \otimes W$$

for any three vector spaces V, U and W.

From now on we write $V \otimes U \otimes W$ for $V \otimes (U \otimes W)$ since the order of taking tensor products doesn't matter. Exercise 6.5 above also allows us to define recursively tensor powers of a vector space V. We define

$$V^{\otimes 0} := \mathbb{R},$$
$$V^{\otimes 1} := V \quad \text{and}$$
$$V^{\otimes n} := V^{\otimes (n-1)} \otimes V \quad \text{for } n > 1.$$

It is not hard to generalize the relationship between bilinear maps and tensor products to the relationship between n-linear maps and n-fold tensor products. For example:

Exercise 6.6. Prove that given a n-linear map

$$f : \overbrace{V \times \cdots \times V}^{n} \to U,$$

then there exists a unique linear map $\overline{f} : V^{\otimes n} \to U$ with

$$\overline{f}(v_1 \otimes \cdots \otimes v_n) = f(v_1, \ldots, v_n).$$

for all $(v_1, \ldots, v_n) \in V \times \cdots \times V$.

Moreover, given $a \in V^{\otimes n}$ and $b \in V^{\otimes m}$, $a \otimes b$ is in $V^{\otimes n} \otimes V^{\otimes m} \simeq V^{\otimes (n+m)}$. This gives us an $\mathbb{R}$-bilinear map,

$$V^{\otimes n} \times V^{\otimes m} \to V^{\otimes (n+m)}, \quad (a, b) \mapsto a \otimes b.$$

Note that if $n = 0$ the map above is simply

$$\mathbb{R} \times V^{\otimes m} \to V^{\otimes m}, \quad (a, t) \mapsto at.$$

(cf. Exercise 6.4).

Definition 6.11. An *algebra* over $\mathbb{R}$ is a vector space A together with a bilinear map $A \times A \to \mathbb{R}$, $(a, a') \mapsto aa'$ ("multiplication"). An algebra A is said to be an *algebra with unity* if there is an element $1 \in A$ such that $1 \cdot a = a$ for all $a \in A$. An algebra A is associative if the multiplication is associative.

Remark 6.12. Note that in any algebra A, $0a = a0 = 0$ for all $a \in A$ (this is because multiplication is required to be bilinear).

Remark 6.13. If A is an algebra with 1 then there is an injection $\mathbb{R} \to A$, $x \mapsto x1$. We will always identify $\mathbb{R}$ with its image in A.

Example 6.14. A Lie algebra is an algebra. It is not associative and does not have 1 (why not?).

Example 6.15. The space $M_n(\mathbb{R})$ of $n \times n$ matrices forms an algebra under matrix multiplication. It is an algebra with unity: the identity matrix I is the unity.

Definition 6.16. An algebra A is *graded* if

$$A = \sum_{i=0}^{\infty} A_i \qquad \text{direct sum}$$

and if for any $a \in A_i$ and $b \in A_j$ we have $a \cdot b \in A_{i+j}$. We will refer to the elements of A_k as elements of *degree* k.

Given a vector space V we construct the corresponding *tensor algebra* $\mathcal{T}(V)$ as follows. As a vector space $\mathcal{T}(V)$ is the direct sum:

$$\mathcal{T}(V) = \mathbb{R} \oplus V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes n} \oplus \cdots = \sum_{i=0}^{\infty} V^{\otimes i}.$$

Thus the elements of $\mathcal{T}(V)$ are finite sums $a_{i_1} + a_{i_2} + \cdots a_{i_k}$, $a_{i_j} \in V^{\otimes i_j}$. We define the multiplication on $\mathcal{T}$ by extending the multiplication

$$V^{\otimes n} \times V^{\otimes m} \to V^{\otimes (n+m)} \quad (a, b) \mapsto a \otimes b.$$

bilinearly to all of $\mathcal{T}(V)$. The tensor algebra $\mathcal{T}(V)$ of a vector space V is a graded associative algebra with 1. Note that by construction the elements of $\mathcal{T}(V)$ are sums of products of elements of V, that is, $\mathcal{T}(V)$ is *generated* by V.

6.2. **The Grassmann (exterior) algebra and alternating maps.** We have seen that tensor products are intimately related to multi-linear maps. Exterior (Grassmann) algebras are just as intimately related to alternating multilinear maps. Recall that an n-linear map $f : V \times \cdots \times V \to U$ is *alternating* if it changes sign whenever we switch to adjacent entries:

$$f(v_1, \dots, v_i, v_{i+1}, \dots, v_n) = -f(v_1, \dots, v_{i+1}, v_i, \dots, v_n)$$

for all $(v_1, \dots, v_n) \in V \times \cdots \times V$ and any index i.

Example 6.17. The determinant

$$\det : \overbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}^{n \text{ factors}} \to \mathbb{R}, \quad (v_1, \dots, v_n) \to \det(v_1 | \dots | v_n)$$

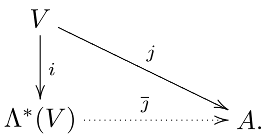
is an alternating map.

Example 6.18. Consider a vector space V and $a, b \in V^*$. Define the bilinear map $a \wedge b$ by

$$(a \wedge b)(v_1, v_2) := a(v_1)b(v_2) - a(v_2)b(v_1), \quad v_1, v_2 \in V.$$

The map $a \wedge b$ ("a wedge b") is alternating.

Definition 6.19 (Grassmann (exterior) algebra)**.** Let V be a finite dimensional vector space over $\mathbb{R}$. The *Grassmann (exterior) algebra* $\Lambda^*(V)$ is an algebra over $\mathbb{R}$ with unity together with an injective linear map $i : V \to \Lambda(V)$ called the *structure map* which has the following universal property: If A is an algebra over $\mathbb{R}$ with unity and $j : V \to A$ is a linear map such that $j(v) \cdot j(v) = 0$ for all $v \in V$, then there is a unique algebra map [8] $\bar{\jmath} : \Lambda^*(V) \to A$ such that the following diagram commutes:



Proposition 6.20. *If the exterior algebra $\Lambda^*(V)$ exists, it is unique (up to isomorphism).*

Proof. This is a formal exercise and is left to the reader. □

Proposition 6.21. *For every vector space V the exterior algebra $\Lambda^*(V)$ exists.*

Proof. Let I be the two-sided ideal in the tensor algebra $\mathcal{T}(V)$ generated by the set $\{v \otimes v : v \in V\}$. Note that $\mathbb{R} \cap I = 0$ and $V \cap I = 0$ for degree reasons. Define

$$\Lambda^*(V) := \mathcal{T}(V)/I,$$

the quotient of the tensor algebra by the ideal I. Then $\Lambda^*(V)$ is an algebra — it inherits the multiplication from $\mathcal{T}(V)$. The induced multiplication in $\Lambda^*(V)$ is denoted by $\wedge$ ("wedge"). Since the tensor algebra is graded, so is I, and

$$I = (I \cap V^{\otimes 2}) \oplus (I \cap V^{\otimes 3}) \oplus \cdots$$

Since $V \cap I = 0$, the composite $i : V \to \mathcal{T}(V) \to \mathcal{T}(V)/I = \Lambda^*(V)$ is an injection. Note that any element of $\Lambda^*(V)$ is a finite linear combination of products of elements of V.

Now that we have constructed the exterior $\Lambda^*(V)$, let us prove the universal property. Suppose that A is an algebra and that we are given a linear map $j : V \to A$ with $j(v) \cdot j(v) = 0$ for all $v \in V$. Consider the map $b : V \times V \to A$ given by $b(v, w) = j(v) \cdot j(w)$. Since the map b is bilinear, there is a unique linear map $j^{(2)} : V \otimes V \to A$ with $j^{(2)}(v \otimes w) = j(v) \cdot j(w)$. Similarly, for all positive integers k, we have k-linear maps $j^{(k)} : V^{\otimes k} \to A$ with

$$j^{(k)}(v_1 \otimes \cdots \otimes v_k) = j(v_1) \cdots j(v_k).$$

In addition, we define $j^{(0)}(a) = a \cdot 1_A$, for all $a \in \mathbb{R}$. In this way, we obtain an algebra map $\tilde{\jmath} : \mathcal{T}(V) \to A$. By assumption, $\tilde{\jmath}(v \otimes v) = 0$ for all $v \in V$. Therefore $\tilde{\jmath}$ vanishes on the ideal I. This implies that $\tilde{\jmath}$ descends to an algebra map $\bar{\jmath} : \Lambda^*(V) = \mathcal{T}/I \to A$ with $\bar{\jmath}(v) = j(v)$ for all $v \in V$. Since an algebra map is uniquely determined on generators, and since V generated $\Lambda^*(V)$, the map $\bar{\jmath}$ is unique. □

[8]A map $f : A \to B$ between two algebras is an *algebra map* if f is linear and preserves multiplication: $f(a_1 a_2) = f(a_1) f(a_2)$

Remark 6.22. For any $v \in V$, we have $v \wedge v = 0$ in the exterior algebra $\Lambda^*(V)$. Also,

$$0 = (v_1 + v_2) \wedge (v_1 + v_2) = v_1 \wedge v_1 + v_1 \wedge v_2 + v_2 \wedge v_1 + v_2 \wedge v_2$$

gives that

$$v_1 \wedge v_2 = -v_2 \wedge v_1;$$

That is, the wedge product is *skew-commutative.*

Remark 6.23. Let $\Lambda^k(V) = \mathcal{T}^k(V)/(\mathcal{T}^k(V) \cap I)$. The vector space $\Lambda^k(V)$ is called the k^{th} *exterior power* of V. Then

$$\Lambda^*(V) = \sum_{k=0}^{\infty} \Lambda^k(V),$$

where

$$\Lambda^0(V) = \mathbb{R} \quad \text{and} \quad \Lambda^1(V) = V.$$

Also, if $\alpha \in \Lambda^k(V)$ and $\beta \in \Lambda^l(V)$, then $\alpha \wedge \beta \in \Lambda^{k+1}(V)$. Thus, $\Lambda^*(V)$ is a graded algebra with 1.

Remark 6.24. We know that if $\{v_1, \ldots, v_n\}$ is a basis for V, then $\{v_i \otimes v_j\}$ is a basis for $V \otimes V$. By induction, $\{v_{i_1} \otimes \cdots \otimes v_{i_k}\}$ is a basis for $V^{\otimes k}$. Thus, $\{v_{i_1} \wedge \cdots \wedge v_{i_k}\}$ generates $\Lambda^k(V) = V^{\otimes k}/(I \cap V^{\otimes k})$. Since $\wedge$ is skew-commutative, however, we can reduce this generating set to a smaller one:

$$\{v_{i_1} \wedge \cdots \wedge v_{i_k} \mid i_1 < \cdots < i_k\}, \tag{6.1}$$

This implies that

$$\Lambda^l(V) = 0 \quad \text{whenever} \quad l > \dim V.$$

We will see below that the set (6.1) is a basis of $\Lambda^k(V)$.

We now investigate the connection between the k-th exterior power $\Lambda^k(V)$ of a vector space V and alternating maps.

Proposition 6.25 (Universal property of k-th exterior power of a vector space)**.** *Let U and V be vector spaces. If $f : \overbrace{V \times \cdots \times V}^{k} \to U$ is alternating then there is a unique linear map $\overline{f} : \Lambda^k(V) \to U$ with*

$$\overline{f}(v_1 \wedge \cdots \wedge v_k) = f(v_1, \ldots, v_k).$$

Proof. By the universal property of $V^{\otimes k}$, there is a unique linear map $\tilde{f} : V^{\otimes k} \to U$ such that $\tilde{f}(v_1 \otimes \cdots \otimes v_k) = f(v_1, \ldots, v_k)$. Since f is alternating, $f\Big|_{I \cap V^{\otimes k}} = 0$, where I is the ideal defined in the construction of $\Lambda^*(V)$. This gives us the linear map $\overline{f} : \Lambda^k(V) = V^{\otimes k}/(I \cap V^{\otimes k}) \to U$ with the desired property. □

Corollary 6.25.1. *The space of k-linear alternating maps $\{f : V \times \cdots \times V \to U \mid f$ is alternating$\}$ is isomorphic to the space* $\mathrm{Hom}(\Lambda^k(V), U)$.

Lemma 6.26. *Let V be an n-dimensional vector space. Then $\Lambda^n(V)$ is 1-dimensional.*

Proof. We may assume that $V = \mathbb{R}^n$. Let $e_1, \ldots, e_n$ be the standard basis. Then $e_1 \wedge \cdots \wedge e_n$ spans $\Lambda^n(V)$. We need to show that $e_1 \wedge \cdots \wedge e_n \neq 0$. The determinant $\det : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ is 1 on the identity matrix $I = (e_1| \ldots |e_n)$: $\det(e_1| \ldots |e_n) = 1$. Hence the induced linear map $\overline{\det} : \Lambda^n(\mathbb{R}^n) \to \mathbb{R}$ is 1 on $e_1 \wedge \cdots \wedge e_n$. Therefore $e_1 \wedge \cdots \wedge e_n \neq 0$. □

Corollary 6.26.1. *If $\{f_1, \ldots, f_n\}$ is a basis for a vector space V, then $\{f_{i_1} \wedge \cdots \wedge f_{i_k} : 1 \leq i_1 < \cdots < i_k \leq n\}$ is a basis for its k-th exterior power $\Lambda^k(V)$.*

Proof. By Remark 6.24 the above set generates $\Lambda^k(V)$. So we only need to check independence. Suppose

$$0 = \sum_{i_1 < \cdots < i_k} a_{i_1,\ldots,i_k} f_{i_1} \wedge \cdots \wedge f_{i_k} \quad \text{for some } a_{i_1,\ldots,i_k} \in \mathbb{R}$$

Pick a sequence $j_1 < j_2 < \cdots < j_k$. Let $j_{k+1} < \cdots < j_n$ be the remaining indices. Then

$$\begin{aligned} & (\sum a_{i_1,\ldots,i_k} f_{i_1} \wedge \cdots \wedge f_{i_k}) \wedge f_{j_{k+1}} \wedge \cdots \wedge f_{j_n} \\ = \quad & a_{j_1,\ldots,j_k} f_{j_1} \wedge \cdots f_{j_k} \wedge f_{j_{k+1}} \wedge \cdots \wedge f_{j_n}, \end{aligned}$$

since $a_{i_1,\ldots,i_k} f_{i_1} \wedge \cdots f_{i_k} \wedge f_{j_{k+1}} \wedge \cdots \wedge f_{j_n} = 0$ whenever $i_s = j_r$ for some s, r. This gives $a_{j_1,\ldots,j_k} = 0$.

Also, $f_{j_1} \wedge \cdots \wedge f_{j_k} \wedge f_{j_{k+1}} \wedge \cdot \wedge f_{j_n} = \pm f_1 \wedge \cdot f_n \neq 0$. Hence $f_{j_1} \wedge \cdots \wedge f_{j_k} \neq 0$. □

Corollary 6.26.2. *For any finite dimensional vector space V*

$$\dim \Lambda^k(V) = \begin{pmatrix} \dim V \\ k \end{pmatrix} = \frac{(\dim V)!}{k!(\dim V - k)!}.$$

Lemma 6.27. *Let $A : V \to W$ be a linear map. Then there is a unique linear map $\Lambda^k(A) : \Lambda^k(V) \to \Lambda^k(W)$ such that*

$$(\Lambda^k(A))(v_1 \wedge \cdots \wedge v_k) = Av_1 \wedge \cdots \wedge Av_k$$

for all $v_1, \ldots, v_k \in V$.

Proof. Consider the map $b : V \times \cdots \times V \to \Lambda^k(W)$ given by

$$b(v_1, \ldots, v_k) = Av_1 \wedge \cdots \wedge Av_k.$$

Since $\wedge$ is skew-commutative, b is an alternating map. By Proposition 6.25 there exists a unique linear map $\Lambda^k(A) : \Lambda^k(V) \to \Lambda^k(W)$ with the required properties. □

Exercise 6.7. Let $A : V \to W$ be a linear map as above. Choose bases of V and W and the corresponding bases of $\Lambda^k(V)$ and of $\Lambda^k(W)$. Show that the entries of the matrix representing $\Lambda^k(A)$ are polynomial in the entries of the matrix representing A.

6.3. **Pairings.**

Definition 6.28. Let V and W be two vector spaces. A *pairing* is a bilinear map $\langle \cdot, \cdot \rangle : V \times W \to \mathbb{R}$.

Example 6.29. Let V be a vector space and V^* be its dual. The evaluation map

$$V^* \times V \to \mathbb{R} \quad \langle \ell, v \rangle = \ell(v)$$

is a pairing.

Definition 6.30. A pairing $\langle \cdot, \cdot \rangle : V \times W \to \mathbb{R}$ is *non-degenerate* if

$$\langle v_0, w \rangle = 0 \;\; \forall w \in W \Rightarrow v_0 = 0$$
$$\langle v, w_0 \rangle = 0 \;\; \forall v \in V \Rightarrow w_0 = 0.$$

Example 6.31. The evaluation map

$$V^* \times V \to \mathbb{R} \quad \langle \ell, v \rangle = \ell(v)$$

is a non-degenerate pairing. In a sense it is the only nondegenerate pairing:

Proposition 6.32. *If $b : V \times W \to \mathbb{R}$ is a nondegenerate pairing, then $V \simeq W^*$ and $W \simeq V^*$.*

Proof. Consider $b_1^\# : V \to W^*$ given by

$$(b_1^\#(v))(w) = b(v, w).$$

The map $b_1^\#$ is linear, and

$$\ker b_1^\# = \{v_0 \in V : b_1^\#(v_0) = 0\} = \{v_0 \in V : b(v_0, w) = 0 \; \forall w\} = \{0\}.$$

Thus $\dim V \leq \dim W^* = \dim W$. By the same argument, we have $\dim W \leq \dim V^* = \dim V$. Therefore $\dim V = \dim W$. Hence $b_1^\#$ is an isomorphism.

By the same argument, $b_2^\# : W \to V^*$ given by $w \mapsto b(\cdot, w)$ is an isomorphism as well. □

Proposition 6.33. *There is a nondegenerate pairing*

$$\langle \cdot, \cdot \rangle : \Lambda^k(V^*) \times \Lambda^k(V) \to \mathbb{R}$$

with

$$\langle v_1^* \wedge \cdots \wedge v_k^*, v_1 \wedge \cdots \wedge v_k \rangle = \det \Big(v_i^*(v_j) \Big).$$

Hence

$$\Lambda^k(V^*) \simeq (\Lambda^k(V))^*.$$

Proof. Consider $b : \overbrace{V^* \times \cdots \times V^*}^{k} \times \overbrace{V \times \cdots \times V}^{k} \to \mathbb{R}$ given by

$$b(l_1, \ldots, l_k, v_1, \ldots, v_k) = \det \Big(l_i(v_j) \Big).$$

For a fixed $(l_1, \ldots, l_k) \in V^* \times \cdots \times V^*$, b is alternating in the v's. So there is a map $\bar{b} : (V^* \times \cdot \times V^*) \times \Lambda^k(V) \to \mathbb{R}$ with

$$(l_1, \ldots, l_k, v_1 \wedge \cdots \wedge v_k) \mapsto \det \Big(l_i(v_j) \Big).$$

Similarly, for a fixed $v_1 \wedge \cdots \wedge v_k \in \Lambda^k(V)$, $\bar{b}$ is alternating in the l's, which means that there is a map $\tilde{b} : \Lambda^k(V^*) \times \Lambda^k(V) \to \mathbb{R}$ with the desired property.

To check non-degeneracy evaluate the pairing on the respective bases. □

Combining the proposition above with Corollary 6.25.1 we get:

Corollary 6.33.1. *The space of k-linear alternating maps $\{f : V \times \cdots \times V \to \mathbb{R} \mid f$ is alternating $\}$ is isomorphic to the k-th exterior power $\Lambda^k(V^*)$.*

Remark 6.34. Explicitly $\ell_1 \wedge \cdots \wedge \ell_k \in \Lambda^k(V^*)$ defines a k-linear alternating map by

$$\ell_1 \wedge \cdots \wedge \ell_k \ (v_1, \cdots, v_k) = \det(\ell_i(v_j))$$

for all $v_1, \ldots, v_k \in V$. In particular

$$\ell_1 \wedge \ell_2 \ (v_1, v_2) = \ell_1(v_1)\ell_2(v_2) - \ell_1(v_2)\ell_2(v_1)$$

Exercise 6.8. Suppose that V is an n-dimensional vector space. Given a linear map $A : V \to V$, we get a map $\Lambda^n(A) : \Lambda^n(V) \to \Lambda^n(W)$, and since $\dim \Lambda^n(V) = 1$, the map $\Lambda^n(A)$ is multiplication by a scalar. Show that this scalar is $\det A$.

7. Differential forms and integration

7.1. **Motivation.** Suppose we want to integrate a function f over a manifold M. We start with an easiest case: the support of f is contained inside a coordinate chart $\phi : U \to \mathbb{R}^m$. We could then try to define

$$\int_M f = \int_U f := \int_{\phi(U) \subset \mathbb{R}^m} (f \circ \phi^{-1})(x)\, dx.$$

Right away we would then run into a problem when we try to compute this integral with respect to a different coordinate chart. Recall the change of variables formula for integrals:

Lemma 7.1. *Let $F : U \to V$ be diffeomorphism between two open subsets of $\mathbb{R}^m$ and $f \in C^\infty(V)$ an integrable function. Then $y \mapsto f(F(y))\, |\det dF_y|$ is an integrable function on U and*

$$\int_{V=F(U)} f(y)\, dy = \int_U f(F(x))\, |\det dF_x|\, dx \tag{7.1}$$

Now suppose $\psi : U \to \mathbb{R}^m$ is another coordinate chart on M with the same domain. By our definition of $\int_M f$ we would want

$$\int_M f = \int_{\psi(U)} (f \circ \psi^{-1})(y)\, dy.$$

But the change of variables formula (7.1) gives us

$$\begin{aligned} \int_{\psi(U)} ((f \circ \psi^{-1})(y)\, dy &= \int_{\phi(U)} \Big((f \circ \psi^{-1}) \circ (\psi \circ \phi^{-1}) \Big)(x)\, |\det(d(\psi \circ \phi^{-1})_x)|\, dx \\ &= \int_{\phi(U)} (f \circ \phi^{-1})(x)\, |\det(d(\psi \circ \phi^{-1})_x)|\, dx \end{aligned}$$

Since there is no reason for $|\det(d(\psi \circ \phi^{-1})_x)|$ to be the constant function 1, the integral of f over M is ill-defined.

One solution is to integrate something other than functions. If μ is one of those somethings and F is a diffeomorphism, then μ should transform under F by the rule

$$\mu \rightsquigarrow (\mu \circ F)\,\det(dF).$$

This will be made more precise shortly. Additionally we will need to confine ourselves to manifolds with atlases ϕ_α with the property that the differentials $d(\phi_\alpha \circ \phi_\beta^{-1})$ all have positive determinants. Such manifolds are called orientable. It turns out that what one integrates over manifolds are differential forms and we now proceed to define them.

Again, let M be a manifold. Recall that we made the disjoint union of its cotangent spaces $\bigsqcup_{q\in M} T_q^*M$ into a manifold, the cotangent bundle T^*M of M. Moreover, we defined the manifold structure on T^*M in such a way that the natural projection

$$\pi : T^*M \to M, \quad T_q^*M \ni \eta \mapsto q \in M$$

is smooth. Similarly one can make the disjoint union of kth exterior powers of the cotangent spaces of M into a manifold $\Lambda^k(T^*M)$:

$$\Lambda^k(T^*M) = \bigsqcup_{q\in M} \Lambda^k(T_q^*M).$$

Moreover, the natural projection

$$\pi : \Lambda^k(T^*M) \to M, \quad \Lambda^k(T_q^*M) \ni \nu \mapsto q \in M$$

can be arranged to be smooth. We defer the details of this construction for section 8, where we will carry it out for arbitrary vector bundles and not just for the cotangent bundle.

The preimages of points $\pi^{-1}(q)$ under $\pi : \Lambda^k(T^*M) \to M$ are called fibers of π. By design they are vector spaces $\Lambda^k(T_q^*M)$. Recall that for any vector space V, the 0th exterior power $\Lambda^0(V)$ is just the real numbers and the 1st exterior power $\Lambda^1(V)$ is the vector space V itself. It will turn out that $\Lambda^0(T^*M) = M \times \mathbb{R}$ and $\Lambda^1(T^*M) = T^*M$.

Definition 7.2. A *smooth k-form* μ (a.k.a. a *differential form of degree k*) is a smooth map $\mu : M \to \Lambda^k(T^*M)$, $q \mapsto \mu_q$ so that

$$\mu_q \in \Lambda^k(T_q^*M)$$

for all $q \in M$.

The last condition can be stated as: $\pi \circ \mu : M \to M$ is the identity map. The smoothness condition will be discussed a few paragraphs down.

Remark 7.3. By definition a 0 form on M is a smooth map $\mu : M \to M \times \mathbb{R}$ such that $\mu_q = (q, f(q))$ for all $q \in M$, where $f(q) \in \mathbb{R}$ depends on q. In other words, 0 forms are nothing but functions. And the smoothness of 0 forms is the smoothness of functions.

Notation. We denote the space of differential k-forms on a manifold M by $\Omega^k(M)$. We denote the space of all differential forms by $\Omega^*(M)$. Thus

$$\Omega^*(M) = \Omega^0(M) \oplus \Omega^1(M) \oplus \cdots \oplus \Omega^k(M) \oplus \cdots$$

Let us try to get some feel for differential forms by considering the special case where the manifold M is an open subset of $\mathbb{R}^m$. We denote the standard coordinates on M by $x_1, \ldots, x_m$. Then for every $q \in M$, the differentials $(dx_1)_q, \ldots, (dx_m)_q$ form a natural basis of the cotangent space $T_q^*M \simeq (\mathbb{R}^m)^*$. Hence the set

$$\{(dx_{i_1})_q \wedge \ldots \wedge (dx_{i_k})_q \mid 1 \le i_1 < \cdots < i_k \le m\}$$

is a basis of the kth exterior power $\Lambda^k(T_q^*M)$. At this point it is convenient to have a bit more notation at our disposal: if I is an ordered k-tuple $i_1 < \cdots < i_k$ then

$$dx_I := dx_{i_1} \wedge \ldots \wedge dx_{i_k}. \tag{7.2}$$

We write $|I|$ to indicate the size of the tuple I: if I is a k-tuple, then $|I| = k$. With this notation, a typical k-form μ on an open subset of $\mathbb{R}^m$ has the following expression:

$$\mu = \sum_{|I|=k} a_I\, dx_I \quad \left(= \sum_{i_1<\cdots<i_k} a_{i_1\cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \right). \tag{7.3}$$

It makes sense to describe μ as smooth if a_I's are smooth functions on M.

This tells us what smoothness of a differential k form μ on an arbitrary manifold M should mean. Namely, in a coordinate chart $\phi = (x_1, \ldots, x_m) : U \to \mathbb{R}^m$

$$\mu|_U = \sum_{|I|=k} a_I \, dx_I \tag{7.4}$$

for some functions $a_I : U \to \mathbb{R}$. Then the form μ is smooth if and only if the functions a_I are all smooth. This begs the obvious question of whether smoothness depends on the choices of coordinate charts. In other words, it is not entirely clear whether this definition of smoothness is consistent. The answer is that there is no problem. The issue is closely tied up with the issue of making $\Lambda^k(T^*M)$ into a smooth manifold, which we punt for the time being. But see subsection 8.2 below.

Since differential forms take values in vector spaces, two forms of the same degree can be added pointwise. It also makes sense to multiply a k-form by a function. This is completely analogous to vector fields being a module over the space of smooth functions.

One can also "multiply" differential forms. Namely, if $\mu \in \Omega^k(M)$ and $\nu \in \Omega^l(M)$ are two differential forms, then at every point $q \in M$, the wedge (exterior) product

$$\mu_q \wedge \nu_q$$

makes sense since $\mu_q \in \Lambda^k(T_q^*M)$ and $\nu_q \in \Lambda^l(T_q^*M)$. This defines the exterior product on differential forms:

$$\wedge : \Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M), \quad (\mu, \nu) \mapsto \mu \wedge \nu,$$

with

$$(\mu \wedge \nu)_q := \mu_q \wedge \nu_q \text{ for all } q \in M.$$

Note that if $\mu \in \Omega^0(M)$, that is, if μ is a function, then $\mu \wedge \nu = \mu \nu$. That is, wedging a function with a differential form is the same as multiplying the differential form by the function.

7.2. **Pullback of differential forms.** In order to discuss integration of differential forms we need to discuss their pullback under smooth maps. We start by discussing the underlying linear algebra. Recall that by Lemma 6.27 if $A : W \to V$ is a linear map, then there exists a unique linear map

$$\Lambda^k(A) : \Lambda^k(W) \to \Lambda^k(V), \quad \text{with} \quad \Lambda^k(A)(w_1 \wedge \cdots \wedge w_k) = Aw_1 \wedge \cdots \wedge Aw_k$$

for all $w_1, \ldots, w_k \in W$. In fact, by the universal property of the exterior algebra, we have more than just a collection of linear maps $\Lambda^k(A)$, $k = 0, 1, \ldots$. Namely, the linear map $A : W \to V$ defines a linear map $A : W \to \Lambda^*V$ with $Aw \wedge Aw = 0$ for all $w \in W$. Hence, by the universal property of $\Lambda^*(W)$ there is a unique *algebra* map

$$\Lambda^*(A) : \Lambda^*(W) \to \Lambda^*(V) \quad \text{with } \Lambda^*(A)(w_1 \wedge \cdots \wedge w_k) = Aw_1 \wedge \cdots \wedge Aw_k$$

for all $w_1, \ldots, w_k \in W$ and for all $k > 0$. Note that $\Lambda^0(A) : \Lambda^0(W) = \mathbb{R} \to \mathbb{R} = \Lambda^0(V)$ is the identity map. This is the reason why the pull-back of a 0-form, thought of as a function, is composition (see below).

If $F : M \to N$ is a smooth map between two manifolds, it defines a *pullback map* $F^* : C^\infty(N) \to C^\infty(M)$ on functions by

$$F^*f := f \circ F \quad \text{for any } f \in C^\infty(N).$$

I claim that F^* extends to a map of algebras $F^* : \Omega^*(N) \to \Omega^*(M)$. Indeed, given $F : M \to N$ we have linear maps

$$dF_q : T_qM \to T_{F(q)}N, \quad q \in M,$$

and therefore dual maps

$$dF_q^* : T^*_{F(q)}N \to T_q^*M,$$

which, in turn, induce maps on the exterior powers

$$\Lambda^k(dF_q^*) : \Lambda^k(T^*_{F(q)}N) \to \Lambda^k(T_q^*M)$$

with

$$\Lambda^k(dF_q^*)(\nu_1 \wedge \cdots \wedge \nu_k) = (dF_q^*\nu_1) \wedge \cdots \wedge (dF_q^*\nu_k)$$

for $\nu_1, \ldots, \nu_k \in T^*_{F(q)}N$. Therefore if $\mu : N \to \Lambda^k(T^*N)$ is a k-form we *define* its pullback $F^*\mu \in \Omega^k(M)$ by

$$(F^*\mu)_q := \Lambda^k(dF_q^*)(\mu_{F(q)}) \quad \text{for all } q \in M. \tag{7.5}$$

This looks a bit convoluted but has a simple (simpler?) interpretation. Recall that for any vector space V we have a canonical isomorphism between the kth exterior $\Lambda^k(V^*)$ of its dual and the space of alternating k-linear maps $f : V \times \cdots \times V \to \mathbb{R}$. The identification in question is given by

$$(\nu_1 \wedge \cdots \wedge \nu_k)(v_1, \ldots, v_k) := \det(\nu_i(v_j))$$

for all $\nu_1, \ldots, \nu_k \in V^*$ and all $v_1, \ldots, v_k \in V$ (cf. Remark 6.34). Hence, if $A : W \to V$ is a linear map and $A^* : V^* \to W^*$ its dual, then

$$\begin{aligned} \left(\Lambda^k(A^*)(\nu_1 \wedge \cdots \wedge \nu_k)\right)(w_1, \cdots, w_k) &= (A^*\nu_1 \wedge \cdots \wedge A^*\nu_k)(w_1, \cdots, w_k) \\ &= \det((A^*\nu_i)(w_j)) \\ &= \det(\nu_i(Aw_j)) \\ &= (\nu_1 \wedge \cdots \wedge \nu_k)(Aw_1, \ldots, Aw_k). \end{aligned}$$

Hence for any $\mu \in \Lambda^k(V^*)$

$$(\Lambda^k(A^*)\mu)(w_1, \ldots, w_k) = \mu(Aw_1, \ldots, Aw_k).$$

Therefore, the pullback of a differential form $\mu \in \Omega^k(N)$ by $F : M \to N$ is given by

$$(F^*\mu)_q(v_1, \ldots, v_k) := \mu_{F(q)}(dF_q v_1, \ldots, dF_q v_k) \tag{7.6}$$

for all $q \in M$, $v_1, \ldots, v_k \in T_qM$. So why did we define the pullback by (7.5) and not by (7.6)? The reason is that the first definition tells us that pullback automatically respects exterior multiplication of forms:

$$(F^*\mu) \wedge (F^*\nu) = F^*(\mu \wedge \nu) \tag{7.7}$$

for any two differential forms μ and ν on N. We will see later on that this is useful.

Remark 7.4. It is easy to see that if $F : M \to N$ and $G : N \to Z$ are two smooth maps then

$$(G \circ F)^*\mu = F^*(G^*\mu)$$

for any form $\mu \in \Omega^*(Z)$.

Remark 7.5. If N is a submanifold of a manifold M and $\mu \in \Omega^*(M)$ is a differential form on M, the *restriction* $\mu|_N$ of μ to N is, by definition, the pullback of μ to N by the inclusion map $\iota : N \to M$.

Before we can get back to our original goal of integrating forms on manifolds, we need to take care of a preliminary observation and some definitions.

Lemma 7.6. *Let $U, V \subset \mathbb{R}^m$ be two open subsets and $F : U \to V$ a diffeomorphism. Then for any smooth function $f \in C^\infty(V)$*

$$(F^*(f\, dx_1 \wedge \cdots \wedge dx_m))_q = f(F(q))\,(\det dF_q)\,(dx_1 \wedge \cdots \wedge dx_m)_q$$

for all $q \in U$.

Proof. Let $\{e_1, \ldots, e_m\}$ be the standard basis of $\mathbb{R}^m$. Then for any point q, $\{(dx_1)_q, \ldots, (dx_m)_q\}$ is the dual basis. Hence

$$\begin{aligned} (F^*(f\, dx_1 \wedge \cdots \wedge dx_m))_q(e_1, \ldots, e_m) &= (f\, dx_1 \wedge \cdots \wedge dx_m))_{F(q)}(dF_q e_1, \ldots, dF_1 e_m) \\ &= f(F(q)) \det\left((dx_i)_{F(q)}(dF_q e_j)\right) \\ &= f(F(q)) \cdot \det(dF_q) \cdot 1 \\ &= f(F(q)) \det(dF_q)\left((dx_1 \wedge \cdots \wedge dx_m)_q(e_1, \ldots, e_m)\right) \end{aligned}$$

□

Definition 7.7. The *support of a k-form $\mu \in \Omega^k(M)$* on a manifold M is the closure of the set of points where μ is non-zero:

$$\operatorname{supp}\mu = \overline{\{q \in M \mid \mu_q \neq 0\}}$$

We denote the space of compactly supported k-forms on M by $\Omega^k_c(M)$.

Definition 7.8. A manifold M is *orientable* if there is an atlas $\{\phi_\alpha : U_\alpha \to \mathbb{R}^m\}$ so that for any two indices α and β

$$\det(d(\phi_\alpha \circ \phi_\beta^{-1})_q) > 0 \tag{7.8}$$

for all $q \in \phi_\beta(U_\alpha \cap U_\beta)$.

A choice of such an atlas is an *orientation* of M.

Two atlases on M define the *same orientation* if their union is an atlas satisfying (7.8).

Example 7.9. The identity map $id : \mathbb{R}^n \to \mathbb{R}^n$ defines an orientation of $\mathbb{R}^n$ called *the standard orientation.* The map $\phi : \mathbb{R}^n \to \mathbb{R}^n$, $\phi(x_1, x_2, \ldots, x_n) = (-x_1, x_2, \ldots, x_n)$ defines a *different* orientation.

Remark 7.10. It is not at all obvious at this point, but a given connected orientable manifold can have only two orientations.

Example 7.11. It should not be too hard to see that an n-sphere S^n is orientable. Somewhat harder is the fact that the real projective space $\mathbb{R}P^n$ is orientable if and only if n is odd. Klein bottle and Möbius strip are not orientable.

7.3. **Integration.** We now proceed with defining integration of compactly supported m-forms over oriented manifolds of dimension m. Given an oriented manifold M of dimension m and a compactly supported form $\mu \in \Omega^m_c(M)$ of top degree we want to define a number $\int_M \mu$ in a reasonable way. For example we want the integration map

$$\int_M : \Omega^k_c(M) \to \mathbb{R}, \quad \mu \mapsto \int_M \mu$$

to be linear.

If $\mu \in \Omega^m_c(\mathbb{R}^m)$ then

$$\mu = f\, dx_1 \wedge \cdots \wedge dx_m$$

for some compactly supported function f. We *define*

$$\int_{\mathbb{R}^m} f(x)\, dx_1 \wedge \cdots \wedge dx_m := \int_{\mathbb{R}^m} f(x)\, dx$$

where the right hand side is the Riemann integral of the compactly supported function f over $\mathbb{R}^m$. (Note that $dx_2 \wedge dx_1 \wedge \cdots \wedge x_m = -dx_1 \wedge dx_2 \wedge \cdots \wedge x_m$, hence $\int_{\mathbb{R}^m} f(x)\, dx_2 \wedge dx_1 \wedge \cdots \wedge x_m = -\int_{\mathbb{R}^m} f(x)\, dx$ and so on.) The definition naturally extends to arbitrary open subsets of $\mathbb{R}^m$: if $U \subset \mathbb{R}^m$ is open and $\mu = f\, dx_1 \wedge \cdots \wedge x_m \in \Omega^m_c(U)$ then

$$\int_U \mu := \int_U f(x)\, dx.$$

Clearly the map $\int_U : \Omega^m_c(U) \to \mathbb{R}$ is linear. In particular if $\mu = 0$, then $\int_U \mu = 0$ as well.

Next we consider the change of variables formula for the integration of m forms over open subsets of $\mathbb{R}^m$.

Definition 7.12. A diffeomorphism $F : U \to V$ between two open subsets of $\mathbb{R}^m$ is *orientation-preserving* if $\det(dF_x) > 0$ for all $x \in U$.

Lemma 7.13 (change of variables formula for differential forms)**.** *Let $F : U \to V$ be an orientation-preserving diffeomorphism between two open subsets of $\mathbb{R}^m$ and let $\omega \in \Omega^m_c(V)$ be a compactly supported form of top degree. Then*

$$\int_U F^*\omega = \int_{F(U)} \omega. \tag{7.9}$$

Proof. We know that $\omega = f\,dx_1 \wedge \cdots \wedge dx_m$ for some $f \in C_c^\infty(V)$ and that

$$\int_V \omega = \int_V f(x)\,dx.$$

On the other hand, by Lemma 7.6,

$$F^*\omega = (f \circ F) \cdot \det dF \cdot dx_1 \wedge \cdots \wedge dx_m.$$

Hence,

$$\int_U F^*\omega = \int_U (f \circ F)(x)\,\det dF_x\,dx$$

Since $\det(dF_x) = |\det(dF_x)|$ for all $x \in U$ by assumption, we have

$$\begin{aligned}\int_U F^*\omega &= \int_U (f \circ F)(x)\,|\det dF_x|\,dx \\ &= \int_{F(U)} f(y)\,dy \quad \text{by (7.1)} \\ &= \int_V \omega.\end{aligned}$$

□

Theorem 7.14. *Let M be an oriented m-dimensional manifold. There exists a unique linear map (integration)*

$$\int_M : \Omega_c^m(M) \to \mathbb{R}, \quad \mu \mapsto \int_M \mu$$

such that if $\phi : U \to \mathbb{R}^m$ is a coordinate chart (in an atlas defining the orientation of M) and $\omega \in \Omega_c^m(U)$ a compactly supported form of top degree, then

$$\int_M \omega = \int_{\phi(U)} (\phi^{-1})^*\omega$$

Proof. We need to check that integration of forms is well-defined and unique (linearity follows from familiar properties of the Riemann integrals). We do this in two steps.

Step I. We check that if the support of ω is in an open subset $U \subset M$ and $\phi : U \to \mathbb{R}^m$, $\psi : U \to \mathbb{R}^m$ are two different charts on M defining the same orientation, then

$$\int_{\phi(U)} (\phi^{-1})^*\omega = \int_{\psi(U)} (\psi^{-1})^*\omega.$$

Since $\phi^{-1} = \psi^{-1} \circ (\psi \circ \phi^{-1})$,

$$\begin{aligned}(\psi^{-1})^*\omega &= (\psi^{-1} \circ (\psi \circ \phi^{-1}))^*\omega \\ &= (\psi \circ \phi^{-1})^*((\psi^{-1})^*\omega) \quad \text{by Remark 7.4}\end{aligned}$$

By the change of variables formula (7.9),

$$\begin{aligned}\int_{\phi(U)} (\phi^{-1})^*\omega &= \int_{\phi(U)} (\psi \circ \phi^{-1})^*((\psi^{-1})^*\omega) \\ &= \int_{\psi(U)} (\psi^{-1})^*\omega.\end{aligned}$$

Step II. We now deal with the general case. Let $\omega \in \Omega_c^m(M)$ be an arbitrary compactly supported form on the manifold M of degree $m = \dim M$. Let $\{\phi_\alpha : U_\alpha \to \mathbb{R}^m\}$ be an atlas on M giving it its orientation. Since $\operatorname{supp}\omega$ is compact, the there are finitely many sets $U_1, \ldots, U_n$ with $U_1 \cup \ldots U_n \supset \operatorname{supp}\omega$. Let $U_0 := M \smallsetminus \operatorname{supp}\omega$. Then $U_0, U_1, \ldots, U_n$ is a cover of M. Let $\{\rho_i\}_{i=0}^n$ be a partition of unity subordinate to this cover. Note that $\rho_0\omega \equiv 0$. Define the integral of ω over M by

$$\int_M \omega := \sum_{i=1}^n \int_{\phi_i(U_i)} (\phi_i^{-1})^*(\rho_i\omega). \tag{7.10}$$

We need to show that our definition of $\int_M \omega$ does not depend on the choices we made. Accordingly, suppose that $\{\psi_\beta : V_\beta \to \mathbb{R}^m\}$ is another atlas giving M the same orientation, $V_1, \dots, V_\ell$ a cover of $\operatorname{supp}\omega$, $V_0 = M \smallsetminus \operatorname{supp}\omega$ and $\{\tau_j\}_{j=0}^{\ell}$ is a partition of unity subordinate to the cover $V_0, V_1, \dots, V_\ell$ of M. By step I, for all indices $i > 0$ and $j > 0$

$$\int_{\psi_j(U_i\cap V_j)} (\psi_j^{-1})^*(\tau_j\rho_i\omega) = \int_{\phi_i(U_i\cap V_j)} (\phi_i^{-1})^*(\tau_j\rho_i\omega). \tag{7.11}$$

Therefore,

$$\begin{aligned}
\sum_i \int_{\phi_i(U_i)} (\phi_i^{-1})^*(\rho_i\omega) &= \sum_i \int_{\phi_i(U_i)} (\phi_i^{-1})^*(\rho_i(\sum_j \tau_j\omega)) \\
&= \sum_{i,j} \int_{\phi_i(U_i\cap V_j)} (\phi_i^{-1})^*(\rho_i\tau_j\omega) \\
&= \sum_{i,j} \int_{\psi_j(U_i\cap V_j)} (\psi_j^{-1})^*(\rho_i\tau_j\omega) \quad \text{by (7.11)} \\
&= \sum_j \int_{\psi_j(V_j)} (\psi_j^{-1})^*(\tau_j\omega)
\end{aligned}$$

Therefore the integral of ω over M is well-defined. ☐

The following lemma is very useful for carrying out integration.

Lemma 7.15. *Let M be an oriented manifold of dimension m, $\omega \in \Omega_c^m(M)$ a compactly supported form and $N \subset M$ an embedded submanifold of codimension 1 or greater. Then*

$$\int_M \omega = \int_{M\smallsetminus N} \omega.$$

Proof. We may assume that M is an open subset of $\mathbb{R}^m$ (why?). In this case the result follows easily from the properties of Riemann integrals of functions. ☐

To compute any integrals of forms, we also need to have a good way of computing pull-backs of forms. We have already seen that if $f : N \to \mathbb{R}$ is a 0-form (i.e., a function) and $F : M \to N$ a smooth map of manifolds then $F^*f = f \circ F$.

Exercise 7.1. Let $F : M \to N$ be a smooth map of manifolds and $f : N \to \mathbb{R}$ a smooth function. Then df is a 1-form on N and

$$F^*df = d(f \circ F).$$

Solution: for any point $q \in M$ and any $v \in T_qM$, $(F^*df)_q(v) = df_{F(q)}(dF_q(v)) = d(f \circ F)_q(v)$ by the chain rule.

Exercise 7.2. Compute the integral of (the restriction of) the one form $xdy - ydx$ over the circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2\}$ (pick any orientation of the circle you want).

Solution: Consider the map $F : (0, 2\pi) \to S^1$ given by $F(t) = (\cos t, \sin t)$. Note that the image of F is all of S^1 except for one point. Therefore, by Lemma 7.15, $\int_{S^1} xdy - ydx = \int_{F((0,2\pi))} xdy - ydx$. Also $F : (0, 2\pi) \to S^1$ is an open embedding, hence the inverse of F is a coordinate chart on S^1. Therefore,

$$\int_{F((0,2\pi))} xdy - ydx = \int_{(0,2\pi)} F^*(xdy - ydx).$$

Since pull-back respects exterior multiplication,

$$F^*(xdy - ydx) = (F^*x)(F^*dy) - (F^*y)(F^*dx).$$

But $F^*x = \cos t$ and $F^*y = \sin t$, while by Exercise 7.1 $F^*dx = d(F^*x) = d\cos t = -\sin t\, dt$ and, similarly, $F^*dy = d\sin t = \cos t\, dt$. Therefore

$$\int_{S^1} xdy - ydx = F^*(xdy - ydx) = \cos t\, d\sin t - \sin t\, d\cos t = \cos^2 t\, dt + \sin^2 t\, dt = dt.$$

We conclude that

$$\int_{S^1} xdy - ydx = \int_{(0,2\pi)} dt = 2\pi.$$

Exercise 7.3. Compute the pull-back of $dx\wedge dy$ by the map $F:(0,\infty)\times\mathbb{R}\to\mathbb{R}^2$, $F(r,\theta)=(r\cos\theta, r\sin\theta)$.

Solution: $F^*(dx\wedge dy)=F^*dx\wedge F^*dy=d(F^*x)\wedge d(F^*y)=d(r\cos\theta)\wedge d(r\sin\theta)=(\cos\theta dr-r\sin\theta d\theta)\wedge(\sin\theta dr+r\cos\theta d\theta)=r\cos^2\theta\, dr\wedge d\theta-r\sin^2\theta\, d\theta\wedge dr=rdr\wedge d\theta$.

The definition of orientability of a manifold that we used above is convenient for *defining* integration. It is inconvenient for everything else. The following criterion is useful.

Proposition 7.16. *An m-dimensional manifold M is orientable if and only if there is a form ν on M of degree m so that*

$$\nu_q\neq 0 \quad \textit{for all } q\in M.$$

Remark 7.17. A nowhere vanishing form of top degree on a manifold M as in Proposition 7.16 is called a *volume form.*

Proof of Proposition 7.16. Suppose M is orientable and $\{\phi_\alpha:U_\alpha\to\mathbb{R}^m\}$ is an atlas giving M an orientation. Let $\{\rho_\alpha\}$ be a partition of unity subordinate to the cover $\{U_\alpha\}$ of M. Define an m-form ν on M by

$$\nu=\sum\rho_\alpha(\phi_\alpha^*(dx_1\wedge\ldots\wedge dx_m))$$

We need to check that ν vanishes nowhere. Fix a point $q\in M$. Then $\rho_\alpha(q)\neq 0$ for finitely many α, say $\alpha_1,\ldots\alpha_k$. Therefore

$$\begin{aligned}((\phi_{\alpha_1}^{-1})^*\nu)_{\phi_{\alpha_1}(q)}&=\sum_{i=1}^k\Big((\phi_{\alpha_1}^{-1})^*(\rho_{\alpha_i}\phi_{\alpha_i}^*(dx_1\wedge\ldots\wedge dx_n))\Big)_{\phi_{\alpha_1}(q)}\\&=\sum_{i=1}^k\rho_{\alpha_i}(q)\Big((\phi_{\alpha_i}\circ\phi_{\alpha_1}^{-1})^*(dx_1\wedge\ldots\wedge dx_n)\Big)_{\phi_{\alpha_1}(q)}\\&=\left(\sum_{i=1}^k\rho_{\alpha_i}(q)\,\det\Big(d(\phi_{\alpha_i}\circ\phi_{\alpha_1}^{-1})_{\phi_{\alpha_1}(q)}\Big)\right)\ (dx_1\wedge\ldots\wedge dx_n)_{\phi_{\alpha_1}(q)}\neq 0\end{aligned}$$

since $\det\Big(d(\phi_{\alpha_i}\circ\phi_{\alpha_1}^{-1})_{\phi_{\alpha_1}(q)}\Big)>0$ and $\rho_{\alpha_i}(q)>0$.

Conversely suppose $\nu\in\Omega^m(M)$ is a volume form. We want to find an atlas $\{\phi_\alpha:U_\alpha\to\mathbb{R}^m\}$ so that

$$\det d(\phi_\alpha\circ\phi_\beta^{-1})_q>0$$

for all q and all α, β. Let $\{\psi_\beta:V_\beta\to\mathbb{R}^m\}$ be an arbitrary atlas on M. It is no loss of generality to assume that all the sets V_β are connected. Then for each index β

$$(\psi_\beta^{-1})^*\nu=f_\beta\,dx_1\wedge\cdots\wedge dx_m,$$

with $f_\beta(x)\neq 0$ for all $x\in\psi_\beta(V_\beta)$. Since V_β is connected f_β is either strictly positive or strictly negative. If $f_\beta>0$, keep the chart ψ_β. Otherwise replace it by $T\circ\psi_\beta$ where $T:\mathbb{R}^m\to\mathbb{R}^m$ is the diffeomorphism given by

$$T(x_1,x_2,\cdots,x_m)=(-x_1,x_2,\cdots,x_m).$$

□

Exercise 7.4. Suppose that M and N are orientable manifolds. Prove that their product $M\times N$ is orientable.

Exercise 7.5. Show that the tangent bundle TM is always orientable, regardless of whether or not the manifold M is.

Exercise 7.6. Evaluate $\int_S\omega|_S$ where S is the helicoid in $\mathbb{R}^3$ parameterized by $\phi(s,t)=(s\cos t, s\sin t, t)$, $0<s<1$, $0<t<4\pi$, and $\omega=z\ dx\wedge dy+3\ dz\wedge dx-x\ dy\wedge dz$. Use the orientation of S defined by ϕ (that is, $\phi^{-1}:S\to\mathbb{R}^2$ is a coordinate chart on S).

8. Vector bundles

Informally a vector bundle is a collection of vector spaces parameterized by points in a manifold. You have already seen two example: the tangent bundle and the cotangent bundle. Here is the formal definition.

Definition 8.1. A *real vector bundle E of rank k over a manifold M* is a manifold E together with a smooth map $\pi : E \to M$ so that

1. for each $x \in M$ the *fiber* $E_x := \pi^{-1}(x)$ is a real vector space of dimension k and
2. for all $x \in M$, there is an open neighborhood U of $x \in M$ and a diffeomorphism $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ such that $pr \circ \psi = \pi$, where pr is the natural projection from $U \times \mathbb{R}^k$ to U, $pr(q, v) = q$. That is, the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\psi} & U \times \mathbb{R}^k \\ & {\scriptstyle \pi} \searrow & \downarrow {\scriptstyle pr} \\ & & U \end{array}$$

commutes. Hence ψ maps the fiber E_y to $\{y\} \times \mathbb{R}^k$ for all $y \in U$. Additionally we require that the restrictions

$$\psi|_{E_y} : E_y \to \{y\} \times \mathbb{R}^k$$

are vector space isomorphisms for all $y \in U$.

Definition 8.2.

- The manifold E is called *the total space* of the bundle $\pi : E \to M$.
- The manifold M is called the *base* of the bundle $\pi : E \to M$.
- The maps $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ are called *local trivializations* of the bundle $\pi : E \to M$.

Example 8.3. The projection $\pi : M \times \mathbb{R}^k \to M$, $\pi(m, v) = m$ is a vector bundle of rank k.

Example 8.4. I claim that the tangent bundle $\pi : TM \to M$ is a vector bundle of rank $\dim M$. Let us construct local trivializations. Given a point $q \in M$ choose a coordinate chart $\phi = (x_1, \ldots, x_m) : U \to \mathbb{R}^m$ with $q \in U$. The map

$$\psi : TM|_U \equiv \pi^{-1}(U) \to U \times \mathbb{R}^m, \quad \psi(v) = (\pi(v), (dx_1(v), \ldots, dx_m(v))$$

is a local trivialization. Note that its inverse $\psi^{-1} : U \times \mathbb{R}^m \to TM|_U$ is given by

$$\psi^{-1}(p, (a_1, \ldots, a_m)) = \sum a_i \frac{\partial}{\partial x_i}\Big|_p.$$

Example 8.5. The cotangent bundle $T^*M \to M$ is also a vector bundle over M of rank $\dim M$.

It is useful to be able to say when two vector bundles are "the same."

Definition 8.6. Let $\pi_E : E \to M$ and $\pi_F : F \to M$ be two vector bundles over a manifold M. A smooth map $f : E \to F$ is a *vector bundle map* if $f(E_x) \subset F_x$ for all x and if the map $f|_{E_x} : E_x \to F_x$ is linear.

A vector bundle map $f : E \to F$ is an *isomorphism* of vector bundles if it is invertible and if $f^{-1} : F \to E$ is a vector bundle map.

Definition 8.7. A vector bundle $E \to M$ of rank k is *trivial* if there is a vector bundle isomorphism $E \to M \times \mathbb{R}^k$.

Example 8.8. The vector field $X = x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}$ on $\mathbb{R}^2$ is tangent to the unit circle S^1 and is not zero anywhere. Therefore the map $f : S^1 \times \mathbb{R} \to TS^1$, $f(q, t) = tX_q$ is an invertible vector bundle map. Convince yourself that the inverse $f^{-1} : TS^1 \to S^1 \times \mathbb{R}$ is smooth.

Exercise 8.1. Show that the tangent bundle of the n-torus $\mathbb{T}^n := \overbrace{S^1 \times \cdots \times S^1}^{n}$ is trivial.

Exercise 8.2. Let $E \to M$, $F \to M$ be two vector bundles over a manifold M. Show that if $f : E \to F$ is a vector bundle map and f has a set-theoretic inverse f^{-1}, then f^{-1} is a vector bundle map.

Hints. Consider first the case where E and F are trivial bundles: $E = M \times \mathbb{R}^k$, $F = M \times \mathbb{R}^l$. Prove that the map $\mathrm{inv} : \mathrm{GL}(\mathbb{R}, k) \to \mathrm{GL}(\mathbb{R}, k)$ given by $A \mapsto A^{-1}$ is smooth. Use trivializations to reduce everything to the trivial bundles case.

Exercise 8.3. If $\pi : E \to M$ is a vector bundle and $N \subset M$ is an embedded submanifold, show that $\pi : E|_N := \pi^{-1}(N) \to N$ is a vector bundle over N, called the *restriction* of E to N.

Hints: to show that $E|_N$ is a submanifold of E prove that $\pi : E \to M$ is transverse to N. The local trivializations of $E|_N$ are the restrictions of the local trivializations of E.

Example 8.9 (tautological line bundle)**.** (This is a sketch with no actual proofs.) Recall that the complex projective space $\mathbb{C}P^n$ is the set of all complex lines in $\mathbb{C}^{n+1}$. We identify lines $\mathbb{C}P^n$ with equivalence classes of nonzero vectors $[v]$. The equivalence relation is given by $v \simeq v'$ if and only if v and v' are collinear: $v = \lambda v'$ for some $0 \neq \lambda \in \mathbb{C}$. We define the *tautological complex line bundle* $\pi : L \to \mathbb{C}P^n$ as follows. We let

$$L = \{(l, v) \in \mathbb{C}P^n \times \mathbb{C}^{n+1} : v \in l\}.$$

and define $\pi : L \to \mathbb{C}P^n$ by

$$\pi(l, v) = l.$$

Thus the fiber $\pi^{-1}(l)$ consists of all vectors $v \in \mathbb{C}^{n+1}$ that lie on the line l, that is, of the complex line l itself. Hence the name. I claim that $\pi : L \to \mathbb{C}P^n$ is indeed a real vector bundle of rank 2 (2 because $\mathbb{C}$ is a 2-dimensional vector space over $\mathbb{R}$).

Why is L a manifold? The relationship $v \in l$ is really a collection of algebraic equations: if $v \in [w]$ then $(v_1, \ldots, v_{n+1}) = \lambda(w_1, \ldots, w_{n+1})$ for some $0 \neq \lambda \in \mathbb{C}$. Therefore $v_i/w_i = \lambda = v_j/w_j$ for all i and j and hence

$$v_j w_i = v_i w_j \quad \text{for all } i, j.$$

From this one can deduce that L is indeed a manifold (in other words I am not really giving you a proof). To construct local trivializations let

$$U_i = \{[w] \in \mathbb{C}P^n : w_i \neq 0\}.$$

Define $\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}$ by

$$\psi_i([w], v) = ([w], v_i).$$

8.1. **Sections.**

Definition 8.10 (Section)**.** A *section* $s : M \to E$ of a vector bundle $\pi : E \to M$ is a C^∞ map such that $\pi \circ s = id_M$. That is, a section is a smooth map from M to E such that $s(x) \in E_x$ for all x.

Notation. We denote the set of smooth sections of a bundle $\pi : E \to M$ by $\Gamma(E)$.

Example 8.11.

- The space of sections $\Gamma(TM)$ of the tangent bundle of a manifold M is the space of vector fields on a manifold M.
- The space of sections $\Gamma(T^*M)$ of the cotangent bundle of a manifold M is the space of 1-forms on a manifold M. That is, $\Gamma(T^*M) = \Omega^1(M)$.
- The space of sections $\Gamma(\Lambda^k(T^*M))$ of the kth exterior power of the cotangent bundle (which we have not constructed yet) is the space of k-forms $\Omega^k(M)$.
- The space of sections $\Gamma(M \times \mathbb{R})$ of the trivial bundle $M \times \mathbb{R} \to \mathbb{R}$ is the space of smooth maps of the form $m \mapsto (m, f(m))$, where $f : M \to R$ is a smooth function. Thus $\Gamma(M \times \mathbb{R})$ "is" $C^\infty(M)$.

Lemma 8.12. *Let $E \to M$ be a vector bundle over a manifold M. The space of sections $\Gamma(E)$ is a vector space over $\mathbb{R}$ under pointwise addition and multiplication by scalars. Moreover, if $s : M \to E$ is a section and $f \in C^\infty(M)$ is a function then we can define a new section $fs : M \to E$ by*

$$(fs)_q = f(q)s_q$$

for $q \in M$. Thus the space of sections $\Gamma(E)$ is a module over the space of functions $C^\infty(M)$.

Proof. The only possible worry is this: suppose $f, \bar{f} \in C^\infty(M)$ are two smooth functions and $s, \bar{s} \in \Gamma(E)$ are two smooth sections. Is the section $fs + \bar{f}\bar{s}$ smooth? Since every bundle is locally trivial and since smoothness is a local condition, we may assume that E is the trivial bundle $M \times \mathbb{R}^k \to M$. In this case $\Gamma(E)$ "is" the space of smooth maps from M to $\mathbb{R}^k$, and the lemma is clearly true for these maps — they do form a module over $C^\infty(M)$. □

Remark 8.13. The map that assigns to every point $q \in M$ the origin 0_q in the fiber E_q is smooth. It's called the *zero section* and is often denoted by 0.

Definition 8.14 (local section)**.** A *local section* of a vector bundle $\pi : E \to M$ is a section of $\pi^{-1}(U) =: E\Big|_U \to U$ for some open $U \subset M$. Equivalently, a local section is a smooth map $s : U \to E$ such that $\pi \circ s = id_U$.

Example 8.15. If $(U, x_1, \dots, x_n)$ is a coordinate chart on a manifold M, then for each index i, the map $q \mapsto \frac{\partial}{\partial x_i}\Big|_q$ is a local section of the tangent bundle $TM \to M$. Similarly $dx_i : U \to T^*M$ is a local section of the cotangent bundle.

Exercise 8.4. Let $E \to M$ be a vector bundle, $x \in M$ a point and $v \in E_x$. Show that there is a *global section* s with $s_x = v$.

8.2. **Frames and local frames.** We now address the issue raised in section 7: a k-form $\mu \in \Omega^k(M)$ on a manifold M is smooth if for any coordinate chart $(x_1, \dots, x_m) : U \to \mathbb{R}^m$ on M, we have

$$\mu = \sum_{|I|=k} a_I \, dx_I \quad \left(= \sum_{i_1 < \cdots < i_k} a_{i_1 \cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \right),$$

where a_I's are *smooth* functions on U. To this end we define frames on a vector bundle.

Definition 8.16. Let $E \to M$ be a vector bundle of rank k. A collection $s_1, \dots, s_k \in \Gamma(E)$ of sections is a *frame* of E if for each point $x \in M$ the vectors $\{s_1(x), \cdots, s_k(x)\}$ form a basis of the fiber E_x.

Similarly, a collection of local sections $s_1, \dots, s_k : U \to E$ is a *local frame* of E if for each point $x \in U$ the vectors $\{s_1(x), \cdots, s_k(x)\}$ form a basis of the fiber E_x.

Example 8.17. A nowhere zero vector field X on a circle S^1 is a frame of the tangent bundle $TS^1 \to S^1$.

Proposition 8.18. *A vector bundle $E \to M$ of rank k is trivial if and only if it has a frame of k sections $s_1, \dots, s_k \in \Gamma(E)$.*

Proof. Suppose that E is a trivial vector bundle over a manifold M. Then we have a global trivialization $\psi : \pi^{-1}(M) \to M \times \mathbb{R}^k$. Define

$$s_i(x) = \psi_x^{-1}(e_i)$$

where $\{e_1, \dots, e_k\}$ is the canonical basis for $\mathbb{R}^k$. Then the collection $\{s_1, \dots, s_k\}$ satisfies the desired properties.
Conversely, suppose that we have smooth sections $s_1, \dots, s_k$ that form a basis of E_x at every x. Then a global trivialization is given by

$$(q, v_1, \dots, v_k) \mapsto \sum_i v_i s_i(q).$$

□

Exercise 8.5. A section s of a vector bundle $E \to M$ is smooth if and only if for each point $q \in M$ there is a neighborhood U of q, a local frame $s_1, \dots, s_k : U \to E$ and smooth functions $f_1, \dots f_k \in C^\infty(U)$ so that

$$s = f_1 s_1 + \cdots + f_k s_k.$$

Exercise 8.6. Show that the discussion of smoothness of k-forms following (7.4) is correct: if $(x_1, \dots, x_m) : U \to \mathbb{R}^m$ is a coordinate chart on a manifold M, then $\{dx_1, \dots, dx_m\}$ is a local frame of T^*M over U. Hence

$$\{dx_I \mid |I| = k\}$$

is a local frame of the kth exterior power $\Lambda^k(T^*M)$ of the cotangent bundle. By Exercise 8.5, a section $\omega \in \Gamma(\Lambda^k(T^*M))$ is smooth on U if and only if there are smooth functions $a_I : U \to \mathbb{R}$ such that

$$\omega|_U = \sum a_I dx_I.$$

8.3. **Vector bundles via transition maps.** The goal of this section is to design a way of tearing vector bundles apart and then putting them back together in a new way. This will allow us to carry over the operations of direct sum $\oplus$, tensor $\otimes$, exterior power Λ^k, taking duals and so on from vector spaces to vector bundles.

Suppose that $\pi : E \to M$ is a vector bundle of rank k and $\{U_\alpha\}$ is a cover of M such that $E|_{U_\alpha}$ are trivial and let $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ denote the local trivializations. If $U_\alpha \cap U_\beta \neq \emptyset$, we have a map

$$\psi_\beta \circ \psi_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^k \to (U_\alpha \cap U_\beta) \times \mathbb{R}^k$$

Since the trivializations ψ_α maps fibers E_y linearly to fibers $\{y\} \times \mathbb{R}^k$ the composition $\psi_\beta \circ \psi_\alpha^{-1}$ maps linearly $\{y\} \times \mathbb{R}^k$ to $\{y\} \times \mathbb{R}^k$ for all $y \in U_\alpha \cap U_\beta$. Hence the map $\psi_\beta \circ \psi_\alpha^{-1}$ has to be of the form

$$\psi_\beta \circ \psi_\alpha^{-1}(y, v) = (y, \psi_{\beta\alpha}(y)v).$$

for some function

$$\psi_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(\mathbb{R}^k)$$

Note that $\psi_{\beta\alpha}$ is smooth, because for every basis vector e_j of $\mathbb{R}^k$, the map

$$q \mapsto \psi_{\beta\alpha}(q)e_j$$

is smooth. Such maps are called a *transition maps* for the bundle $\pi : E \to M$. It is not hard to see that the set of transition maps $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(\mathbb{R}^k)$ for the bundle $E \to M$ relative to the cover $\{U_\alpha\}$ satisfy the following three conditions called the *cocycle conditions*:

(1) $\psi_{\alpha\alpha} = id_{U_\alpha}$ for all α.
(2) $\psi_{\alpha\beta} \cdot \psi_{\beta\alpha} = id_{U_\alpha \cap U_\beta}$ for all pairs of indices α, β (the dot denotes the multiplications in $\mathrm{GL}(\mathbb{R}^k)$).
(3) $\psi_{\alpha\beta} \cdot \psi_{\beta\gamma} \cdot \psi_{\beta\gamma} = id_{U\alpha \cap U_\beta \cap U_\gamma}$ for all triples of indices α, β and γ.

Note that (2) implies that $\psi_{\beta\alpha} = \psi_{\alpha\beta}^{-1}$. The transition maps determine the vector bundle E.

Theorem 8.19. *Let M be a manifold, $\{U_\alpha\}$ an open cover, and $\{\psi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(\mathbb{R}^k)\}$ a collection of smooth maps satisfying the cocycle conditions. Then there is a vector bundle E over M of rank k with transition maps $\{\psi_{\alpha\beta}\}$.*

Sketch of proof. Consider the disjoint union $\overline{E}$ of the trivial bundles $U_\alpha \times \mathbb{R}^k$:

$$\overline{E} = \bigsqcup_\alpha (U_\alpha \times \mathbb{R}^k).$$

Define a relation on $\overline{E}$ by

$$U_\alpha \times \mathbb{R}^k \ni (q, v) \sim (q', v') \in U_\beta \times \mathbb{R}^k \text{ if and only if } q = q' \text{ and } \psi_{\beta\alpha}(v) = v'.$$

The cocycle conditions guaranty that $\sim$ is an equivalence relation. Let

$$E = \overline{E}/\sim,$$

and write $[q, v]$ for the equivalence class of (q, v). Define the projection $\pi : E \to M$ by

$$\pi([q, v]) = q.$$

Then

$$\pi^{-1}(U_\alpha) = \{[q, v] \mid (q, v) \in U_\alpha \times \mathbb{R}^k\}.$$

Define the trivializations $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ by

$$\psi_\alpha([q, v]) = (q, v).$$

It's not hard to check that the maps ψ_α are well-defined and that the corresponding transitions maps are the maps $\phi_{\alpha\beta}$ we started out with. It remains to check that E can be given the structure of a manifold so that all the maps in sight are smooth. But this is not bad. Let's examine what we have.

We have a topological space E covered by open sets $\pi^{-1}(U_\alpha)$. For each α we have a homeomorphism $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$. This suggests a way to get coordinate charts on our topological space E: compose homeomorphisms ψ_α with charts on $U_\alpha \times \mathbb{R}^k$. This gives us a cover of E by open sets and a collection of homeomorphism from these sets to open subsets of $\mathbb{R}^n$, where $n = \dim M + k$. This is an atlas because the maps

$$\psi_\beta \circ \psi_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^k \to (U_\alpha \cap U_\beta) \times \mathbb{R}^k, \quad (q, v) = (q, \psi_{\beta\alpha}(q)v).$$

are smooth. Note that here we are given that the maps $\psi_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(\mathbb{R}^k)$ are smooth and we are using it to conclude that $\psi_\beta \circ \psi_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^k \to (U_\alpha \cap U_\beta) \times \mathbb{R}^k$ are smooth. □

Remark 8.20. Naturally a different choice of a cover of M and a different choice of trivializations gives rise to a different collection of transition maps. And we should worry whether two different sets of data (open cover, transitions maps) give rise to the same bundle. But this would take us too far afield.

As a first application of Theorem 8.19 we construct the *direct sum* $E \oplus F \to M$ of two vector bundles $\pi_E : E \to M$ and $\pi_F : F \to M$ over a manifold M. The direct sum $E \oplus F$ (also known as Whitney sum) should be a vector bundle with with the fiber $(E \oplus F)_q = E_q \oplus F_q$ for $q \in M$. We define it by way of the transition maps.

Pick an open cover $\{U_\alpha\}$ of M such that $E|_{U_\alpha}$ and $F|_{U_\alpha}$ are trivial. Let $\psi^E_{\alpha\beta} : U\alpha \cap U_\beta \to \mathrm{GL}(\mathbb{R}^k)$ and $\psi^F_{\alpha\beta} : U\alpha \cap U_\beta \to \mathrm{GL}(\mathbb{R}^l)$ be the associated transition maps (thus E is of rank k and F is of rank l). Define the maps

$$\psi^{E\oplus F}_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(\mathbb{R}^k \oplus \mathbb{R}^l)$$

by

$$\psi^{E\oplus F}_{\alpha\beta}(q) = \left(\begin{array}{c|c} \psi^E_{\alpha\beta}(a) & 0 \\ \hline 0 & \psi^F_{\alpha\beta}(a) \end{array}\right).$$

It is not hard to check that the maps $\psi^{E\oplus F}_{\alpha\beta}$ are smooth and satisfy cocycle conditions. Therefore, by Theorem 8.19 there is a vector bundle $E \oplus F \to M$ with transition maps $\psi^{E\oplus F}_{\alpha\beta}$. Its fibers are isomorphic to the direct sum of the corresponding fibers of E and F.

It was worth reflecting on what made the construction above work. It is simply the fact that the map

$$\mathrm{GL}(\mathbb{R}^k) \times \mathrm{GL}(\mathbb{R}^l) \to \mathrm{GL}(\mathbb{R}^{k+l}), \quad (A, B) \to A \oplus B := \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right)$$

is smooth (as a map between open subspaces of $\mathbb{R}^{k^2} \times \mathbb{R}^{l^2}$ and $\mathbb{R}^{(k+l)^2}$) and the fact that under this map the compositions go to compositions:

$$(A \circ A') \oplus (B \circ B') = (A \oplus B) \circ (A' \oplus B').$$

There are many more examples of maps of this sort. For instance, consider the map that takes a matrix $A \in \mathrm{GL}(\mathbb{R}^k)$ to its inverse transpose:

$$(\ ^{-1})^* : A \mapsto (A^{-1})^* \in \mathrm{GL}((\mathbb{R}^k)^*).$$

The map is smooth (since the entries of the matrix $(A^{-1})^*$ are rational functions of the entries of the matrix A), and $((AB)^{-1})^* = (A^{-1})^*(B^{-1})^*$. Now let $E \to M$ be a vector bundle of rank k and $\{U_\alpha\}$ an open cover of M such that $E|_{U_\alpha}$ are trivial. Let $\psi_{\beta\alpha} : U_\beta \cap U_\alpha \to \mathrm{GL}(\mathbb{R}^k)$ be the corresponding transition maps. Then the maps $\psi^*_{\beta\alpha} : U_\beta \cap U_\alpha \to \mathrm{GL}((\mathbb{R}^k)^*)$ defined by

$$\psi^*_{\beta\alpha}(x) = \left((\psi_{\beta\alpha}(x))^{-1}\right)^*$$

are smooth and satisfy the cocycle conditions. By Theorem 8.19 there exists a vector bundle $E^* \to M$ whose transitions maps are precisely $\{\psi^*_{\beta\alpha}\}$. The bundle E^* is called the *dual bundle* of E. Its fibers E^*_q are vector spaces dual to the fibers E_q of E. We have seen this construction in one special case: the cotangent bundle T^*M is the dual bundle of the tangent bundle.

The maps $(A, B) \mapsto A \oplus B$ and $A \mapsto (A^{-1})^*$ are what is known as smooth functors. They allowed us to define direct sum of two bundles and the dual bundle, respectively. Here are a few more examples of the functors that will be very useful for us. Let V and W be finite-dimensional vector spaces, $A \in \mathrm{GL}(V)$ and $B \in \mathrm{GL}(W)$ over M. Then

- $(A,B) \mapsto A \otimes B \in \mathrm{GL}(V \otimes W)$
- $A \mapsto \Lambda^k(A) \in \mathrm{GL}(\Lambda^k(V))$ and
- $(A,B) \mapsto \mathrm{Hom}(A,B) \in \mathrm{GL}(\mathrm{Hom}(V,W)$, $\mathrm{Hom}(A,B)T := B \circ T \circ A^{-1}$

are smooth functors.[9] Indeed, pick bases of V and W. Then the entries of the matrix representing $A \otimes B$ are products of entries of matrices representing A and B. The entries of the matrix representing $\Lambda^k(A)$ are polynomial in the entries of the matrix representing A. [If this is confusing, work out the following simple example and you'll see what I mean. Let $V = \mathbb{R}^3$, $k = 2$ and compute $\Lambda^2(A)(e_i \wedge e_j)$ in terms of the basis $\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$.] Similarly the entries of matrix representing $\mathrm{Hom}(A,B)$ are polynomial in the entries of A and B. This allows us, given two vector bundles $E \to M$ and $F \to M$ to construct the bundles

- $E \otimes F \to M$
- $\Lambda^k(E) \to M$ and
- $\mathrm{Hom}(E,F) \to M$.

Exercise 8.7. Check that the bundle E^* and $\mathrm{Hom}(E, M \times \mathbb{R})$ are isomorphic.

Exercise 8.8. Let $E \to M$ and $F \to M$ be two vector bundles. Convince yourself that a section of $\mathrm{Hom}(E,F)$ "is" a vector bundle map from E to F.

Show that $E^* \otimes F$ is isomorphic to $\mathrm{Hom}(E,F)$.

Exercise 8.9. Compute transition maps for the tautological real line bundle $L \to \mathbb{R}P^n$:

$$L = \{(l,v) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} \mid v \in l\}.$$

Compute transition maps for $L \otimes L$. (Hint: write down the isomorphism $\mathbb{R} \otimes \mathbb{R} \to \mathbb{R}$.) Compute the transitions maps for $L^{\otimes k} := \overbrace{L \otimes \cdots \otimes L}^{k}$, $k > 1$.

Exercise 8.10. Let $\pi_E : E \to M$ and $\pi_F : F \to M$ be two vector bundles over M.
(a) Show that $E \times F$ is a vector bundle over $M \times M$.
(b) Explain why $G = \{(e,f) \in E \times F : \pi_E(e) = \pi_F(f)\}$ can be considered a vector bundle over M.
(c) Show that, as a vector bundle over M, G is isomorphic to the Whitney sum $E \oplus F$.

9. Exterior differentiation, contractions and Lie derivatives of forms

9.1. **Exterior differentiation.** In this section we first learn how to differentiate differential forms. We define an operator d of exterior differentiation that raises the degree of the form by 1. It is an generalization of div, grad and curl operators of vector calculus.

Theorem 9.1. *For every manifold M, there is a unique $\mathbb{R}$-linear operator*

$$d_M : \Omega^*(M) \to \Omega^{*+1}(M)$$

with the following properties :

1. *d_M raises the degrees by 1: $d_M(\Omega^k(M)) \subset \Omega^{k+1}(M)$;*
2. *$d_M f = df$ for all $f \in C^\infty(M)$, that is, d_M extends the operator d, which takes functions to 1-forms, to forms of arbitrary degree;*
3. *the operator d_M commutes with restrictions to open sets: for all open sets $U \subset M$ and all $\omega \in \Omega^*(M)$, $(d_M\omega)|_U = d_U(\omega|_U)$;*
4. *the operator d_M is a super-derivation: $d_M(\omega \wedge \eta) = (d_M\omega) \wedge \eta + (-1)^k \omega \wedge (d_M\eta)$ for $\omega \in \Omega^k(M), \eta \in \Omega^l(M)$;*
5. *$d_M \circ d_M = 0$.*

Remark 9.2. Note that any open set $U \subset M$ is a manifold, so the theorem asserts that there is an operator $d_U : \Omega^*(U) \to \Omega^*(U)$ with properties (1) – (5), hence property (3) makes sense.

[9] Note that $\Lambda^0(A) = 1 \in \mathrm{GL}(\Lambda^0(V)) = \mathrm{GL}(\mathbb{R})$.

Proof of Theorem 9.1. We prove uniqueness of the operator d_M first. We then construct the operator locally, on open sets. By uniqueness, these locally defined operators patch together into a global operator. This would prove existence.

Suppose the operator d_M with the desired properties exist. Fix a coordinate chart $(x_1, \ldots, x_m) : U \to \mathbb{R}^m$ on M. Then for all $\alpha \in \Omega^k(M)$, $\alpha|_U = \sum_{|I|=k} a_I dx_I$, where $a_I \in C^\infty(U)$ (cf. (7.2) and (7.3)). We claim that

$$(d_M\alpha)|_U = \sum_{|I|=k} da_I \wedge dx_I. \tag{9.1}$$

This would prove uniqueness since the right hand side is defined independently of d_M. We prove (9.1) in four steps. By property (3) of d_M,

$$(d_M\alpha)|_U = d_U(\alpha|_U).$$

By properties (2) and (5)

$$d_U(dx_i) = d_U(d_U x_i) = (d_U \circ d_U)x_i = 0.$$

Hence, by property (4)

$$d_U(dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}) = d_U(dx_{i_1}) \wedge (dx_{i_2} \wedge \cdots \wedge dx_{i_k}) - dx_{i_1} \wedge d_U(dx_{i_2} \wedge \cdots \wedge dx_{i_k})$$

Since $d_U(dx_i) = 0$, induction on k then gives:

$$d_U(dx_I) = d_U(dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}) = 0.$$

Hence for any multi-index I,

$$d_U(a_I\, dx_I) = da_I \wedge dx_I.$$

Linearity of d_U finishes the proof of (9.1).

To prove existence of d_M we run equation (9.1) backwards. Given a coordinate chart $(x_1, \ldots, x_m) : U \to \mathbb{R}^m$ on M we *define* an operator $d_U : \Omega^k(U) \to \Omega^{k+1}(U)$ by

$$d_U(\sum_{|I|=k} a_I dx_I) = \sum_{|I|=k} da_I \wedge dx_I \tag{9.2}$$

(in particular, if $k = 0$, then $d_U a = da$). Suppose, for the moment, that d_U defined by (9.2) satisfies properties (1) – (5). Then by uniqueness, for any two coordinate charts $(x_1, \ldots, x_m) : U \to \mathbb{R}^m$ and $(y_1, \ldots, y_m) : V \to \mathbb{R}^m$ and any k-form $\alpha \in \Omega^k(M)$

$$(d_U\alpha|_U)|_{U\cap V} = (d_V\alpha|_V)|_{U\cap V}.$$

Consequently $d_M : \Omega^*(M) \to \Omega^{*+1}(M)$, given by

$$(d_M\alpha)|_U = d_U(\alpha|_U)$$

for all coordinate charts U, is well-defined. Since d_Us have properties (1) – (5), so does d_M (check that).

It remain to prove that the map d_U given by (9.2) has the desired properties. Clearly d_U is $\mathbb{R}$-linear and raises degrees by 1. Property (2) holds by definition.

To prove (3) we want to show that for any open subset $W \subset U$ and any k-form $\alpha = \sum a_I dx_I \in \Omega^k(U)$

$$(d_U\alpha)|_W = d_W(\alpha|_W)$$

(Note that $(x_1, \ldots, x_m)|_W : W \to \mathbb{R}^m$ is also a coordinate chart). Let $j : W \hookrightarrow U$ denote the inclusion. For any smooth function $f \in C^\infty(U)$, $j^*f = f|_W$. Hence, by Exercise 7.1,

$$d(f|_W) = (df)|_W.$$

Therefore

$$\begin{aligned}(d_U\alpha)|_W &= \left(\sum da_I dx_I\right)|_W = \sum da_I|_W \wedge dx_I|_W \\ &= \sum d(a_I|_W) \wedge dx_I|_W = d_W\left(\sum a_I|_W\, dx_I|_W\right) \\ &= d_W(\alpha|_W).\end{aligned}$$

To prove (4) it's enough to show that for any $a_I\, dx_I \in \Omega^k(U)$ and any $b_J\, dx_J \in \Omega^*(U)$

$$d_U(a_I\, dx_I \wedge b_J\, dx_J) = d_U(a_I\, dx_I) \wedge b_J\, dx_J + (-1)^k a_I\, dx_I \wedge d_U(b_I\, dx_J). \tag{9.3}$$

We compute:

$$\begin{aligned}
d_U(a_I\,dx_I \wedge b_J\,dx_J) &= d_U(a_I b_J\,dx_I \wedge dx_J)\\
&= d(a_I b_J) \wedge dx_I \wedge dx_J\\
&= (b_J da_I + a_I db_J) \wedge dx_I \wedge dx_J\\
&= da_I \wedge dx_I \wedge b_J\,dx_J + (-1)^k a_I dx_I \wedge db_J \wedge dx_J\\
&= d_U(a_I\,dx_I) \wedge b_J\,dx_J + (-1)^k (a_I\,dx_I) \wedge d_U(b_J\,dx_J).
\end{aligned}$$

This proves (4). Similarly, if $\alpha = a_I dx_I \in \Omega^k(U)$ then

$$\begin{aligned}
d_U(d_U\alpha)) &= d_U(da_I \wedge dx_I)\\
&= d_U\left(\sum_{i=1}^m \frac{\partial a}{\partial x_i} dx_i \wedge dx_I\right)\\
&= \left(\sum_{i,j} \frac{\partial^2 a}{\partial x_j \partial x_i} dx_j \wedge dx_i\right) \wedge dx_I
\end{aligned}$$

Now, for $i = j$, $dx_i \wedge dx_j = 0$ so we are only summing over indices i and j with $i \neq j$. Each unordered pair i, j with $i \neq j$ contributes two terms to the sum: $\frac{\partial^2 a}{\partial x_j \partial x_i} dx_j \wedge dx_i$ and $\frac{\partial^2 a}{\partial x_i \partial x_j} dx_i \wedge dx_j$. These two terms cancel since $dx_j \wedge dx_i = -dx_j \wedge dx_i$ while the mixed partials commute. Therefore

$$d_U(d_U\alpha) = 0.$$

By linearity this is true for all k forms on the coordinate patch U. This proves property (5) and we are done. □

Notation. From now on we drop the subscript $_M$ from d_M and simply write d instead.

Example 9.3. The exterior derivative of a form is easy to compute: let $\alpha = dz + xdy$ be a 1-form on $\mathbb{R}^3$. Then

$$d\alpha = d(dz) + d(xdy) = 0 + dx \wedge dy = dx \wedge dy.$$

9.2. **Contractions of forms and vector fields.** To relate the exterior derivate operation to the standard calculus operation of div, grad and curl we need to define contractions of forms with vector fields.

Let u be a vector in a finite dimensional vector space V. Then u defines a linear map

$$\iota(u) : \Lambda^k(V^*) \to \Lambda^{k-1}(V^*)$$

by

$$(\iota(u)\eta)(v_1, \ldots, v_{k-1}) = \eta(u, v_1, \ldots, v_{k-1})$$

for any $\eta \in \Lambda^k(V^*)$ and any $v_1, \ldots, v_{k-1} \in V$. Here, of course, we think η as k-linear alternating maps from $V \times \cdots \times V$ to $\mathbb{R}$. We refer to $\iota(u)\eta$ as the *contraction* of u with η. Note that if $\eta \in \Lambda^1(V^*) = V^*$, then $\iota(u)\eta$ is simply the number $\eta(u)$. If $\eta \in \Lambda^0(V^*) = \mathbb{R}$, then we *define* $\iota(u)\eta := 0$ (and tacitly define $\Lambda^{-1}(V^*) = 0$).

Similarly, if $X \in \Gamma(TM)$ is a vector field on a manifold M and $\alpha \in \Omega^k(M)$ is a k form with $k > 0$ we define the *contraction of X with α* to be the $k-1$ form $\iota(X)\alpha$ given, for any point $q \in M$, by

$$(\iota(X)\alpha)_q = \iota(X_q)\alpha_q.$$

Here, on the right we are contracting a vector $X_q \in T_qM$ with $\alpha_q \in \Lambda((T_qM)^*)$. In particular, if α is a 1-form, $\iota(X)\alpha = \alpha(X)$. And again, if α is a 0-form, then $\iota(X)\alpha = 0$ (and the space of (-1)-forms is 0).

Example 9.4. Suppose $l_1, l_2 \in V^*$, so that $l_1 \wedge l_2 \in \Lambda^2(V^*)$. Let $u \in V$ be a vector. Then, for any $v \in V$,

$$\begin{aligned}
(\iota(u)(l_1 \wedge l_2))(v) &= (l_1 \wedge l_2)(u, v)\\
&= l_1(u)l_2(v) - l_1(v)l_2(u)\\
&= (l_1(u)l_2 - l_2(u)l_1)(v).
\end{aligned}$$

Hence

$$\iota(u)(l_1 \wedge l_2) = l_1(u)l_2 - l_2(u)l_1.$$

This example suggests a general way of computing contractions.

Lemma 9.5. *If $l_1, \dots, l_k \in V^*$, $u \in V$, then*

$$\iota(u)(l_1 \wedge \ldots \wedge l_k) = \sum_{j=1}^{k} (-1)^{j-1} (\iota(u) l_j)(l_1 \wedge \ldots \wedge \widehat{l_j} \wedge \ldots \wedge l_k),$$

where $\widehat{l_j}$ means that l_j is omitted from the expression.

Proof. For any $k-1$ vectors $v_1, \dots, v_{k-1} \in V$,

$$\begin{aligned}
(\iota(u) l_1 \wedge \ldots \wedge l_k)(v_1, \ldots, v_{k-1}) &= \det \begin{pmatrix} l_1(u) & l_1(v_1) & \ldots & l_1(v_{k-1}) \\ \vdots & & & \vdots \\ l_k(u) & l_k(v_1) & \ldots & l_k(v_{k-1}) \end{pmatrix} \\
&= \sum_{j=1}^{k} (-1)^{j-1} l_j(u) \det A_j \\
&= \sum_{j=1}^{k} (-1)^{j-1} l_j(u) (l_1 \wedge \ldots \wedge \widehat{l_j} \wedge \ldots \wedge l_k)(v_1, \ldots, v_{k-1}),
\end{aligned}$$

where A_j is the matrix obtained from the matrix

$$\begin{pmatrix} l_1(u) & l_1(v_1) & \ldots & l_1(v_{k-1}) \\ \vdots & & & \vdots \\ l_k(u) & l_k(v_1) & \ldots & l_k(v_{k-1}) \end{pmatrix}$$

by deleting the first column and jth row. □

Corollary 9.5.1. *Let V be a vector spaces, $u \in V$ a vector and $\alpha \in \Lambda^r(V^*)$ and $\beta \in \Lambda^s(V^*)$ be two exterior forms. Then*

$$\iota(u)(\alpha \wedge \beta) = (\iota(u)\alpha) \wedge \beta + (-1)^r \alpha \wedge (\iota(u)\beta).$$

Proof. It's enough to consider the case of $\alpha = l_1 \wedge \ldots \wedge l_r$ and $\beta = l_{r+1} \wedge \ldots \wedge l_{r+s}$ for some $l_1, \ldots, l_{r+s} \in V^*$. Then

$$\begin{aligned}
\iota(u)(\alpha \wedge \beta) &= \iota(u))(l_1 \wedge \ldots \wedge l_r \wedge l_{r+1} \wedge \ldots \wedge l_{r+s}) \\
&= \sum_{j=1}^{r+s} (-1)^{j-1} (\iota(u) l_j)(l_1 \wedge \ldots \wedge \widehat{l_j} \wedge \ldots \wedge l_{r+s}) \\
&= \left(\sum_{j=1}^{r} (-1)^{j-1} (\iota(u) l_j)(l_1 \wedge \ldots \wedge \widehat{l_j} \wedge l_r \right) \wedge l_{r+1} \wedge \ldots \wedge l_{r+s} \\
&+ l_1 \wedge \ldots \wedge l_r \wedge \left(\sum_{j=r+1}^{r+s} (-1)^{j-1} (\iota(u) l_j)(l_{r+1} \wedge \ldots \wedge \widehat{l_j} \wedge l_{r+s}) \right) \\
&= (\iota(u)\alpha) \wedge \beta + \alpha \wedge \left(\sum_{j'=1}^{r+s} (-1)^r (-1)^{j'-1} (\iota(u) l_{j'+r})(l_{r+1} \wedge \ldots \wedge \widehat{l}_{j'+r} \wedge \ldots \wedge l_{r+s} \right) \\
&= (\iota(u)\alpha) \wedge \beta + (-1)^r \alpha \wedge (\iota(u)\beta).
\end{aligned}$$

□

Corollary 9.5.2. *Let M be a manifold, $X \in \Gamma(TM)$ a vector field and $\alpha \in \Omega^r(V^*)$ and $\beta \in \Omega^s(V^*)$, be two differential forms. Then*

$$\iota(X)(\alpha \wedge \beta) = (\iota(X)\alpha) \wedge \beta + (-1)^r \alpha \wedge (\iota(X)\beta).$$

Example 9.6. Let $W = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$ be a vector field on $\mathbb{R}^3$ and let $\omega = dx \wedge dy \wedge dz$ (ω is the standard volume form on $\mathbb{R}^3$). Then

$$\begin{aligned}\iota(W)\omega &= \iota(W)(dx \wedge dy \wedge dz)\\ &= dx(W)\, dy \wedge dz - dy(W)\, dx \wedge dz + dz(W)\, dx \wedge dy\\ &= x\, dx \wedge dy - y\, dy \wedge dz + z\, dx \wedge dy\end{aligned}$$

Exercise 9.1. In $\mathbb{R}^3$, the standard inner product $(\cdot, \cdot)$ defines an isomorphism $\mathbb{R}^3 \to (\mathbb{R}^3)^*$, $v \mapsto (v, \cdot)$, which in turn induces an isomorphism of spaces of sections

$$A : \Gamma(T\mathbb{R}^3) \to \Omega^1(\mathbb{R}^3), \quad A(X) = (X, \cdot).$$

The standard volume form $\mu = dx_1 \wedge dx_2 \wedge dx_3$ defines an isomorphism $\mathbb{R}^3 \to \Lambda^2((\mathbb{R}^3)^*)$ by $v \mapsto \iota(v)\mu$, which also induces an isomorphism

$$B : \Gamma(T\mathbb{R}^3) \mapsto \Omega^2(\mathbb{R}^3) \quad B(X) = \iota(X)\mu.$$

Finally, the map

$$C : C^\infty(\mathbb{R}^3) \to \Omega^3(\mathbb{R}^3) \quad C(f) = f\mu$$

is also an isomorphism. (Check these facts!)

Show that the standard vector calculus notions of div, grad, and curl can be defined as

(1) $\mathrm{grad}(f) = A^{-1}(df)$ for any smooth function f on $\mathbb{R}^3$.
(2) $\mathrm{curl}(X) = B^{-1}(d(A(X)))$ for any vector field X on $\mathbb{R}^3$.
(3) $\mathrm{div}(X) = C^{-1}(d(B(X)))$ for any vector field X on $\mathbb{R}^3$.

9.3. **Lie derivatives of differential forms.** In order to understand divergence of a vector field on a manifold we need to define Lie derivatives of differential forms. This is fairly easy to do, but then the definition is hard to compute with. Cartan's formula makes computation of Lie derivatives of forms easy, but it requires understanding of the interaction between exterior differentiation and pull-backs. Which is why we address the pull-backs first.

Lemma 9.7. *Exterior differentiation d commutes with pull-backs. That is to say, let $F : M \to N$ be a smooth map between two manifold and $\alpha \in \Omega^*(N)$ a differential form. Then*

$$d(F^*\alpha) = F^*(d\alpha). \tag{9.4}$$

Proof. We know that equation (9.4) holds if α is a zero-form, that is, a function (cf. Exercise 7.1).

We now argue that for any coordinate chart $(x_1, \ldots, x_n) : U \to \mathbb{R}^n$ on N and for any k-form α on N, $k > 1$, we have

$$(F^*(d\alpha))|_{F^{-1}(U)} = d\left(F^*\alpha|_{F^{-1}(U)}\right). \tag{9.5}$$

Equation 9.5 is enough to prove the lemma. Now, $(F^*d\alpha)|_{F^{-1}(U)} = F^*(d\alpha|_U)$ and $\alpha|_U = \sum_{|I|=k} a_I dx_I$ for all multi-indices I of size k and some functions $a_I \in C^\infty(U)$. Therefore

$$d\alpha|_U = \sum_I da_I \wedge dx_I,$$

and

$$(F^*(d\alpha))|_{F^{-1}(U)} = \sum_I F^*da_I \wedge F^*dx_I.$$

Since (9.4) holds for functions,

$$F^*da_I = d(F^*a_I).$$

Similarly,

$$F^*dx_I = d(F^*x_{i_1}) \wedge \ldots \wedge d(F^*x_{i_k})$$

for all $I = (i_1, \ldots, i_k)$. Therefore

$$(F^*(d\alpha))|_{F^{-1}(U)} = \sum_I d(F^*a_I) \wedge d(F^*x_{i_1}) \wedge \ldots \wedge d(F^*x_{i_k})$$

We now argue that the right hand side of the equation above is $d\left((F^*\alpha)|_{F^{-1}(U)}\right)$. Properties (4) and (5) of the exterior derivative d and induction on k shows that for any k functions $f_1, \ldots f_k$,

$$d(df_1 \wedge \ldots \wedge df_k) = 0.$$

Hence for any functions $f_0, f_1, \ldots, f_k$,

$$d(f_0 df_1 \wedge \ldots \wedge df_k) = df_0 \wedge df_1 \wedge \ldots \wedge df_k.$$

In particular,

$$d(F^*a_I) \wedge d(F^*x_{i_1}) \wedge \ldots \wedge d(F^*x_{i_k}) = d\Big(F^*a_I\ d(F^*x_{i_1}) \wedge \ldots \wedge d(F^*x_{i_k})\Big) = d\left(F^*(a_I dx_{i_1} \wedge \ldots \wedge dx_{i_k})\right).$$

Therefore,

$$(F^*(d\alpha))|_{F^{-1}(U)} = \sum_I d\Big(F^*\left(a_I dx_{i_1} \wedge \ldots \wedge dx_{i_k}\right)\Big) = d(F^*\left(\sum_I a_I dx_I\right)) = d\left(F^*(\alpha|_U)\right) = d(F^*\alpha|_{F^{-1}(U)})$$

and we are done. □

Definition 9.8. Let X be a vector field on a manifold M and $\omega \in \Omega^k(M)$ a k-form. Let ϕ_t denote the local flow of X. The *Lie derivative* $L_X\omega$ of ω with respect to X is defined by

$$(L_X\omega)_q = \frac{d}{dt}\Big|_t (\phi_t^*\omega)_q$$

for any point $q \in M$.

Note that by definition of the flow ϕ_t, the Lie derivative of a 0-form $f \in C^\infty(M)$ is

$$\begin{aligned}(L_X f)(q) &= \frac{d}{dt}\Big|_{t=0} (\phi_t^* f)(q) \\ &= \frac{d}{dt}\Big|_{t=0} (f \circ \phi_t)(q) = X_q(f)\end{aligned}$$

As was mentioned above, the goal of this subsection is to prove Cartan's formula for Lie derivatives.

Theorem 9.9 (Cartan's Formula)**.** *Suppose that X is a vector field on a manifold M and $\omega \in \Omega^*(M)$ a differential form. Then*

$$L_X\omega = d(\iota(X)\omega) + \iota(X)d\omega.$$

We prove the theorem in a sequence of lemmas.

Lemma 9.10. *Let X be a vector field on a manifold M. The Lie derivative L_X is a derivation on the space of forms $\Omega^*(M)$ which commutes with the exterior differentiation d. That is to say,*

1. *$L_X : \Omega^*(M) \to \Omega^*(M)$ is $\mathbb{R}$-linear.*
2. *$L_X(\omega \wedge \eta) = (L_X\omega) \wedge \eta + \omega \wedge (L_X\eta)$ for all $\omega, \eta \in \Omega^*(M)$.*
3. *$L_X(d\omega) = d(L_X\omega)$ for all $\omega \in \Omega^*(M)$*

Proof. The first property of the Lie derivative is easy to see: pull-backs and differentiation are both linear. Let us prove (2). Since pull-back respects exterior multiplication,

$$\frac{d}{dt}\Big|_{t=0} (\phi_t^*(\omega \wedge \eta)) = \frac{d}{dt}\Big|_{t=0} (\phi_t^*\omega) \wedge (\phi_t^*\eta).$$

Since exterior multiplication is bilinear,

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\Big((\phi_t^*\omega) \wedge (\phi_t^*\eta)\Big) &= (\frac{d}{dt}\Big|_{t=0}\phi_t^*\omega) \wedge (\phi_0^*\eta) + (\phi_0^*\omega) \wedge \frac{d}{dt}\Big|_{t=0}(\phi_t^*\eta) \\ &= (L_X\omega) \wedge \eta + \omega \wedge (L_X\eta).\end{aligned}$$

This proves that the Lie derivative is a derivation. We now prove that it commutes with the exterior multiplication. For any form ω

$$\begin{aligned} L_X(d\omega) &= \frac{d}{dt}\Big|_{t=0}(\phi_t^*(d\omega)) \\ &= \frac{d}{dt}\Big|_{t=0} d(\phi_t^*\omega) \\ &= d(\frac{d}{dt}\Big|_{t=0}\phi_t^*\omega) \quad \text{(since mixed partials commute)} \\ &= d(L_X\omega). \end{aligned}$$

□

Lemma 9.11. *Let X be a vector field on a manifold M. Let*

$$Q = d\iota(X) + \iota(X)d : \Omega^*(M) \to \Omega^*(M).$$

The operator Q is also a derivation on the space of forms $\Omega^*(M)$ which commutes with the exterior differentiation d.

Proof. It's clear that Q is $\mathbb{R}$-linear. We check that Q commutes with d:

$$\begin{aligned} Q \circ d &= d\ \iota(X)\ d + \iota(X)\ d\ d \\ &= d\ \iota(X)\ d \quad \text{(since } d \circ d = 0) \\ &= d\ d\ \iota(X) + d\ \iota(X)\ d = d \circ Q. \end{aligned}$$

Now we need to check that Q is a derivation. Accordingly, let $\omega \in \Omega^k(M)$, $\eta \in \Omega^l(M)$ be two forms on M. Then

$$\begin{aligned} Q(\omega \wedge \eta) &= d(\iota(X)(\omega \wedge \eta)) + \iota(X)(d(\omega \wedge \eta)) \\ &= d[(\iota(X)\omega) \wedge \eta + (-1)^k \omega \wedge (\iota(X)\eta)] + \iota(X)[d\omega \wedge \eta + (-1)^k \omega \wedge d\eta] \\ &= d(\iota(X)\omega) \wedge \eta + (-1)^{k-1}(\iota(X)\omega) \wedge d\eta + (-1)^k d\omega \wedge \iota(X)\eta + (-1)^k(-1)^k \omega \wedge d\iota(X)\eta \\ &+ (\iota(X)d\omega) \wedge \eta + (-1)^{k+1} d\omega \wedge \iota(X)\eta + (-1)^k(\iota(X)\omega) \wedge d\eta + (-1)^k(-1)^k \omega \wedge \iota(X)d\eta \\ &= Q(\omega) \wedge \eta + \omega \wedge Q(\eta) \end{aligned}$$

□

Proof of Cartan's formula. If $f \in \Omega^0(M)$ is a function, then $\iota(X)f = 0$ by definition. Hence

$$Q(f) = (\iota(X)d)f = \iota(X)\, df = df(X),$$

while

$$L_X f = X(f) = df(X).$$

We conclude that L_X and Q agree on functions.

To prove Cartan's formula it is enough to prove

$$(L_X\omega)|_U = (Q\omega)|_U,$$

where $(x_1, \ldots, x_m) : U \to \mathbb{R}^m$ be a coordinate chart. But both L_X and Q commute with restrictions, so it's enough to prove that

$$L_X(\omega|_U) = Q(\omega|_U).$$

Thus, we may further assume that $\omega = a_I dx_I = a_I dx_{i_1} \wedge \ldots \wedge dx_{i_n}$ for some function a_I and multi-index I. Both the Lie derivative L_X and Q are derivations that commute with d, so

$$\begin{aligned} L_X(a_I dx_{i_1} \wedge \ldots \wedge dx_{i_n}) &= (L_X a_I) d(L_X x_{i_1}) \wedge \ldots \wedge (L_X dx_{i_n}) \\ &= (Q a_I) d(Q x_{i_1}) \wedge \ldots \wedge d(Q x_{i_n}) \\ &= Q(a_I dx_{i_1} \wedge \ldots \wedge dx_{i_n}). \end{aligned}$$

□

Exercise 9.2. Let M be an orientable m-dimensional manifold and $\mu \in \Omega^m(M)$ a nowhere zero form of top degree. Show that for any vector field X on M the Lie derivative $L_X\mu$ satisfies

$$L_X\mu = f\mu$$

for some function $f \in C^\infty(M)$, which depends on X. We define the *divergence* of X with respect to μ to be this function f and denote it by $\mathrm{div}_\mu(X)$. Thus,

$$L_X\mu = \mathrm{div}_\mu(X)\,\mu.$$

Show that for $M = \mathbb{R}^m$ and $\mu = dx_1 \wedge \ldots \wedge dx_m$

$$\mathrm{div}_\mu(\sum_i v^i \frac{\partial}{\partial x_i}) = \sum_i \frac{\partial v^i}{\partial x_i}.$$

Exercise 9.3. Consider polar coordinates (r, θ) on $\mathbb{R}^2$. The "function" θ is defined up to a constant. Show that $d\theta$ is a well-defined 1-form on $\mathbb{R}^2 - \{0\}$ and that

$$d\theta = \frac{1}{x^2 + y^2}(x\ dy - y\ dx).$$

Exercise 9.4. (1) Consider the two-form $\omega = x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2$ in $\mathbb{R}^3$. Compute $d\omega$.
(2) Compute

$$\iota(\sum_{i=1}^3 x_i \frac{\partial}{\partial x_i}) dx_1 \wedge dx_2 \wedge dx_3.$$

(3) Compute $L_X(dx_1 \wedge dx_2 \wedge dx_3)$, where $X = \sum_{i=1}^3 x_i \frac{\partial}{\partial x_i}$.

Exercise 9.5. Consider $k : \mathbb{R}^2 \to \mathbb{R}^2$ given by $(u, v) \mapsto (u^2 + 1, uv)$. Compute $k^*\Big((xy - y)dx \wedge dy\Big)$.

Exercise 9.6. Let X and Y be vector fields and α a 1-form on a manifold M. Prove that
1) $L_X(\iota(Y)\alpha) = \iota(Y)(L_X\alpha) + \alpha(L_XY)$.
2) Using (1), show that $d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y])$.

9.4. **de Rham cohomology.** One of the most interesting applications of Cartan's formula is the proof of smooth homotopy invariance of de Rham cohomology. We start by defining de Rham cohomology.

Definition 9.12. Let M be a manifold. A form $\alpha \in \Omega^k(M)$ is *closed* if $d\alpha = 0$. A form $\beta \in \Omega^k(M)$ is *exact* if there is a $k - 1$ form γ with $\beta = d\gamma$.

Note that since $d^2 = 0$, any exact form is closed. The converse need not be true. The difference between the spaces of closed and exact forms is measured by the de Rham cohomology.

Definition 9.13. Let M be a manifold. The kth de Rham cohomology $H^k(M)$ is defined by

$$\begin{aligned} H^k(M) :&= \{\text{closed } k\text{-forms}\}/\{\text{exact } k\text{-forms}\} \\ &= \ker(d : \Omega^k(M) \to \Omega^{k+1}(M)) \,/\, \mathrm{Im}(d : \Omega^{k-1}(M) \to \Omega^k(M)). \end{aligned}$$

$H^k(M)$ is a vector space over the reals. Thus $H^k(M)$ is the space of equivalence classes $[\alpha]$ of closed k-forms: two closed k-forms α and α' are equivalent if and only if $\alpha - \alpha' = d\gamma$ for some $k - 1$ form γ.

Remark 9.14. By definition $\Omega^{-1}(M) = 0$ so

$$\begin{aligned} H^0(M) &= \{f \in C^\infty(M) \mid df = 0\} \\ &= \text{ locally constant functions on } M \\ &= \mathbb{R}^k, \end{aligned}$$

where k is the number of connected components of M. In particular $H^0(\text{point}) = \mathbb{R}$.

Definition 9.15. We define the de Rham cohomology $H^*(M)$ to be the direct sum of the de Rham cohomology groups:

$$H^*(M) := H^0(M) \oplus \cdots \oplus H^k(M) \oplus \cdots$$

It takes a bit of work to compute the de Rham cohomology of just about anything. Here is an important, but not very exciting, example of a computation directly from the definition.

Example 9.16. Let M be a connected zero dimensional manifold, that is, a single point. Then $\Omega^k(M) = 0$ for $k > 0$. Hence $H^k(M) = 0$ for $k > 0$. On the other hand $H^0(M) = \mathbb{R}$ since a point has one connected component.

Lemma 9.17. *The de Rham cohomology $H^*(M)$ has a well-defined the multiplication given by*

$$[\alpha] \wedge [\beta] := [\alpha \wedge \beta],$$

which makes $H^*(M)$ into a ring.

Proof. We need to show that the space of exact forms is an ideal in the algebra of closed forms. That is, if $d\alpha = 0$ then $d\beta \wedge \alpha$ is exact for any β. But

$$d(\beta \wedge \alpha) = d\beta \wedge \alpha \pm \beta \wedge d\alpha = d\beta \wedge \alpha + 0,$$

and we are done. □

Lemma 9.18. *Let $F : M \to N$ be a smooth map. Then for each positive integer k the pull-back map*

$$F^* : \Omega^*(N) \to \Omega^*(M)$$

gives rise to a well-defined ring homomorphism.

$$F^* : H^*(N) \to H^*(M), \quad F^*[\alpha] := [F^*\alpha].$$

Moreover, if $id_M : M \to M$ is the identity map then $id_M^* : H^*(M) \to H^*(M)$ is also the identity map. Additionally, for any two maps $F : M \to N$, $G : N \to Z$ we have

$$(G \circ F)^* = F^* \circ G^*.$$

Proof. If $d\alpha = 0$, then $dF^*\alpha = F^*d\alpha = F^*0 = 0$. Therefore F^* maps closed forms to closed forms. For the same reason, F^* maps exact forms to exact forms. Consequently the pullback on forms gives rise to a well-defined pullback of cohomology classes. Since

$$F^*(\alpha \wedge \beta) = (F^*\alpha) \wedge (F^*\beta),$$

the map on cohomology is a ring homomorphism. The rest of the lemma is left as an exercise. □

Definition 9.19 (Homotopy)**.** Two maps $f_1, f_0 : M \to N$ of manifolds are (smoothly) *homotopic* if there is a smooth map

$$F : (a, b) \times M \to N,$$

where (a, b) is an open interval containing $[0, 1]$, so that

$$F(0, x) = f_0(x) \quad \text{for all } x \in M \quad \text{and}$$
$$F(1, x) = f_1(x) \quad \text{for all } x \in M.$$

Example 9.20. The maps $f_1 : \mathbb{R}^n \to \mathbb{R}^n$, $f_1(x) = x$ and $f_0 : \mathbb{R}^n \to \mathbb{R}^n$, $f_0(x) = 0$ are smoothly homotopic: let $F(t, x) = tx$.

Lemma 9.21 (homotopy invariance of de Rham cohomology)**.** *If two smooth maps $f_1, f_0 : M \to N$ are homotopic, then $f_0^*, f_1^* : H^*(N) \to H^*(M)$ are the same map:*

$$f_0^*[\alpha] = f_1^*[\alpha] \quad \textit{for all } [\alpha] \in H^*(N).$$

To prove Lemma 9.21, we need the following simple observation.

Lemma 9.22. *Let $\{\phi_t\}$ denote the flow of a vector field X on a manifold M. For any k-form α on M,*

$$\left.\frac{d}{dt}\right|_{t=\tau} \phi_t^*\alpha = \phi_\tau^*(L_X\alpha).$$

Proof. For any map $f : M \to M$,

$$\frac{d}{dt}\Big|_{t=0} f^* \phi_t^* \alpha = f^* \left(\frac{d}{dt}\Big|_{t=0} \phi_t^* \alpha \right)$$

since for any point $q \in M$ the map $\Lambda(df_q^*) : \Lambda^k(T^*_{f(q)}M \to \Lambda^k(T_q^* M)$ is linear. Therefore

$$\frac{d}{dt}\Big|_{t=\tau} \phi_t^* \alpha = \frac{d}{dt}\Big|_{t=0} \phi_{\tau+t}^* \alpha = \frac{d}{dt}\Big|_{t=0} \phi_\tau^*(\phi_t^* \alpha) = \phi_\tau^* \left(\frac{d}{dt}\Big|_{t=0} \phi_t^* \alpha \right) = \phi_\tau^*(L_X \alpha).$$

□

Proof of Lemma 9.21. Let $F : (a, b) \times M \to N$ denote the homotopy between f_1 and f_0. It is no loss of generality to assume that the interval (a, b) is all of $\mathbb{R}$. (If (a, b) is not all of $\mathbb{R}$, let $\rho : \mathbb{R} \to [0, 1]$ be a smooth function with $\operatorname{supp} \rho \subset (a, b)$ and $\rho|_{[0,1]} = 1$. Define the map $\bar{F} : \mathbb{R} \times M \to N$ by

$$\bar{F}(t, x) = \begin{cases} F(\rho(t)t, x) & t \in (a, b) \\ F(0, x) & t \notin (a, b) \end{cases}$$

The map $\bar{F}$ is a homotopy between f_1 and f_0.) Let $i_0 : M \hookrightarrow \mathbb{R} \times M$ denote the embedding given by

$$i_0(x) = (0, x)$$

and let $\phi_t : \mathbb{R} \times M \to \mathbb{R} \times M$ be given by

$$\phi_t(s, x) = (s + t, x).$$

Then $f_1 = F \circ \phi_1 \circ i_0$ and $f_0 = F \circ \phi_0 \circ i_0$. Therefore, since $f_t^* = i_0^* \circ \phi_t^* \circ F^*$, f_1^* and f_0^* are the same map on cohomology if and only if $\phi_1, \phi_0 : \mathbb{R} \times M \to \mathbb{R} \times M$ induce the same map in cohomology. The collection of maps $\{\phi_t\}$ is the flow of the vector field $X = \frac{\partial}{\partial t}$ on $\mathbb{R} \times M$. Therefore, for any k-form $\alpha \in \Omega^k(\mathbb{R} \times M)$,

$$\begin{aligned} \phi_1^* \alpha - \phi_0^* \alpha &= \int_0^1 \frac{d}{dt}(\phi_t^* \alpha)\, dt \\ &= \int_0^1 \phi_t^*(L_X \alpha)\, dt = \int_0^1 \phi_t^*(d\iota(X) + \iota(X) d)\alpha\, dt \\ &= d(\int_0^1 \phi_t^*(\iota(X)\, \alpha)\, dt) + \int_0^1 \phi_t^*(\iota(X)\, d\alpha)\, dt \\ &= d\kappa(\alpha) + \kappa(d\alpha), \end{aligned}$$

where

$$\kappa(\alpha) := \int_0^1 \phi_t^* \left(\iota(X)\, \alpha\right) dt.$$

Therefore, for any $\alpha \in \Omega^k(\mathbb{R} \times M)$ with $d\alpha = 0$,

$$\phi_1^* \alpha - \phi_0^* \alpha = d(\kappa(\alpha)).$$

Hence

$$[\phi_1^* \alpha] = [\phi_0^* \alpha]$$

and we are done. □

Corollary 9.22.1 (Poincaré lemma)**.**

$$H^k(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & k > 0 \end{cases}$$

Proof. Let $\imath : \{0\} \to \mathbb{R}^n$ be the inclusion and $p : \mathbb{R}^n \to \{0\}$ be the map that sends every point to 0. We want to show that $p^* : H^*(\{0\}) \to H^*(\mathbb{R}^n)$ is an isomorphism. It's enough to show that $\imath_0^* : H^*(\mathbb{R}^n) \to H^*(\{0\})$ and p^* are the inverses of each other. Define $f_t : \mathbb{R}^n \to \mathbb{R}^n$ by

$$f_t(x) = tx.$$

The $F(t, x) = f_t(x)$ is a homotopy between f_0 and f_1. Hence $f_0^* = f_1^*$ as maps on $H^*(\mathbb{R}^n)$. Moreover,

$$p \circ \imath = id_{\{0\}} \quad \text{and} \quad \imath \circ p = f_0.$$

Therefore

$$id_{H^*(\{0\})} = (p \circ \imath)^* = \imath^* \circ p^*$$

and

$$id_{H^*(\mathbb{R}^n)} = f_1^* = f_0^* = (\imath \circ p)^* = p^* \circ \imath^*.$$

Therefore p^* and $\imath^*$ *are* inverses of each other, and $H^*(\{0\})$ and $H^*(\mathbb{R}^n)$ are isomorphic. □

10. Stokes's theorem

There are two slightly different (but equivalent) ways of stating Stokes's theorem: for manifolds with boundary and for regular domains. Recall that manifolds are locally homeomorphic to open subsets of $\mathbb{R}^n$. *Manifolds with boundary* are locally homeomorphic to opens subsets of the half-space

$$H^n := \{x \in \mathbb{R}^n \mid x_1 \leq 0\}.$$

Technically it is slightly easier to work with regular domains, which is what we will do. Any regular domain is a manifold with boundary. And conversely, any manifold with boundary is a regular domain in some larger manifold. We will not prove the last two statements.

Definition 10.1. Let M be a manifold of dimension m. A closed subset $D \subset M$ is a *regular domain* (or alternatively, a *domain with smooth boundary*) if for any point $p \in D$, there is a coordinate chart $\phi = (x_1, \ldots, x_m) : U \to \mathbb{R}^m$ on M such that $p \in U$ and

$$\phi(U \cap D) = \phi(U) \cap \{x \in \mathbb{R}^n \mid x_1 \leq 0\} = \phi(U) \cap H^n.$$

Such a chart ϕ is *adapted* to the domain D.

Example 10.2. The unit disk

$$D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$$

is a regular domain in $\mathbb{R}^2$. For example, if $p = (1, 0)$, we may take $U = \{(x, y) \mid x \geq 0\}$ and $\phi(x, y) = (x - \sqrt{1 - y^2}, y)$.

Recall that the *interior* $int(Y)$ of a subset Y of a topological space X is the union of all open subsets of X which are contained in Y. We define the *boundary* ∂D of a regular domain D in a manifold M to be the points in D that are *not* in the interior of D. Alternatively, $q \in \partial D$ if and only if any open set containing q contains points of D and points of $M \smallsetminus D$.

Lemma 10.3. *Let D be a regular domain in a manifold M. The boundary ∂D is a submanifold of M of codimension 1.*

Proof. Let $\phi : U \to \mathbb{R}^m$ be a chart adapted to D. The $\phi(U \cap int(D)) \subset \{x \in \mathbb{R}^m \mid x_1 < 0\}$ and $\phi(U \cap \partial D) \subset \{x \in \mathbb{R}^m \mid x_1 = 0\}$. If $\psi : V \to \mathbb{R}^m$ is another coordinate chart adapted to D, then ψ also maps $V \cap int(D)$ to an open subset of $\{x \in \mathbb{R}^m \mid x_1 < 0\}$ and $V \cap \partial D$ to an open subset of $\{x \in \mathbb{R}^m \mid x_1 = 0\}$. Therefore

$$\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$$

maps smoothly $\phi(U \cap V \cap \partial D) = \phi(U \cap V) \cap \{x_1 = 0\}$ to $\psi(U \cap V) \cap \{x_1 = 0\}$.

It follows that the collection of charts $\{\phi_\alpha : U_\alpha \to \mathbb{R}^m\}$ of M which are adapted to D give rise to an atlas $\{\phi_\alpha|_{\partial D} : \partial D \cap U_\alpha \to \{x_1 = 0\} \simeq \mathbb{R}^{m-1}\}$ on ∂D. □

Lemma 10.4. *If D is a regular domain in an orientable manifold M then $int(D)$ and ∂D are orientable.*

Proof. An open subset of an orientable manifold is orientable. Hence $int(D)$ is orientable. We now address the orientability of ∂D. If $\phi : U \to \mathbb{R}^m$, $\psi : V \to \mathbb{R}^m$ are two charts adapted to D, then $\psi \circ \phi^{-1}$ maps the $\phi(U \cap V) \cap \{x_1 < 0\}$ to $\psi(U \cap V \cap \{x_1 < 0\}$. Therefore, at the points of $\phi(U \cap V) \cap \{x_1 = 0\}$ the differential

$d(\psi \circ \phi^{-1})$ maps the vectors that point into $\{x_1 < 0\}$ to vectors that point into $\{x_1 < 0\}$. In other words, at a point $q = (0, x_2, \ldots, x_{m-1})$ the differential has the form

$$d(\psi \circ \phi^{-1})_q = \left(\begin{array}{c|ccc} a & 0 & \ldots\ldots\ldots & 0 \\ \hline \vdots & & & \\ \vdots & & d(\psi \circ \phi^{-1})|_{\{0\}\times\mathbb{R}^{m-1}} & \\ \vdots & & & \end{array} \right)$$

for some smooth function $a = a(x_2, \ldots, x_{m-1}) > 0$. Hence if $\det d(\psi \circ \phi^{-1})_q > 0$ then $\det d(\phi \circ \psi^{-1})_q|_{\{0\}\times\mathbb{R}^{m-1}} > 0$ as well. Therefore, if M is orientable, then so is the boundary ∂D of a regular domain D in M. $\square$

We rephrase the lemma above in terms of volume forms (cf. Proposition 7.16).

Proposition 10.5. *Let D be a regular domain in a manifold M and μ a non-vanishing top form on M. Then there is a vector field N defined on M near ∂D which points out of D. Moreover,*

$$\nu = (\iota(N)\mu)|_{\partial D}$$

is an orientation on ∂D.

Proof. If $D = \{x \in \mathbb{R}^m : x_1 \leq 0\}$, take $N = \frac{\partial}{\partial x_1}$. In general, cover ∂D by adapted charts $\{\phi_i : U_i \to \mathbb{R}^m\}$ (Since all our manifolds are second countable, by passing to a subcover we may assume that the cover is countable). On each U_i there is a vector field $N_i \in \Gamma(TU_i)$ such that N_i points outward. Pick a partition of unity ρ_i subordinate to $\{U_i\}$, and let $N = \sum \rho_i N_i$. The vector field N defined on $\cup U_i$ is the desired vector field. Note that in adapted coordinates $(x_1, \ldots, x_m) : U_i \to \mathbb{R}^m$ it has the form

$$N = b_1 \frac{\partial}{\partial x_1} + \cdots + b_m \frac{\partial}{\partial x_m}$$

for some functions $b_1, \ldots, b_m$ with $b_1 > 0$.

We now argue that

$$\nu := (\iota(N)\mu)|_{\partial D}$$

is a nowhere vanishing form on ∂D. We argue in adapted coordinates $(x_1, \ldots, x_m)$. The form μ satisfies

$$\mu = f(x_1, \ldots, x_m) dx_1 \wedge \ldots \wedge dx_m$$

for some nowhere zero function f. Consequently

$$\iota(N)\mu = \sum_{j=1}^{m} (-1)^{j-1} f b_j \, dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_m$$

(recall that $\widehat{dx_j}$ means that dx_j is omitted). Since for $j > 1$,

$$dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_m|_{x_1=0} = 0$$

we have

$$(\iota(N)\mu)|_{\partial D} = (f b_1)|_{x_1=0} dx_2 \wedge \ldots \wedge dx_m$$

with $f b_1 \neq 0$. $\square$

Definition 10.6. Let M be an oriented manifold and $D \subset M$ a regular domain. We will refer to the orientation on the boundary ∂M defined by the orientation of M as in Proposition10 above as the *induced orientation.*

Theorem 10.7 (Stokes's Theorem)**.** *Let M be an oriented m-dimensional manifold, $D \in M$ a regular domain, and $\omega \in \Omega_c^{m-1}(M)$ a compactly supported form of degree one less than the dimension of M. Then*

$$\int_{int(D)} d\omega = \int_{\partial D} \omega|_{\partial D}.$$

Here $int(D)$ and ∂D are both given the orientation induced by the one on M.

Proof. First, consider the case that $M = \mathbb{R}^m$ and $D = \{x \in \mathbb{R}^m \mid x_1 \leq 0\}$. It doesn't matter what orientation we choose on M; we just have to be consistent in orienting $int(D)$ and ∂D. Choose the orientation on $\mathbb{R}^m$ defined by the standard volume form $\mu = dx_1 \wedge \ldots \wedge dx_m$. Let $N = \frac{\partial}{\partial x_1}$ so that $\iota(N)\mu|_{\partial D} = dx_2 \wedge \ldots \wedge dx_m$. Let $\omega \in \Omega_c^{m-1}(\mathbb{R}^m)$ be a compactly supported form. Then

$$\omega = \sum_j (-1)^{j-1} f_j dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_m$$

for some compactly supported functions f_j. Note that

$$\omega|_{\partial D} = \left(\sum_j (-1)^{j-1} f_j dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_m \right) \Bigg|_{x_1=0} = f_1(0, x_2, \ldots, x_m)\, dx_2 \wedge \ldots \wedge dx_m.$$

On the other hand,

$$\begin{aligned} d\omega &= \sum_j (-1)^{j-1} \frac{\partial f}{\partial x_j} dx_j \wedge dx_1 \wedge \ldots \wedge d\hat{x}_j \wedge \ldots \wedge dx_m \\ &= \sum_j \frac{\partial f_j}{\partial x_j} dx_1 \wedge \ldots \wedge dx_m. \end{aligned}$$

Now,

$$\int_D d\omega = \sum_j \int_{\{x_1 \leq 0\}} \frac{\partial f}{\partial x_j} dx_1 \wedge \ldots \wedge dx_m = \sum_j \int_{\{x_1 \leq 0\}} \frac{\partial f}{\partial x_j} dx_1 \ldots dx_m.$$

Since the supports of f_j's are compact, there is an $R > 0$ such that

$$\operatorname{supp}(f_j) \subset \{x \in \mathbb{R}^n \mid -R \leq x_j \leq R\}$$

for all j. For $j > 1$,

$$\begin{aligned} \int_{\{x_1<0\}} \frac{\partial f}{\partial x_j} dx &= \int_{\{x_1<0\}} \left(\int_{\mathbb{R}} \frac{\partial f}{\partial x_j} dx_j \right) dx_1 \ldots \widehat{dx_j} \ldots dx_m \\ &= \int_{\{x_1<0\}} \left(\int_{-R}^{R} \frac{\partial f}{\partial x_j} dx_j \right) dx_1 \ldots \widehat{dx_j} \ldots dx_m \\ &= 0, \end{aligned}$$

since

$$\int_{-R}^{R} \frac{\partial f}{\partial x_j} dx_j = f(x_1, \ldots, x_{j-1}, R, x_{j+1}, \ldots, x_m) - f(x_1, \ldots, x_{j-1}, -R, x_{j+1}, \ldots, x_m) = 0 - 0 = 0.$$

For $j = 1$, we have

$$\begin{aligned} \int_{\{x_1<0\}} \frac{\partial f_1}{\partial x_1} dx &= \int_{\mathbb{R}^{m-1}} \left(\int_{-\infty}^{0} \frac{\partial f_1}{\partial x_1} dx_1 \right) dx_2 \ldots dx_m = \int_{\mathbb{R}^{m-1}} \left(\int_{-R}^{0} \frac{\partial f_1}{\partial x_1} dx_1 \right) dx_2 \ldots dx_m \\ &= \int_{\mathbb{R}^{m-1}} (f_1(0, x_2, \ldots, x_m) - 0)\, dx_2 \ldots dx_m \\ &= \int_{\mathbb{R}^{m-1}} f_1(0, x_2, \ldots, x_m\, dx_2 \wedge \ldots \wedge dx_m = \int_{\partial D} \omega|_{\partial D}. \end{aligned}$$

Therefore

$$\int_{int(D)} d\omega = \int_{\partial D} \omega|_{\partial D}$$

in the special case of $M = \mathbb{R}^m$, $D = \{x_1 < 0\}$.

We now consider a slightly more general case: D is a regular domain in an oriented manifold M of dimension m, $\phi : U \to \mathbb{R}^m$ a chart adapted to D and $\omega \in \Omega_c^{m-1}(M)$ with $\operatorname{supp} \omega \subset U$. Then

$$\int_{int(D)} d\omega = \int_{int(D)\cap U} d\omega = \int_{\phi(int(D))} \phi^*(d\omega) = \int_{\{x_1<0\}} d(\phi^*\omega)$$
$$= \int_{\partial\{x_1\le 0\}} \phi^*\omega|_{\partial\{x_1\le 0\}} \quad \text{(here we used the special case above)}$$
$$= \int_{\phi(U\cap\partial D)} \phi^*\omega|_{\phi(U\cap\partial D)} = \int_{U\cap\partial D} \omega|_{U\cap\partial D} = \int_{\partial D} \omega|_{\partial D}.$$

Finally we remove the restriction on the support of ω. Let $\omega \in \Omega_c^{m-1}(M)$ be an arbitrary compactly supported form. Cover $D \cap \operatorname{supp}\omega$ by finitely many charts $\{\phi_\alpha : U_\alpha \to \mathbb{R}^m\}$ adapted to the domain D and giving D its orientation (we now have to make sure that changes of coordinates between charts preserve orientation). It is no loss of generality to assume that $M = \cup_\alpha U_\alpha$ (after all, we are going to be only interested in $\omega|_D$ and $\operatorname{supp}\omega|_D \subset \cup_\alpha U_\alpha$.) Let $\{\rho_\alpha\}$ be a partition of unity subordinate to the cover. Then $\sum \rho_\alpha\omega = \omega$ and $\operatorname{supp}(\rho_\alpha\omega) \subset U_\alpha$. By the previous discussion

$$\int_{int(D)} d(\rho_\alpha\omega) = \int_{\partial D} \rho_\alpha\omega$$

for each index α. Therefore

$$\int_{int(D)} d\omega = \int_D \sum d(\rho_\alpha\omega) = \sum \int_D d(\rho_\alpha\omega) = \sum \int_{\partial D} \rho_\alpha\omega = \int_{\partial D} \sum \rho_\alpha\omega = \int_{\partial D} \omega.$$

□

Exercise 10.1. Let M be an m-dimensional compact oriented manifold, $D \subset M$ a domain with smooth boundary, $f \in C^\infty(M)$, and $\omega \in \Omega^{m-1}(M)$. Show that

$$\int_D f\, d\omega = \int_{\partial D} f\omega - \int_D df \wedge \omega.$$

Exercise 10.2.
Let M be an m-dimensional oriented manifold and $\mu \in \Omega^n(M)$ a nowhere vanishing form. Recall that for any vector field X on M,

$$L_X\mu = \operatorname{div}_\mu(X)\,\mu,$$

where $\operatorname{div}_\mu(X)$ is the divergence of μ with respect to X (cf. Exercise 9.2). Show that if $D \subset M$ is a regular domain then

$$\int_D \operatorname{div}_\mu(X)\,\mu = \int_{\partial D} \iota(X)\mu$$

for any vector field X with compact support.

Exercise 10.3. What is the integral of $x\,dy - y\,dx$ over ∂D, where D is the unit disk in $\mathbb{R}^2$ (and $\mathbb{R}^2$ is given the standard orientation)?

11. Connections on vector bundles

11.1. **Connections.** If X is a vector field on an open subset U of $\mathbb{R}^m$, then X is determined by m-tuple $(a_1, \dots a_m)$ of functions:

$$X = \sum_i a_i \frac{\partial}{\partial x_i}$$

Therefore we know how to take directional derivatives of X at a point $q \in U$ in the direction of a vector $v \in T_qU = \mathbb{R}^m$ — we simply differentiate the coefficients:

$$(D_vX)_q = \sum_i (D_va_i)_q \frac{\partial}{\partial x_i}|_q$$

where $D_v a_i$ is the directional derivative of the function a_i in the direction v. Consequently we know when a vector field does not change along a curve γ:

$$D_{\dot\gamma} X = 0.$$

Covariant derivatives generalize the directional derivatives allowing us to differentiate vector fields on arbitrary manifolds and, more generally, sections of arbitrary vector bundles.

Definition 11.1 (Covariant derivative of sections of a vector bundle)**.** Let $\pi : E \to M$ be a vector bundle. A *covariant derivative* (also knows as a *connection*) is an $\mathbb{R}$-bilinear map

$$\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E), \quad (X, s) \mapsto \nabla_X s$$

such that

(1) $\nabla_{fX} s = f \nabla_X s$
(2) $\nabla_X (fs) = X(f) \cdot s + f \nabla_X s$.

for all $f \in C^\infty(M)$, all $X \in \Gamma(TM)$, and all $s \in \Gamma(E)$.

Example 11.2. Let $U \subset \mathbb{R}^m$ be an open set and $E = TU \to U$ the tangent bundle. Define a connection D on $TU \to U$ by

$$D_X(\sum a_i \frac{\partial}{\partial x_i}) = \sum X(a_i) \frac{\partial}{\partial x_i}.$$

I leave it to the reader to check that this is indeed a connection.

Remark 11.3. Lie derivative $(X, Y) \mapsto L_X Y$ is *not* a connection on the tangent bundle (why not?).

Example 11.4. Let $\pi : E \to M$ be a trivial bundle of rank k. Then there exist global sections $\{s_1, \ldots, s_k\}$ of E such that $\{s_j(x)\}$ is a basis for E_x for all points $x \in M$ ($\{s_i\}$ is a *frame* of $E|_U$). So for any $s \in \Gamma(E)$, we have $s = \sum_j f_j s_j$, for some C^∞ functions f_j. We define a bilinear map $\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$ by

$$\nabla_X s = \nabla_X(\sum_j f_j s_j) := \sum_j X(f_j) s_j.$$

It is easy to check that ∇ is indeed a connection on E:

$$\nabla_{fX} s = \nabla_{fX}(\sum_j f_j s_j) = \sum_j f X(f_j) s_j = f \sum_j f_j s_j = f \nabla_X s;$$

and

$$\nabla_X(fs) = \nabla_X(f \sum_j f_j s_j) = \sum_j X(f f_j) s_j = X(f) \sum_j f_j s_j + f \sum_j X(f_j) s_j = X(f)s + f \nabla_X s.$$

Lemma 11.5. *Any convex linear combination of two connections on a vector bundle $E \to M$ is a connection. More precisely, let ∇^1, ∇^2 be two connections on E and $\rho_1, \rho_2 \in C^\infty(M)$ be two functions with $\rho_1 + \rho_2 = 1$. Then*

$$\Gamma(TM) \times \Gamma(E) \ni (X, s) \mapsto \nabla_X s := \rho_1 \nabla^1_X s + \rho_2 \nabla^2_X s \in \Gamma(E)$$

is a connection.

Proof. Exercise. Check that the two properties of the connection hold. □

As a corollary we get:

Proposition 11.6. *Any vector bundle $\pi : E \to M$ has connection.*

Proof. Choose a cover $\{U_\alpha\}$ on M such that $E|_{U_\alpha}$ is trivial. Let ∇^α be a connection on $E|_{U_\alpha}$, as in Example 11.4. Let $\{\rho_\beta\}$ be a partition of unity subordinate to $\{U_\alpha\}$. Then $\operatorname{supp} \rho_\beta \subset U_\alpha$ for some $\alpha = \alpha(\beta)$. Define a map $\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$ by

$$\nabla_X s = \sum_\beta \rho_\beta (\nabla^\alpha_{X_U} s|_{U_\alpha}).$$

This is indeed a connection, since a convex linear combination of any finite number of connections is a connection — see Lemma 11.5 above. □

Proposition 11.7. *Let ∇ be a connection on a vector bundle $\pi : E \to M$. Then ∇ is local: for any open set U and any vector fields X and Y, and for any sections s and s' of E such that $X|_U = Y|_U$ and $s|_U = s'|_U$, we have*

$$(\nabla_X s)|_U = (\nabla_Y s')|_U.$$

Proof. Since ∇ is bilinear, it is enough to show two things:
(a) if $X|_U = 0$, then $(\nabla_X s)|_U = 0$ for any $s \in \Gamma(E)$; and
(b) if $s|_U = 0$, then $(\nabla_X s)|_U = 0$ for any $X \in \Gamma(TM)$.
Fix a point $x_0 \in U$. Then there is a smooth function $\rho : U \to [0, 1]$ with $\operatorname{supp} \rho \subset U$ and $\rho|_V = 1$ for some open neighborhood V of x_0. If $X|_U = 0$ then $\rho X = 0$, and hence for any section s of E,

$$0 = (\nabla_{\rho X} s)(x_0) = \rho(x_0)(\nabla_X s)(x_0) = (\nabla_X s)(x_0).$$

Since $x_0 \in U$ is arbitrary, (a) follows. If $s|_U = 0$ then $\rho s = 0$ on M. This in turn implies that

$$0 = (\nabla_X \rho s)(x_0) = (X(\rho)s + \rho\nabla_X s)(x_0) = 0 + \rho(x_0)(\nabla_X s)(x_0) = (\nabla_X s)(x_0).$$

□

Remark 11.8. It follows that if ∇ is a connection on a vector bundle $E \to M$ then ∇ induces a connection

$$\nabla^U : \Gamma(TU) \times \Gamma(E_U) \to \Gamma(E|_U)$$

on the restriction $E|_U$ for any open set $U \subset M$. Namely, for any $x_0 \in U$ let $\rho : U \to [0, 1]$ be a bump function as in the proof above. Then for any $X \in \Gamma(TU)$ and any $s \in \Gamma(E|_U)$ we have $\rho X \in \Gamma(TM)$ and $\rho s \in \Gamma(E)$ (with ρX and ρs extended to all of M by 0). We define:

$$({\nabla^U}_X s)(x_0) = (\nabla_{\rho X} \rho s)(x_0).$$

By Proposition 11.7, the right hand side does not depend on the choice of the function ρ. We leave it to the reader to check that ∇^U is a connection.

Definition 11.9 (Christoffel symbols)**.** Let $E \to M$ be a vector bundle with a connection ∇. Let $(x_1, \ldots, x_n) : U \to \mathbb{R}^m$ be a coordinate chart on M small enough so that $E|_U$ is trivial. Let $\{s_\alpha\}$ be a frame of $E|_U$: for each $x \in U$ we require that $\{s_\alpha(x)\}$ is a basis of the fiber E_x. Then any local section $s \in \Gamma(E|_U)$ can be written as a linear combination of s_α's. In particular, for each index i and β

$$\nabla^U_{\frac{\partial}{\partial x_i}} s_\beta = \sum_\alpha \Gamma^\alpha_{i\beta} s_\alpha$$

for some functions $\Gamma^\beta_{i\alpha} \in C^\infty(U)$. These functions are the *Christoffel symbols* of the connection ∇ relative to the coordinates $(x_1, \ldots, x_n)$ and the frame $\{s_\alpha\}$.

It follows easily that the Christoffel symbols determine the connection ∇^U on the coordinate chart U. It is customary not to distinguish between ∇ and its restriction ∇^U.

Proposition 11.10. *Let ∇ be a connection on on a vector bundle $\pi : E \to M$. For any $X \in \Gamma(TM)$, any $s \in \Gamma(E)$ and any point q the value of the connection $(\nabla_X s)(q)$ at a point $q \in M$ depends only on the vector X_q (and not on the value of X near q).*

Proof. It's enough to show that if $X_q = 0$ then $(\nabla_X s)(q) = 0$. Since connections are local we can argue in coordinates. Choose a coordinate chart $(x_1, \ldots, x_n) : U \to \mathbb{R}^m$ on M with $q \in U$ such that $E|_U$ is trivial. Pick a local frame $\{s_j\}$ of $E|_U$. Then, if $X = \sum X^i \frac{\partial}{\partial x_i}$, $s = \sum f_j s_j$, and Γ^k_{ij} denote the associated Christoffel symbols,

$$\begin{aligned}
\nabla_X s = \nabla_{\sum X^i \frac{\partial}{\partial x_i}} (\textstyle\sum f_j s_j) &= \textstyle\sum X^i \nabla_{\frac{\partial}{\partial x_i}} (\sum f_j s_j) \\
&= \sum X^i \frac{\partial f_j}{\partial x_i} s_j + \sum X_i f_j \nabla_{\frac{\partial}{\partial x_i}} s_j \\
&= \sum X^i (\sum \frac{\partial f_j}{\partial x_i} s_j + \sum f_j \Gamma^k_{ij} s_k).
\end{aligned}$$

If $X_q = 0$ then $X^i(q) = 0$ for all i. Hence $(\nabla_X s)(q) = 0$ and we are done. □

As a corollary of the proof computation above we get an expression for the connection in terms of the Christoffel symbols.

Corollary 11.10.1. *Let ∇ be a connection on on a vector bundle $\pi : E \to M$ and $(x_1, \ldots, x_n) : U \to \mathbb{R}^m$ a coordinate chart on M with $E|_U$ being trivial. Let $\{s_j\}$ be a frame of $E|_U$. Then*

$$\nabla_{\sum_i X^i \frac{\partial}{\partial x_i}}(\sum_j f_j s_j) = \sum_{i,k} X^i(\frac{\partial f_k}{\partial x_i} + \sum_j f_j \Gamma^k_{ij}) s_k. \tag{11.1}$$

We note one more corollary that will be useful when we try to define connections induced on submanifolds.

Corollary 11.10.2. *Let ∇ be a connection on on a vector bundle $\pi : E \to M$. For any $X \in \Gamma(TM)$, any $s \in \Gamma(E)$ and any point q the value of the connection $(\nabla_X s)(q)$ at a point $q \in M$ depends only on the values of s along the integral curve of X through q*

Proof. By the previous corollary, for $X = \sum_i X^i \frac{\partial}{\partial x_i}$ and $s = \sum_j f_j s_j$

$$(\nabla_X s)(q) = (X f_k)(q)\ s_k(q) + \sum_{i,k,j} X^i(q) f_j(q) \Gamma^k_{ij}(q) s_k(q).$$

And $(X f_k)(q)$ depends only on the values of f_k along the integral curve of X. □

The proof that connections are local has an important generalization to maps of sections of vector bundles.

Definition 11.11. Let $E \to M$ and $F \to M$ be two vector bundles. We say that a map $T : \Gamma(E) \to \Gamma(F)$ is *tensorial* if T is $\mathbb{R}$-linear and for any $f \in C^\infty(M)$

$$T(fs) = fT(s)$$

for all sections $s \in \Gamma(E)$.

Lemma 11.12. *Let $E \to M$ and $F \to M$ be two vector bundles. If $T : \Gamma(E) \to \Gamma(F)$ is tensorial then there is a vector bundle map $\phi : E \to F$ so that*

$$[T(s)](x) = \phi(s(x))$$

for all $s \in \Gamma(E)$ and $x \in M$. And conversely, any vector bundle map $\phi : E \to F$ defines a tensorial map on sections $T_\phi : \Gamma(E) \to \Gamma(F)$ by $T_\phi(s) = \phi \circ s$.

Proof. The proof is in two steps. We first argue that T is local: if $s \in \Gamma(E)$ vanishes on an open set $U \subset M$ then $T(s)$ vanishes on U as well. Pick a point $x \in U$ and a smooth function $\rho \in C^\infty(M)$ with $\operatorname{supp} \rho \subset U$ and $\rho \equiv 1$ on a neighborhood V of x ($V \subset U$, of course). Then ρs is identically zero everywhere. Hence

$$0 = T(\rho s)(x) = \rho(x)\ T(s)(x) = T(s)(x).$$

Since $x \in U$ is arbitrary $T(s)|_U = 0$.

Since T is local and E, F are locally trivial, we may assume that E and F are, in fact, trivial. That is $E = M \times \mathbb{R}^k$ and $F = M \times \mathbb{R}^l$. Moreover the sections of E and F are simply k- and l-tuples of functions. We want to define a vector bundle map $\phi : E \to F$. Then $\phi : M \times \mathbb{R}^k \to M \times \mathbb{R}^l$ has to be of the form

$$\phi(x, v) = (x, A(x)v)$$

where $A : M \to \operatorname{Hom}(\mathbb{R}^k, \mathbb{R}^l)$ is smooth, with the property that

$$T(f_1, \ldots, f_k)(x) = A(x) \begin{pmatrix} f_1(x) \\ \vdots \\ f_k(x) \end{pmatrix}$$

for all $x \in M$. But this is easy: define the jth column of $A(x)$ to be the l-tuple of functions $T(e_j)$, where e_j is the section of E that assigns to every point the jth basis vector $(0, \ldots, 0, 1, 0, \ldots, 0)$ (1 in jth slot). Or, if you prefer, e_j is the k-tuple of functions with jth function being identically 1 and all the others being zero. □

Remark 11.13. Lemma 11.12 above generalizes further: let E_1, $E_2, \dots E_k$ and F be vector bundles over a manifold M and

$$T : \Gamma(E_1) \times \cdots \times \Gamma(E_k) \to \Gamma(F)$$

a k-linear map which is tensorial in each slot:

$$T(f_1 s_1, \dots, f_k s_k) = f_1 \dots f_k T(s_1, \dots, s_k)$$

for all $s_i \in \Gamma(E_i)$ and $f_j \in C^\infty(M)$. Then for every $x \in M$ there is a unique k-linear map

$$T_x : (E_1)_x \times \cdots \times (E_k)_x \to F_x$$

with

$$T_x(s_1(x), \dots, s_k(x)) = [T(s_1, \dots, s_k)](x).$$

Globally this means that there is a vector bundle map

$$\phi : E_1 \otimes \cdots \otimes E_k \to F$$

so that

$$T(s_1, \dots, s_k)(x) = \phi(s_1(x) \otimes \dots \otimes s_k(x))$$

for all $x \in M$ and all sections $s_i \in \Gamma(E_i)$.

Remark 11.14. We add one more layer of abstraction to the remark above: there is a bijection between vector bundle maps $\phi : E \to F$ and sections of the bundle $\mathrm{Hom}(E, F) \simeq E^* \otimes F$. Namely, if $\phi : E \to F$ is a vector bundle map, then $\phi|_{E_x} : E_x \to F_x$ is an element of $\mathrm{Hom}(E_x, F_x) = \mathrm{Hom}(E, F)_x$ for each point $x \in M$. Thus $x \mapsto \phi|_{E_x}$ is a section of the bundle $\mathrm{Hom}(E, F) \to M$.

We summarize the preceding discussion as a proposition.

Proposition 11.15. *Let E_1, $E_2, \dots E_k$ and F be vector bundles over a manifold M. There is a bijection between k-linear tensorial maps*

$$T : \Gamma(E_1) \times \cdots \times \Gamma(E_k) \to \Gamma(F)$$

and the sections of the bundle $E_1^* \otimes \cdots \otimes E_k^* \otimes F \to M$.

Here are a few instances where the above point of view is useful.

Lemma 11.16. *Let ∇^1 and ∇^2 be two connections on a vector bundle $E \to M$. Their difference $\nabla^1 - \nabla^2$ "is" a section of the bundle $T^*M \otimes E^* \otimes E \simeq Hom(TM \otimes E, E)$. Conversely, given a connection ∇ on $E \to M$ and a section A of the bundle* $\mathrm{Hom}(TM \otimes E, E)$ *then the map $\nabla^A : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$ given by*

$$(\nabla^A_X s)(x) := \nabla_X s(x) + A_x(X_x \otimes s(x))$$

is again a connection on E. Here, of course, $x \in M$ is a point, X a vector field on M and s is a section of E. Thus a choice of a connection on $E \to M$ defines a bijection

$$\{\textit{space of all connections on } E \to M\} \leftrightarrow \Gamma(T^*M \otimes E^* \otimes E) = \Gamma(\mathrm{Hom}(TM \otimes E, E)) = \Gamma(T^*M \otimes \mathrm{Hom}(E, E)).$$

Proof. In one direction it's enough to prove that $\nabla^1 - \nabla^2$ is tensorial in both slots. It's obviously tensorial in the vector field slot. The tensoriality in the second slot is an easy computation.

We also leave it to the reader to check that ∇^A as defined above is a connection. □

Definition 11.17. A *connection on a manifold* M is a connection on its tangent bundle $TM \to M$.

Definition 11.18. The *torsion* T^∇ of a connection ∇ on a manifold M is a bilinear map

$$T^\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM), \quad T^\nabla(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y].$$

If $T^\nabla = 0$, the connection ∇ is called *torsion-free*.

Lemma 11.19. *The torsion of a connection is tensorial, hence corresponds to a section of the bundle $T^*M \otimes T^*M \otimes TM$.*

Proof. This is yet another computation left to the reader. □

Definition 11.20. The *curvature* R of a connection ∇ on a vector bundle $E \to M$ is a tri-linear map $\Gamma(TM) \times \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$ defined by

$$R(X,Y)s = \nabla_X(\nabla_Y s) - \nabla_Y(\nabla_X s) - \nabla_{[X,Y]}s.$$

Lemma 11.21. *Curvature is tensorial hence correspond to a section of $T^*M \otimes T^*M \otimes \mathrm{Hom}(E,E) \to M$. Moreover, since $R(X,Y)s = -R(Y,X)s$, it actually corresponds to a section of $\Lambda^2(T^*M) \otimes \mathrm{Hom}(E,E)$.*

Proof. Once again this is a computation. We check tensoriality in one slot and leave the rest to the reader. For all vector fields X, Y, sections s and functions f,

$$\begin{aligned} R(X,Y)(fs) &= \nabla_X(\nabla_Y(fs)) - \nabla_Y(\nabla_X(fs)) - \nabla_{[X,Y]}(fs) \\ &= \nabla_X(Y(f)s + f\nabla_Y s) - \nabla_Y(X(f)s - f\nabla_X s) - ([X,Y]f)s - f\nabla_{[X,Y]}s \\ &= X(Y(f))s + Y(f)\nabla_X s + X(f)\nabla_Y s + f\nabla_X(\nabla_Y s) - Y(X(f))s \\ &\quad -X(f)\nabla_Y s - Y(f)\nabla_X s - f\nabla_X(\nabla_Y s) - ([X,Y]f)s - f\nabla_{[X,Y]}s \\ &= fR(X,Y)s \end{aligned}$$

□

11.2. **Parallel Transport.** In general there is no consistent way of identifying vectors in tangent spaces at different points of a manifold. More generally there is no consistent way of identifying vectors in fibers of a vector bundle above different points of a manifold. However we will see that given a connection ∇ on a vector bundle $\pi : E \to M$, for any curve $\gamma : [a,b] \to M$ there is a family of vector space isomorphisms

$$P_{t_1}^{t_2}(\gamma) = P_{t_1}^{t_2} : E_{\gamma(t_1)} \to E_{\gamma(t_2)},$$

depending smoothly on $t_1, t_2 \in [a,b]$. These isomorphisms $P_{t_1}^{t_2}$ are called *parallel transport* along γ. The connection can then be recovered from parallel transport. We now proceed with the construction.

Definition 11.22. Let $\pi : E \to M$ be a vector bundle and $\gamma : [a,b] \to M$ a curve. A *section σ of $E \to M$ along γ* is a smooth map $s : [a,b] \to E$ so that $\pi(\sigma(t)) = \gamma(t)$ for all $t \in [a,b]$. We denote the space of sections of E along the map γ by $\Gamma(\gamma^*E)$.

Example 11.23. If $s : M \to E$ is a section of E, then $s \circ \gamma$ is a section along γ.

Example 11.24. The derivative $\dot\gamma := d\gamma_t(\frac{d}{dt}|_t)$ is a section of the tangent bundle $TM \to M$ along γ.

Remark 11.25. If $E = TM$ then a section along a curve γ is also known as *a vector field along* γ. It's not true that every section σ along γ is of the form $\sigma = s \circ \gamma$ for some $s \in \Gamma(E)$: if the curve γ crosses itself than $\dot\gamma$ cannot be of the form $X \circ \gamma$ for any vector field X on M.

Remark 11.26. Here's another way to consider sections along a curve γ. Suppose $f : N \to M$ is a smooth map of manifolds and that $\pi : E \to M$ is a vector bundle. Define the *pullback of the bundle E along f* to be the set

$$f^*E = \{(n,e) \in N \times E \mid f(n) = \pi(e)\}.$$

together with the projection $\pi' : f^*E \to N$, $f^*E \ni (n,e) \mapsto n$. A transversality argument shows that f^*E is a submanifold of $N \times E$, so π' is smooth. It's not hard to see that f^*E is a vector bundle of the same rank as E. The point of this construction is that a section of a bundle $E \to M$ along a curve $\gamma : (a,b) \to M$ is simply a section of the pullback bundle $\gamma^*E \to [a,b]$.

Strictly speaking the construction above doesn't apply to maps from closed intervals, since a closed interval is not a manifold. However, a smooth map from a closed interval $[a,b]$ is, by definition, a smooth curve from a slightly larger open interval $(a',b') \supset [a,b]$ and pulling back E to a bundle over (a',b') does make sense.

Definition 11.27. Let $\pi : E \to M$ be a vector bundle and $\gamma : [a,b] \to M$ a smooth curve. A *covariant derivative $\frac{\nabla}{dt}$ along γ* is an $\mathbb{R}$-linear map

$$\frac{\nabla}{dt} : \Gamma(\gamma^*(E)) \to \Gamma(\gamma^*(E)), \quad \sigma \mapsto \frac{\nabla}{dt}\sigma$$

such that for all function $f \in C^\infty([a,b])$ and all sections $\sigma \in \Gamma(\gamma^*(E))$

$$\frac{\nabla}{dt}(f\sigma) = \frac{df}{dt}\sigma + f\frac{\nabla}{dt}\sigma. \tag{11.2}$$

Proposition 11.28. *Given a connection ∇ on a vector bundle $\pi : E \to M$ and a curve $\gamma : [a,b] \to M$, there is a unique covariant derivative $\frac{\nabla}{dt} : \Gamma(\gamma^*(E)) \to \Gamma(\gamma^*(E))$ along γ such that*

$$\frac{\nabla}{dt}(s \circ \gamma)(t) = (\nabla_{\dot\gamma(t)} s)(\gamma(t)). \tag{11.3}$$

for all sections s of the bundle E.

Proof. (Uniqueness) Arguing as in Proposition 11.7, it is not hard to show that $\frac{\nabla}{dt}$ is local: for a section σ of E along γ the value $(\frac{\nabla}{dt}\sigma)(t)$ at a point t depends only on the values of σ near t. Therefore, in order to prove uniqueness it is no loss of generality to assume that the image $\gamma([a,b])$ of γ is contained in an open set U in M with $E|_U$ trivial. Pick a frame $\{s_j\}$ of $E|_U$. Then for any $\sigma \in \Gamma(\gamma^* E)$ there are smooth functions $f_j \in C^\infty([a,b])$ so that

$$\sigma(t) = \sum f_j(t) s_j(\gamma(t))$$

for all $t \in [a,b]$. Then, using (11.2) and (11.3), we get

$$\frac{\nabla}{dt}\sigma(t) = \frac{\nabla}{dt}\left(\sum f_j\,(s_j \circ \gamma)\right)(t) = \sum \frac{df_j}{dt}(t)\, s_j(\gamma(t)) + \sum f_j (\nabla_{\dot\gamma(t)} s_j)(\gamma(t)). \tag{11.4}$$

Since the right hand side of (11.4) depends only on ∇, $\frac{\nabla}{dt}$ is unique.

(Existence) Cover $\gamma([a,b])$ with sets U_j such that $E|_{U_j}$ is trivial. It's enough to construct $\frac{\nabla}{dt}$ on each $\Gamma(\gamma^* E|_{\gamma^{-1}(U_j)})$ for by uniqueness the operators on each $\Gamma(\gamma^* E|_{\gamma^{-1}(U_j)})$ will patch together to a map $\frac{\nabla}{dt}$: $\Gamma(\gamma^* E) \to \Gamma(\gamma^* E)$. Pick a frame $\{s_k^{(j)}\}$ on $E|_{U_j}$ and define $\frac{\nabla}{dt}$ on $\gamma^*(E|_{U_j})$ by (11.4). □

Definition 11.29. We will refer to the covariant derivative $\frac{\nabla}{dt}$ along γ as in the Proposition 11.28 above as being *induced* by the connection ∇.

Definition 11.30. Let $E \to M$ be a vector bundle with a connection ∇, $\gamma : [a,b] \to M$ a curve. A section $\sigma \in \Gamma(\gamma^* E)$ is *parallel* if

$$\frac{\nabla}{dt}\sigma = 0,$$

where $\frac{\nabla}{dt}$ is the covariant derivative along γ induced by ∇.

To define parallel transport along a curve $\gamma : [a,b] \to M$, we want, for every vector $v \in E_{\gamma(a)}$, a section $\sigma^v \in \Gamma(\gamma^*(E))$ such that $\sigma^v(a) = v$ and $\frac{\nabla}{dt}\sigma^v = 0$. We also want the map $v \mapsto \sigma^v$ to be linear. The existence of such sections and linearity in v is the result of the next two lemmas. The first one is the standard result for linear time dependent ODE's.

Lemma 11.31. *Suppose that $B = (B_{jk}(t)) : [c,d] \to \mathbb{R}^{k^2}$ is a smooth curve in the space of $k \times k$ real matrices. Then there is a smooth curve $R : [c,d] \to \mathrm{GL}(\mathbb{R}, k)$ such that $f(t) := R(t) f^0$ is a solution of the ODE*

$$\begin{pmatrix} f_1'(t) \\ \vdots \\ f_k'(t) \end{pmatrix} = B(t) \begin{pmatrix} f_1(t) \\ \vdots \\ f_k(t) \end{pmatrix}, \tag{11.5}$$

with initial conditions $f(c) = f^0$.

Lemma 11.32. *Let $E \to M$ be a vector bundle with a connection ∇ and $\gamma : [a,b] \to M$ be smooth curve. For any vector $v \in E_{\gamma(a)}$ there is a section $\sigma^v \in \Gamma(\gamma^*(E))$ such that $\sigma^v(a) = v$ and $\frac{\nabla}{dt}\sigma^v = 0$. Moreover, the map*

$$E_{\gamma(a)} \to \Gamma(\gamma^* E), \quad v \mapsto \sigma^v$$

is a linear isomorphism.

Proof. As before, it is no loss of generality to assume the image of γ is contained in a coordinate chart $(x_1, \dots, x_m) : U \to \mathbb{R}^m$ with $E|_U$ being trivial. Let $\{s_j\}$ be a frame of $E|_U$ and Γ_{ij}^k the corresponding Christoffel symbols. Suppose σ is a section of E along γ which is parallel and satisfies $\sigma(a) = v$. Then there are smooth functions $f_j \in C^\infty([a,b])$ so that $\sigma = \sum f_j\,(s_j \circ \gamma)$. We argue that the f_j's satisfy a linear ODE as in Lemma 11.31 for some curve B. By (11.4), since $\frac{\nabla}{dt}\sigma = 0$,

$$\sum \frac{df_j}{dt}(t)\, s_j(\gamma(t)) = -\sum f_j\, (\nabla_{\dot\gamma(t)} s_j)(\gamma(t)).$$

We also have $\dot\gamma = \sum_i (\frac{d}{dt}\gamma_i)\frac{\partial}{\partial x_i}$, where $\gamma_i := x_i \circ \gamma$. Therefore

$$\nabla_{\dot\gamma} s_j = \sum \dot\gamma_i\,(\nabla_{\frac{\partial}{\partial x_i}} s_j)\circ\gamma = \sum_{i,j,k} \dot\gamma_i (\Gamma^k_{ij} s_k)\circ\gamma = \sum_k (\sum_i \dot\gamma_i\,(\Gamma^k_{ij}\circ\gamma))\,(s_k\circ\gamma)$$

We conclude that $\sigma = \sum f_j\,(s_j\circ\gamma)$ is parallel if and only if

$$\frac{df_k}{dt}(t) = -\sum_{i,j} f_j(t)\dot\gamma_i(t)\,\Gamma^k_{ij}(\gamma(t)). \tag{11.6}$$

That is, $f = (f_1, \dots, f_k)$ satisfies the ODE (11.5) with

$$B_{jk}(t) = \sum_i \dot\gamma_i(t)(\Gamma^k_{ij}(\gamma(t))$$

By Lemma 11.31 the system of linear equations (11.6) has a solution defined for all time $t \in [a,b]$ which depends linearly on the initial conditions. Therefore the desired parallel transport exists. □

Parallel transport leads to one definition of geodesics.

Definition 11.33. Let ∇ be a connection on the tangent bundle $TM \to M$ of a manifold M. A curve $\gamma : [a,b] \to M$ is a *geodesic* if its velocity field $\dot\gamma(t)$ is parallel:

$$\frac{\nabla}{dt}\dot\gamma = 0. \tag{11.7}$$

Remark 11.34. It will be useful to know what (11.7) means in coordinates. Let $(x_1, \dots, x_m) : U \to \mathbb{R}^m$ be a coordinate chart on our manifold. Define $\gamma_i = x_i \circ \gamma$, $\dot\gamma_i = \frac{d}{dt}\gamma_i$ and $\ddot\gamma_i = \frac{d}{dt}\dot\gamma_i$. Then $\dot\gamma = \sum \dot\gamma_i \frac{\partial}{\partial x_i}$. Hence the functions f_k in (11.6) are $\dot\gamma_k$s. Therefore, in this case, (11.6) reads

$$\ddot\gamma_k = -\sum \dot\gamma_i\dot\gamma_j\Gamma^k_{ij}(\gamma). \tag{11.8}$$

We conclude that a curve γ is a geodesic for a connection ∇ if and only if (11.8) holds in every coordinate chart.

Exercise 11.1. Consider the manifold $\mathbb{R}^n$. We have seen that $D_X Y = \sum X(Y^i)\frac{\partial}{\partial x_i}$ is a connection. Suppose that $\gamma : \mathbb{R} \to \mathbb{R}^n$ is a curve. Let $\frac{D}{dt}$ denote the covariant derivative along γ induced by the connection D on $\mathbb{R}^n$. Show that

$$\frac{D}{dt}\dot\gamma = \ddot\gamma \ (= \frac{d^2\gamma}{dt^2}).$$

Conclude that the geodesics in $\mathbb{R}^n$ with respect to D are straight lines.

12. Riemannian geometry

12.1. **Levi-Civita connection.**

We now specialize the discussion of connections and parallel transport to the case of manifolds with a choice of an inner product on each tangent space.

Definition 12.1 (Riemannian metric)**.** A *Riemannian metric* g on a manifold M assigns smoothly to each point $x \in M$ a positive definite inner product g_x on T_xM.

A *Riemannian manifold* is a manifold M together with a choice of a Riemannian metric g. In other words, it's a pair (M, g).

Remark 12.2. An inner product h on a vector space V is a bilinear map $h : V \times V \to \mathbb{R}$. Hence it is an element of the tensor product $V^* \otimes V^*$. Therefore a Riemannian metric on a manifold M is nothing but a smooth section of the bundle $(T^*M)^{\otimes 2} := T^*M \otimes T^*M \to M$. (Not all sections of $T^*M^{\otimes 2} \to M$ are Riemannian metrics. For instance, the zero section is not. But all symmetric and positive definite sections of $T^*M^{\otimes 2} \to M$ are Riemannian metrics.)

Theorem 12.3. *Any second countable manifold M has a Riemannian metric.*

Proof. Let $\{\phi_i = (x_1^{(i)}, \dots, x_m^{(i)}) : U_i \to \mathbb{R}^m\}$ be a countable collection of coordinate charts that cover M. One each chart U_i define a metric $g^{(i)} = \sum_j dx_j^{(i)} \otimes dx_j^{(i)}$ Let $\{\rho_i\}$ be a partition of unity subordinate to this cover. Define a section g of $T^*M \otimes T^*M \to M$ by

$$g = \sum_i \rho_i g^{(i)}.$$

Then g is a Riemannian metric. $\square$

Fiber metrics. The notion of a Riemannian metric generalizes to arbitrary vector bundles.

Definition 12.4. A *fiber metric* on the vector bundle $E \to M$ assigns smoothly to each point $x \in M$ a positive definite symmetric bilinear form $g_x : E_x \times E_x \to \mathbb{R}$. In particular a fiber product is a section of $E^* \otimes E^* \to M$.

Proposition 12.5. *Every vector bundle $E \to M$ over a paracompact manifold M has a fiber metric.*

Proof. If $\{s_\alpha : U \to E\}$ is a local frame, then

$$g_x(\sum a_\alpha s_\alpha(x), \sum b_\beta s_\beta(x)) = \sum a_\alpha b_\beta \delta_{\alpha\beta}$$

is a fiber metric on $E|_U$. Patch these local fiber metrics together using a partition of unity. $\square$

The next theorem is the fundamental theorem of Riemannian geometry. It says that for every Riemannian manifold (M, g) there is a connection ∇ (which depends on the metric g) with two important properties. Such connection is called the *Levi-Civita connection*.

Theorem 12.6 (existence and uniqueness of the Levi-Civita connection)**.** *On every Riemannian manifold (M, g) there is a unique connection $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ which is*
(1) Torsion-free : $\nabla_X Y - \nabla_Y X = [X, Y]$ for all $X, Y \in \Gamma(TM)$
(2) metric (i.e. compatible with g) : $X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ for all $X, Y, Z \in \Gamma(TM)$.

Proof. (Uniqueness) The proof is a trick. Suppose that ∇ exists. Then for any $X, Y, Z \in \Gamma(TM)$,

$$\begin{aligned} X\, g(Y, Z) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\ Y\, g(Z, X) &= g(\nabla_Y Z, X) + g(Z, \nabla_Y X) \\ -Z\, g(X, Y) &= -\, g(\nabla_Z X, Y) - g(X, \nabla_Z Y) \end{aligned}$$

since the connection is compatible with the metric. Adding up the three equations and using the fact that the connection is torsion free, we get

$$\begin{aligned} X\, g(Y, Z) + Y\, g(Z, X) - Z\, g(X, Y) &= g(\nabla_X Y, Z) + g(\nabla_Y X, Z) + g(Y, \nabla_X Z - \nabla_Z X) + g(X, \nabla_Y Z - \nabla_Z Y) \\ &= g(\nabla_X Y, Z) + g(\nabla_X Y - [X, Y], Z) + g(Y, [X, Z]) + g(X, [Y, Z]) \\ &= 2g(\nabla_X Y, Z) - g([X, Y], Z) + g(Y, [X, Z]) + g(X, [Y, Z]) \end{aligned}$$

Thus, we have

$$2g(\nabla_X Y, Z) = X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) + g([X, Y], Z) - g(Y, [X, Z]) - g(X, [Y, Z]). \tag{12.1}$$

Since Z is arbitrary and g is nondegenerate, the formula above uniquely determines $\nabla_X Y$. This proves uniqueness of a Levi-Civita connection.

It remains to prove existence. The proof is very simple, if one is willing to skip all the details. *Define* an $\mathbb{R}$-trilinear map

$$\Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to C^\infty(M)$$

by sending a triple of vector fields (X, Y, Z) to $1/2$ of the right hand side of (12.1). Since g is nondegenerate this defines an $\mathbb{R}$-bilinear map

$$\Gamma(TM) \times \Gamma(TM) \to \Gamma(TM), \quad (X, Y) \mapsto \text{“}\nabla\text{”}_X Y.$$

It remains to verify that “∇” so defined is a connection, and that it is metric and torsion-free. These minor details are traditionally left to the reader. We will provide a different and more detailed proof below after a brief detour. $\square$

Equation (12.1) has the following interesting consequence:

Lemma 12.7. *The Christoffel symbols of the Levi-Civita connection depend only on the metric and its first partials.*

Proof. Given a coordinate chart $(x_1, \ldots, x_m) : U \to \mathbb{R}^m$ on M, the Christoffel symbols Γ_{ij}^k of the Levi-Civita connection ∇ are defined by

$$\nabla_{\partial_i}\partial_j = \sum_k \Gamma_{ij}^k \partial_k,$$

where $\partial_i = \frac{\partial}{\partial x_i}$. Plugging $X = \partial_i$, $Y = \partial_j$ and $Z = \partial_k$ into (12.1) we get

$$2g(\nabla_{\partial_i}\partial_j, \partial_k) = \partial_i(g(\partial_j, \partial_k)) + \partial_j(g(\partial_j, \partial_i)) - \partial_k(g(\partial_i, \partial_j))$$

since $[\partial_i, \partial_j] = [\partial_j, \partial_k] = [\partial_i, \partial_k] = 0$. Writing $g_{ij} = g(\partial_i, \partial_j)$ etc., we obtain

$$2\sum_l \Gamma_{ij}^l g_{lk} = \partial_i g_{jk} + \partial_j g_{ji} - \partial_k g_{ij}. \tag{12.2}$$

Since g is a metric, the matrix (g_{ij}) is nondegenerate. Let (g^{rs}) denote its inverse, so that

$$\sum_s g^{rs} g_{sk} = \delta_{rk}.$$

Multiplying both sides of (12.2) by g^{sk} and summing over k we get

$$\sum_l \delta_{sl}\Gamma_{ij}^l = \frac{1}{2}\sum_k g^{sk}\left(\partial_i g_{jk} + \partial_j g_{ji} - \partial_k g_{ij}\right),$$

and simplifying

$$\Gamma_{ij}^s = \frac{1}{2}\sum_k g^{sk}\left(\partial_i g_{jk} + \partial_j g_{ji} - \partial_k g_{ij}\right). \tag{12.3}$$

This proves that the Christoffel symbols depend only on the metric and its first order partials. $\square$

Proof of Theorem 12.6 continued. It remains to (re)prove the existence of the Levi-Civita connection. By uniqueness, it is enough to construct a Levi-Civita connection ∇ in each coordinate chart. For then by uniqueness, these coordinate chart connections patch together into a Levi-Civita connection on the whole manifold M. We have shown that if the Levi-Civita connection exists then its Christoffel symbols have to be given by (12.3). Therefore on a chart $(x_1, \ldots, x_m) : U \to \mathbb{R}^m$ we *define* a connection ∇ by

$$\nabla_{X_i\partial_i} Y_j\partial_j = X_i(\partial_i Y_j)\partial_j + X_i Y_j \Gamma_{ij}^k \partial_k$$

with Christoffel symbols Γ_{ij}^k given by (12.3). In the equation above we finally resorted to the Einstein summation convention: we sum on repeated indices and omit the symbol $\sum$. We now check that ∇ is a Levi-Civita connection.

Since $\Gamma_{ij}^k = \Gamma_{ji}^k$ (c.f. (12.3)),

$$\nabla_{\partial_i}\partial_j - \nabla_{\partial_j}\partial_i = \Gamma_{ij}^k\partial_k - \Gamma_{ji}^k\partial_k = 0.$$

Thus, for two vector fields $X = X_i\partial_i$ and $Y = Y_j\partial_j$, we have

$$\begin{aligned}
\nabla_X Y - \nabla_Y X &= \nabla_{X_i\partial_i}(Y_j p_j) - \nabla_{Y_j p_j}(X_i\partial_i) \\
&= X_i(\partial_i Y_j)\partial_j + X_i Y_j \nabla_{\partial_i}\partial_j - Y_j(\partial_j X_i)\partial_i - Y_j X_i \nabla_{\partial_j}\partial_i \\
&= X_i(\partial_i Y_j)\partial_j - Y_j(\partial_j X_i)\partial_i \\
&= [X_i\partial_i, Y_j\partial_j].
\end{aligned}$$

Thus, ∇ is torsion-free. Compatibility with g is a somewhat longer computation. First, note that

$$\begin{aligned}
g(\nabla_{\partial_i}\partial_j, \partial_k) + g(\partial_j, \nabla_{\partial_i}\partial_k) &= g(\Gamma_{ij}^l\partial_l, \partial_k) + g(\partial_j, \Gamma_{ik}^m\partial_m) \\
&= \Gamma_{ij}^l g_{lk} + \Gamma_{ik}^m g_{jm} \\
&= \partial_i g_{jk},
\end{aligned}$$

where the last equality follows from (12.3). Thus, we have for vector fields $X = X_i\partial_i$,$Y = Y_j\partial_j$ and $Z = Z_k\partial_k$,

$$\begin{aligned}
(X_j\partial_j)g(Y_i\partial_i, Z_k\partial_k) &= X_j\partial_j(Y_iZ_kg_{ik}) \\
&= X_j(\partial_jY_i)Z_kg_{ik} + X_jY_i(\partial_jZ_k)g_{ik} + X_iY_jZ_k(\partial_jg_{ik}) \\
&= g(X_j(\partial_jY_i)\partial_i, Z_k\partial_k) + g(Y_i\partial_i, X_j(\partial_jZ_k)\partial_k) \\
&\quad +X_jY_iZ_k(g(\nabla_{\partial_j}\partial_i, \partial_k) + g(\partial_i, \nabla_{\partial_j}\partial_k)) \\
&= g((X_j\partial_j)Y_i\partial_i, Z_k\partial_k) + g(Y_i\nabla_{X_j\partial_j}\partial_i, Z_k\partial_k) \\
&\quad +g(Y_i\partial_i, (X_j\partial_j)Z_k\partial_k) + g(Y_i\partial_i, Z_k\nabla_{X_j\partial_j}\partial_k) \\
&= g(\nabla_{X_j\partial_j}(Y_i\partial_i), Z_k\partial_k) + g(Y_i\partial_i, \nabla_{X_j\partial_j}(Z_k\partial_k)).
\end{aligned}$$

That is, the connection ∇ is compatible with the metric g. Therefore the connection with Christoffel symbols defined by (12.3) is a Levi-Civita connection. This finishes the proof of existence and uniqueness of the Levi-Civita connection. □

Example 12.8. Consider the manifold $\mathbb{R}^n$. We have seen that $D_XY = \sum X(Y_i)\frac{\partial}{\partial x_i}$ is a connection. An easy computation shows D is the Levi-Civita connection on $\mathbb{R}^n$ with respect to the standard inner product on $\mathbb{R}^n$.

We end this section with a brief discussion of the geometric meaning of a connection being metric.

Definition 12.9. Let $E \to M$ be a vector bundle with a fiber metric g. A connection ∇ on E is *metric* if

$$X(g(s, s')) = g(\nabla_X s, s') + g(s, \nabla_X s')$$

for all vector fields X and sections $s, s' \in \Gamma(E)$.

Definition 12.10. Let V_1, V_2 be two vector spaces with inner products g_1, g_2 respectively. A linear map $A : V_1 \to V_2$ is an *isometry* if

$$g_2(Av, Aw) = g_1(v, w)$$

for all $v, w \in V_1$.

Lemma 12.11. *If a connection ∇ is metric then the associated parallel transport is an isometry.*

Proof. We will only prove the lemma for embedded curves and leave the general case as an exercise. If $\gamma : [a, b] \to M$ is an embedded curve, then locally any section $\sigma : [a, b] \to E$ is of the form $s \circ \gamma$. Let $v, w \in E_{\gamma(a)}$ be two vectors and $\sigma^v, \sigma^w : [a, b] \to E$ two parallel sections with $\sigma^v(a) = a$ and $\sigma^w(a) = w$. We want to prove that the function $t \mapsto g_{\gamma(t)}(\sigma^v(t), \sigma^w(t))$ is constant. For this it's enough to prove that its derivative is zero for all t. This condition is local in t, so we may assume, by above remark, that $\sigma^v = s^v \circ \gamma$ and $\sigma^w = s^w \circ \gamma$ for some (local) sections s^v, s^w of E. Then

$$g_{\gamma(t)}(\sigma^v(t), \sigma^w(t)) = [g(s^v, s^w)](\gamma(t)).$$

Hence

$$\begin{aligned}
\left.\frac{d}{dt}\right|_t g_{\gamma(t)}(\sigma^v(t), \sigma^w(t)) =& \dot\gamma(g(s^v, s^w)) \\
&= g(\nabla_{\dot\gamma} s^v, s^w) + g(s^v, \nabla_{\dot\gamma} s^w) \\
&= g(0, s^w) + g(s^v, 0) = 0.
\end{aligned}$$

□

12.2. **Connections induced on submanifolds.** Let (M, g) be a Riemannian manifold and $N \hookrightarrow M$ an embedded submanifold (think of a surface in $\mathbb{R}^3$). We'll see that the embedding induces a Levi-Civita connection on N in two ways that turn out to be equivalent. It will also turn out that for surfaces in $\mathbb{R}^3$ the curvature of the induced connection is intimately related to Gauss curvature.

Suppose $f : N \to M$ is a map of manifolds. Then we can use f to pull back a metric g on M to a positive semi-definite symmetric bilinear form on N:

$$(f^*g)_x(v, w) = g_{f(x)}(df_x v, df_x w)$$

for all $x \in N$, $v, w \in T_xN$. Moreover, if df_x is *injective* then $(f^*g)_x$ is non-degenerate. Therefore if $f : N \to M$ is an immersion then $g^N := f^*g$ is a metric on N. The metric g^N defines a Levi-Civita connection ∇^N on N.

Suppose now that $f : N \hookrightarrow M$ is an *embedding*. Then there is another way to induce a connection on N from a connection M. First of all, for all point $x \in N$ the tangent space T_xM splits as an orthogonal direct sum with respect to g_x:

$$T_xM = T_xN \oplus (T_xN)^\perp.$$

Hence there is an orthogonal projection

$$\Pi_x : T_xM \to T_xN.$$

Globally $\nu := \sqcup_x (T_xN)^\perp$ is a vector bundle, the *normal bundle* of the embedding of N into M. Hence globally the first equation says that the restriction $TM|_N$ is a direct sum of two bundles:

$$TM|_N = TN \oplus \nu$$

and the second equation says that we have a bundle map

$$\Pi : TM|_N \to TN.$$

Here is how one can see that Π_x depends smoothly on x: Choose coordinates $\phi = (x_1, \dots, x_n, \dots, x_m) : U \to \mathbb{R}^m$ on M near a point $x \in N$ that are adapted to N. That is, $\phi(N \cap U) = \phi(U) \cap \{x_{n+1} = 0, \dots, x_m = 0\}$. Apply Gram-Schmidt to the basis vectors $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \dots, \frac{\partial}{\partial x_m}\}$ to obtain an orthonormal frame $\{e_1(x), \dots, e_n(x), \dots, e_m(x)\}$ on TU. Remember that every tangent space T_xM has an inner product g_x that depends smoothly on x. The Gram-Schmidt is smooth in the inner product. Define the projection Π by

$$\Pi_x(v) = \sum_{i=1}^n g_x(v, e_i(x)) e_i(x)$$

Definition 12.12. Let $N \subset M$ be an embedded submanifold. A vector field $\tilde{X} \in \Gamma(TM)$ is an *extension* of a vector field $X \in \Gamma(TN)$ if

$$X_x = \tilde{X}_x$$

for all $x \in N$. We will also say that $\tilde{X}$ is *tangent to* N.

Lemma 12.13. *Let $N \subset M$ be an embedded submanifold and $X \in \Gamma(TN)$ a vector field. Then for any $x \in N$ there is a neighborhood $U \subset M$ and an extension $\tilde{X} \in \Gamma(TM|_U)$ of $X|_{N \cap U}$.*

Proof. Let $(x_1, \dots, x_n, \dots, x_m) : U \to \mathbb{R}^m$ be coordinates on M adapted to N. Then $X = \sum_{i=1}^n X_i \frac{\partial}{\partial x_i}$, with X_i being smooth functions on $U \cap N$. Extend X_i to all of U by making them constant in $x_{n+1}, \dots, x_m$. This extends X to all of U. □

Lemma 12.14. *Let $N \subset M$ be an embedded submanifold, $X, Y \in \Gamma(TN)$ be two vector fields and $\tilde{X}, \tilde{Y} \in \Gamma(TM)$ their extensions. Then their Lie bracket $[\tilde{X}, \tilde{Y}]$ is tangent to N, hence is an extension of $[X, Y]$.*

Proof. We give two proofs. The first is computational. In coordinates $(x_1, \dots, x_n, \dots, x_m)$ on M adapted to N, $\tilde{X} = \sum_{i=1}^m \tilde{X}_i \frac{\partial}{\partial x_i}$ with $\tilde{X}_i(x) = 0$ for $i > n$ for all $x \in N$. Similarly $\tilde{Y} = \sum_{i=1}^m \tilde{Y}_i \frac{\partial}{\partial x_i}$ with $\tilde{Y}_i(x) = 0$ for $i > n$ for all $x \in N$. Since

$$[\tilde{X}, \tilde{Y}] = \sum_{i,j} \tilde{X}_i \frac{\partial \tilde{Y}_j}{\partial x_i} \frac{\partial}{\partial x_j} - \sum_{i,j} \tilde{Y}_j \frac{\partial \tilde{X}_i}{\partial x_j} \frac{\partial}{\partial x_i}$$

for $i > n$ the coefficient in front of $\frac{\partial}{\partial x_i}$ vanishes at the points of N.

Here is a geometric proof. If $\tilde{X}$ is tangent to N, its flow ϕ_t preserves N (maps it into itself). Hence its differential $d\phi_t$ maps vectors tangent to N to vectors tangent to N. But $\tilde{Y}$ is tangent to N. Hence for any $x \in N$

$$(d(\phi_{-t})\tilde{Y})_x \in T_xN$$

for all t. Differentiating with respect to t we get

$$[\tilde{X}, \tilde{Y}]_x \in T_xN.$$

□

We now define a connection $\bar{\nabla}$ on a manifold N induced by its embedding into a Riemannian manifold (M, g) by

$$\bar{\nabla}_X Y(x) := \Pi_x(\nabla_{\tilde{X}} \tilde{Y}(x)),$$

where $x \in N$ is a point, $X, Y \in \Gamma(TN)$ are two vector fields, $\tilde{X}, \tilde{Y}$ their (local) extensions to M, $\Pi_x : T_xM \to T_xN$ is the orthogonal projection and ∇ is the Levi-Civita connection on (M, g).

We need to make sure that $\bar{\nabla}$ is well-defined, that is, that $\bar{\nabla}_X Y(x)$ does not depend on the choice of the local extensions $\tilde{X}, \tilde{Y}$. By Corollary 11.10.2 $\nabla_{\tilde{X}} \tilde{Y}(x)$ depends only on $\tilde{X}_x = X_x$ and the values of $\tilde{Y}$ along the integral curve of $\tilde{X}$ through x. Therefore $\nabla_{\tilde{X}} \tilde{Y}(x)$ depends only on X_x and the values of Y along the integral curve of X through x. Hence $\bar{\nabla}$ is well-defined. Moreover, $\bar{\nabla}_X Y$ is clearly tensorial in the X slot. To see that it is a connection, let $f \in C^\infty(N)$ be a function and $\tilde{f}$ its (local) extension to M. Then, at the points of N,

$$\begin{aligned}\bar{\nabla}_X(fY) &= \Pi(\nabla_{\tilde{X}}(\tilde{f}\tilde{Y}) = \Pi((\tilde{X}\tilde{f})\tilde{Y} + \tilde{f}\nabla_{\tilde{X}}\tilde{Y})\\ &= (\tilde{X}\tilde{f})\Pi(\tilde{Y}) + \tilde{f}\Pi(\nabla_{\tilde{X}}\tilde{Y}) = (Xf)Y + f\bar{\nabla}_X Y.\end{aligned}$$

We conclude that the induced connection $\bar{\nabla}$ is indeed a connection.

Remark 12.15. The projection Π is really necessary in the definition of the induced connection. This is because even if vector fields $\tilde{X}$ and $\tilde{Y}$ are tangent to a submanifold N there is no reason for their covariant derivative $\nabla_{\tilde{X}} \tilde{Y}$ to be tangent to N. Here is an example:

Let $W = Z = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}$, two vector fields on $M = \mathbb{R}^2$. Let D denote the Levi-Civita connection on $\mathbb{R}^2$ for the standard metric $dx \otimes dx + dy \otimes dy$. Then

$$D_W Z = (W x_2)\frac{\partial}{\partial x_1} + (W(-x_1))\frac{\partial}{\partial x_2} = -x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2}.$$

Let $N = S^1$. Then W and Z are tangent to N, hence are extensions of a vector field on N. But $D_W Z$ is orthogonal to S^1.

Lemma 12.16. *Let (M, g) be a Riemannian manifold and $i : N \hookrightarrow M$ an embedded submanifold. Then the connection $\bar{\nabla}$ induced on N by the Levi-Civita connection ∇ on M is the Levi-Civita connection for the pullback metric $g^N := i^* g$.*

Proof. It is enough to check that

(1) $\bar{\nabla}$ is torsion-free and that
(2) $\bar{\nabla}$ is metric.

For all $X, Y \in \Gamma(TN)$ and their local extensions $\tilde{X}, \tilde{Y} \in \Gamma(TM)$

$$\bar{\nabla}_X Y - \bar{\nabla}_Y X = \Pi(\nabla_{\tilde{X}}\tilde{Y} - \nabla_{\tilde{Y}}\tilde{X}) = \Pi([\tilde{X}, \tilde{Y}]) = \Pi([X, Y]) = [X, Y].$$

To show that $\bar{\nabla}$ is metric we need to check that

$$Z(g^N(X, Y)) = g^N(\bar{\nabla}_Z X, Y) + g^N(X, \bar{\nabla}_Z Y)$$

for any vector fields X, Y, Z on N. At any point of N,

$$\begin{aligned} Z(g^N(X, Y)) &= \tilde{Z}(g(\tilde{X}, \tilde{Y}))\\ &= g(\nabla_{\tilde{Z}}\tilde{X}, \tilde{Y}) + g(\tilde{X}, \nabla_{\tilde{Z}}\tilde{Y})\\ &= g(\bar{\nabla}_Z X + (\nabla_{\tilde{Z}}\tilde{X} - \bar{\nabla}_Z X), Y) + g(X, \bar{\nabla}_Z Y + (\nabla_{\tilde{Z}}\tilde{Y} - \bar{\nabla}_Z Y))\\ &= g(\bar{\nabla}_Z X, Y) + g(X, \bar{\nabla}_Z Y). \end{aligned}$$

since $\nabla_{\tilde{Z}}\tilde{Y} - \bar{\nabla}_Z Y$ and $\nabla_{\tilde{Z}}\tilde{X} - \bar{\nabla}_Z X$ are perpendicular to N. □

12.3. **The second fundamental form of an embedding.** As before let $N \hookrightarrow M$ be an embedded submanifold of a Riemannian manifold (M, g). We want to understand how much N curves in M. We define a tensor, the second fundamental form $I\!I_x : T_xN \times T_xN \to (T_xN)^\perp$ to measure the extrinsic geometry of N in M. We first define

$$I\!I : \Gamma(TN) \times \Gamma(TN) \to \Gamma(TN^\perp)$$

by

$$I\!I(X, Y) = \nabla_{\tilde{X}}\tilde{Y} - \bar{\nabla}_X Y,$$

where, as before, ∇ is the Levi-Civita connection on M, $\bar{\nabla}$ is the induced Levi-Civita connection on N, $\tilde{X}, \tilde{Y} \in \Gamma(TM)$ are local extensions of the vector fields $X, Y \in \Gamma(TN)$.

Proposition 12.17. *The map $I\!I$ defined above is symmetric and tensorial.*

Proof. We first argue that $I\!I$ is symmetric.

$$\begin{aligned} I\!I(X, Y) - I\!I(Y, X) &= (\nabla_{\tilde{X}}\tilde{Y} - \bar{\nabla}_X Y) - (\nabla_{\tilde{Y}}\tilde{X} - \bar{\nabla}_Y X) \\ &= (\nabla_{\tilde{X}}\tilde{Y} - \nabla_{\tilde{Y}}\tilde{X}) - (\bar{\nabla}_X Y - \bar{\nabla}_Y X) \\ &= [\tilde{X}, \tilde{Y}] - [X, Y] = 0. \end{aligned}$$

Next we argue that $I\!I$ is tensorial in the first slot. Let $\tilde{f}$ be a local extension of a function f on N. Then at the points of N,

$$I\!I(fX, Y) = \nabla_{\tilde{f}\tilde{X}}\tilde{Y} - \bar{\nabla}_{fX} Y = \tilde{f}\nabla_{\tilde{X}}\tilde{Y} - f\bar{\nabla}_X Y = f\, I\!I(X, Y).$$

□

It follows that for all points $x \in N$ there is a symmetric bilinear map

$$I\!I_x : T_xN \times T_xN \to (T_xN)^\perp.$$

Remark 12.18. In classical terminology the *first fundamental form* of an embedding is the induced metric.

Next suppose that the embedded submanifold N is a *hypersurface*, that is, that $\dim M - \dim N = 1$. Then the normal bundle $TN^\perp$ has 1-dimensional fibers hence, locally, a frame on $TN^\perp$ is defined by one nowhere zero vector field. By rescaling, if necessary, we may assume that this vector n field has length 1 everywhere:

$$g_x(n_x, n_x) = 1$$

for all points $x \in N$. We furthermore make an extra assumption that unit vector field n normal to N is defined on *all* of N. That is, N is orientable inside M. This is true for the sphere embedded in $\mathbb{R}^3$ but false for the central circle of the Möbius band inside the band. If $N \subset M$ has a globally defined unit normal n, we can write

$$I\!I_x(v, w) = h_x(v, w) n_x$$

for a symmetric bilinear map $h_x : T_xN \times T_xN \to \mathbb{R}$. Unwinding the definitions we see that for any vector fields X, Y on N

$$h(X, Y) = g(\nabla_{\tilde{X}}\tilde{Y}, n).$$

We will refer to $h \in \Gamma(TN^* \otimes TN^*)$ also as the second fundamental form. The second fundamental form h allows us to relate the curvature tensor R of the Levi-Civita connection on M, the *Riemann curvature* of M, and the curvature $\bar{R}$ of the induced connection on N:

Theorem 12.19. *Let $N \hookrightarrow M$ be an embedded orientable hypersurface of a Riemannian manifold (M, g). Let $h \in \Gamma(T^*N^{\otimes 2})$ be the second fundamental form of the embedding. Then for any vector fields $X, Y, Z, W \in TN$*

$$g(R(X, Y)Z, W) = g^N(\bar{R}(X, Y)Z, W) - h(Y, Z)h(X, W) + h(X, Z)h(Y, W), \tag{12.4}$$

where R is the Riemann curvature tensor of M and $\bar{R}$ is the induced Riemann curvature tensor of N.

We prove an easy lemma before tackling the computations involved in the proof of the theorem.

Lemma 12.20. *Let (M, g), ∇, N, n and h be as above. Then*

$$h(X, W) = -g(\nabla_X n, W).$$

for any vector fields $X, W \in \Gamma(TN)$ (here we didn't bother with putting tildes on the extensions).

Proof. The function $g(n, W)$ is identically 0 on N. Hence

$$0 = X(g(n, W)) = g(\nabla_X n, W) + g(n, \nabla_X W)$$

since ∇ is a metric connection. ☐

Proof of Theorem 12.19. Recall that

$$R(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z.$$

$$\begin{aligned}\nabla_X(\nabla_Y Z) &= \nabla_X(\bar{\nabla}_Y Z + h(Y,Z)n) \\ &= \bar{\nabla}_X(\bar{\nabla}_Y Z) + h(X, \bar{\nabla}_Y Z)n + (Xh(Y,Z))n + h(Y,Z)\bar{\nabla}_X n.\end{aligned}$$

Hence

$$g(\nabla_X(\nabla_Y Z), W) = g(\bar{\nabla}_X(\bar{\nabla}_Y Z), W) + h(Y,Z)g(\bar{\nabla}_X n, W) = g(\bar{\nabla}_X(\bar{\nabla}_Y Z), W) - h(Y,Z)h(X,W). \tag{12.5}$$

Similarly,

$$g(\nabla_Y(\nabla_X Z), W) = g(\bar{\nabla}_Y(\bar{\nabla}_X Z), W) - h(X,Z)h(Y,W), \tag{12.6}$$

while

$$g(\nabla_{[X,Y]}Z, W) = g(\bar{\nabla}_{[X,Y]}Z, W). \tag{12.7}$$

Subtracting (12.6) and (12.7) from (12.5) we get (12.4). ☐

Let us see what the theorem tells us about the curvature of oriented surfaces in $\mathbb{R}^3$. If $N \subset \mathbb{R}^3$ is an oriented embedded manifold, then the unit normal field n assigns to every point in N a unit vector in $\mathbb{R}^3$. Hence we can think of n as a map to the unit sphere,

$$n : N \to S^2.$$

This is the *Gauss map*. Since T_xN and $T_{n_x}S^2$ are two planes perpendicular to the same vector n_x, they are the same two plane in $\mathbb{R}^3$. Therefore we may think of the differential dn_x of the Gauss map as a map

$$dn_x : T_xN \to T_xN.$$

Definition 12.21. The *Gauss curvature* κ of an oriented surface N in $\mathbb{R}^3$ is the determinant of the differential of the Gauss map:

$$\kappa(x) = \det dn_x.$$

We compute a few examples of Gauss curvature by brute force.

Example 12.22. Consider

$$N = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = 0\},$$

a plane. The normal vector field $n(x)$ is constant, and so the Gauss curvature $\kappa(x)$ is 0.

Example 12.23. Now let N be a round cylinder:

$$N = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2^2 + x_3^2 = R^2\},$$

Here the unit normal $n(x)$ is constant in the x_1 direction. Hence, $dn_x(e_1) = 0$, and so the Gauss curvature is again zero.

Example 12.24. Let N be the standard round sphere of radius R:

$$N = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = R^2\}.$$

Then the normal vector field n is given by $n(x) = \frac{1}{R}x$, hence

$$dn = \frac{1}{R} \cdot id.$$

Therefore

$$\kappa(x) = \frac{1}{R^2}.$$

Note that the Gauss curvature is constant and positive. Also, the bigger the radius of the sphere the smaller the Gauss curvature. This makes sense since the sphere gets flatter as its radius increases.

In general one computes the Gauss curvature from the first and second fundamental form.Once again we denote the Levi-Civita connection on $\mathbb{R}^3$ by D. Then for any vector v and vector field $Y : \mathbb{R}^3 \to \mathbb{R}^3$

$$D_v Y = dY(v).$$

Hence for any two vector fields X, Y on a surface N,

$$h_x(X_x, Y_x) = -g_x((D_X n)(x), Y_x) = -g_x(dn_x(X_x), Y_x). \tag{12.8}$$

In particular the differential of the Gauss map is completely determined by the induced metric and the second fundamental form. We will see shortly that the Gauss curvature depends only on the metric g and its first and second partials. But first we extract Gauss curvature from the above equation.

Lemma 12.25. *Let g be a positive definite inner product on a vector space V, $h : V \times V \to \mathbb{R}$ a symmetric bilinear map and $S : V \to V$ the linear map uniquely defined by*

$$h(v, w) = g(Sv, w).$$

Let $\{e_i\}$ be a basis of V. Then

$$\det(h(e_i, e_j)) = \det(g(e_i, e_j)) \det S.$$

Proof. The matrix (s_{ki}) of S with respect to the basis $\{e_i\}$ is defined by

$$Se_i = \sum_k s_{ki} e_k.$$

Therefore

$$h(e_i, e_j) = g(Se_i, e_j) = g(\sum_k s_{ki} e_k, e_j) = \sum_k s_{ki} g(e_k, e_j).$$

Therefore the matrix $(h(e_i, e_j))$ is the product of matrices (s_{ki}) and $(g(e_j, e_k)) = (g(e_k, e_j))$. Thus

$$\det(h(e_i, e_j)) = \det(g(e_j, e_k)) \det(s_{ki}).$$

□

Together Lemma 12.25 above and (12.8) tell us how to compute the Gauss curvature: pick a basis $\{e_1, e_2\}$ of the tangent space $T_x N$. Then

$$\kappa(x) = \frac{\det(h(e_i, e_j))}{\det(g(e_i, e_j))}.$$

In particular, if the basis $\{e_1, e_2\}$ is orthonormal with respect to the induced metric g,

$$\kappa(x) = \det(h(e_i, e_j)).$$

We are now ready to prove Gauss' *theorema egregium* ("remarkable theorem") from 1828!

Theorem 12.26. *Let $N \hookrightarrow \mathbb{R}^3$ be an oriented embedded surface. Let $\bar{R}$ denote the Riemann curvature on N. Then the Gauss curvature κ is given by*

$$\kappa(x) = -g^N_x(\bar{R}_x(e_1, e_2)e_1, e_2)$$

where $\{e_1, e_2\}$ is a basis of $T_x N$ orthonormal with respect to the induced metric g^N.

Hence the Gauss metric depends only on the induced metric and its first and second partials and not on the embedding.

Proof. The Riemann curvature of the standard Levi-Civita connection D on $\mathbb{R}^3$ is 0. Hence, by Theorem 12.19

$$g^N_x(\bar{R}_x(e_1, e_2)e_1, e_2) = h_x(e_2, e_1)h_x(e_1, e_2) - h_x(e_1, e_1)h_x(e_1, e_1) = -\det(h_x(e_i, e_j)) = -\kappa(x).$$

The curvature of a connection depends on the Christoffel symbols and their first partials. The Christoffel symbols of a Levi-Civita connection are functions of the metric and its first partials. □

Exercise 12.1. Let $f(x, y)$ be a smooth function on $\mathbb{R}^2$ and N its graph in $\mathbb{R}^3$:

$$N = \{(x, y, f(x, y)) \mid (x, y) \in \mathbb{R}^2\}$$

Show that the Gauss curvature κ is given by

$$\kappa = \frac{f_{xx}f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}$$

where $f_{xy} = \frac{\partial^2 f}{\partial x \partial y}$ and so on.

13. Geodesics as critical points of the energy functional

This section is a brief excursion into the calculus of variations. The basic setup is this. Let M be a manifold. Consider the set of all maps $\mathcal{P}$ from a fixed interval $[a, b]$ to M with fixed end points:

$$\mathcal{P} = \mathcal{P}([a, b], q_1, q_2) = \{\gamma : [a, b] \to M \mid \gamma(a) = q_1, \gamma(b) = q_2\},$$

where $q_1, q_2 \in M$ are two points. Every path $\gamma \in \mathcal{P}$ gives rise to a path $\dot{\gamma} : [a, b] \to TM$. Therefore, a smooth function $L : TM \to \mathbb{R}$ on the tangent bundle of M (a "Lagrangian") defines a map ("action")

$$\mathcal{A} : \mathcal{P} \to \mathbb{R}, \quad \mathcal{A}(\gamma) = \int_a^b L(\dot{\gamma}(t))\, dt.$$

For example, if g is a Riemannian metric on a manifold M then

$$L(x, v) = \frac{1}{2} g_x(v, v) \quad x \in M, v \in T_x M$$

is a Lagrangian and the corresponding action

$$\mathcal{A}_L(\gamma) = \int_a^b \frac{1}{2} g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))\, dt$$

is the "energy" of the path. The term "energy" comes from the fact that for a particle of mass m moving in $\mathbb{R}^3$ the quantity $\frac{1}{2}m(v_1^2 + v_1^2 + v_3^2) = \frac{1}{2}m||v||^2$ is the kinetic energy.

We want to make sense of a path $\gamma \in \mathcal{P}$ being *critical* for a an action $\mathcal{A}_L : \mathcal{P} \to \mathbb{R}$. This is a bit delicate since we have been careless with the topology on $\mathcal{P}$ and since $\mathcal{P}$ is infinite dimensional. The cheapest way to do it is by analogy with a finite dimensional case: a point is critical for a function f if and only if for every path $\sigma(s)$ through the point, we have $\frac{d}{ds}\Big|_{s=0} f(\sigma(s)) = 0$. Now, a path in the space $\mathcal{P}$ through $\gamma^0 \in \mathcal{P}$ is a family of curves γ_s with $\gamma_s|_{s=0} = \gamma^0$, where s varies in some open interval $(-\epsilon, \epsilon)$. We say that γ_s depends smoothly on s if the map

$$(-\epsilon, \epsilon) \times [a, b] \to M, \quad (s, t) \mapsto \gamma_s(t)$$

is smooth.

Definition 13.1. Let $\mathcal{P} = \mathcal{P}([a, b], q_1, q_2)$ be a space of paths in a manifold M and $L : TM \to \mathbb{R}$ a Lagrangian. A path $\gamma^0 \in \mathcal{P}$ is L-*critical* if for any family γ_s of paths through γ^0 we have

$$\frac{d}{ds}\Big|_{s=0} (\mathcal{A}_L(\gamma_s)) = 0,$$

where $\mathcal{A}_L$ is the associated action.

A connection between variational problems and Riemannian geometry is provided by the following theorem.

Theorem 13.2. *Let (M, g) be a Riemannian manifold and $L(x, v) = \frac{1}{2}g_x(v, v)$ the associated Lagrangian. A path γ is L-critical if and only if γ is a geodesic of the Levi-Civita connection.*

We will first prove the theorem above locally, when the image of the path is contained in a coordinate chart. We will then show that any L-critical path is a geodesic. We will not have time to prove the converse. We start by examining what critical paths for an arbitrary Lagrangian look like locally.

Theorem 13.3. *Let $L : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$, $(x, v) \mapsto L(x, v)$ be a Lagrangian. A path $\gamma^0(t) = (\gamma^0_1(t), \ldots, \gamma^0_m(t)) : [a, b] \to \mathbb{R}^m$ is L-critical if and only if it satisfies the Euler-Lagrange equations:*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v_i}(\gamma(t), \dot{\gamma}(t))\right) - \frac{\partial L}{\partial x_i}(\gamma(t), \dot{\gamma}(t)) = 0, \quad 1 \le i \le m. \tag{13.1}$$

Proof. Let $\gamma_s(t) = \gamma(s, t) = (\gamma_1(s, t), \ldots, \gamma_m(s, t))$ be a variation of γ^0. Then $\gamma(0, t) = \gamma^0(t)$ for all t, and $\gamma(s, a) = \gamma^0(a)$, $\gamma(s, b) = \gamma^0(b)$ for all s. Hence

$$h(t) := \frac{\partial}{\partial s}\Big|_{s=0} \gamma(s, t) : [a, b] \to \mathbb{R}^m$$

has to vanish at $t = a$ and at $t = b$. It's important that there are no other restrictions on h: given an arbitrary curve $h : [a, b] \to \mathbb{R}^m$ which vanishes at the endpoints,

$$\gamma(s, t) := \gamma^0(t) + sh(t)$$

is a variation of γ^0. Note further that $\dot{\gamma}_s(t) = \frac{\partial}{\partial t}\Big|_t \gamma(s, t)$ and consequently

$$\frac{\partial}{\partial s}\Big|_{s=0} \dot{\gamma}_s(t) = \frac{\partial^2 \gamma}{\partial s \partial t}\Big|_{(0,t)} = \frac{d}{dt}\Big|_t (\frac{\partial}{\partial s}\Big|_{s=0} \gamma(s, t) = \dot{h}(t).$$

Since γ^0 is L-critical,

$$\begin{aligned} 0 &= \frac{d}{ds}\Big|_{s=0} \int_a^b L(\gamma(t, s), \dot{\gamma}(t, s))\, dt \\ &= \int_a^b \frac{\partial}{\partial s}\Big|_{s=0} L(\gamma(t, s), \dot{\gamma}(t, s))\, dt \\ &= \int_a^b \sum_i \left(\frac{\partial L}{\partial x_i}(\gamma^0, \dot{\gamma}^0)\frac{\partial \gamma_i}{\partial s}\Big|_{s=0} + \frac{\partial L}{\partial v_i}(\gamma^0, \dot{\gamma}^0)\frac{\partial \dot{\gamma}_i}{\partial s}\Big|_{s=0}\right) dt \\ &= \sum_i \int_a^b (\frac{\partial L}{\partial x_i}(\gamma^0, \dot{\gamma}^0) h_i + \frac{\partial L}{\partial v_i}(\gamma^0, \dot{\gamma}^0)\dot{h}_i)\, dt. \end{aligned}$$

Integration by parts gives

$$\int_a^b \frac{\partial L}{\partial v_i}(\gamma^0, \dot{\gamma}^0)\dot{h}_i\, dt = \frac{\partial L}{\partial v_i}(\gamma^0, \dot{\gamma}^0) h_i\Big|_a^b - \int_a^b \frac{d}{dt}\left(\frac{\partial L}{\partial v_i}(\gamma^0(t), \dot{\gamma}^0(t))\right) h_i(t)\, dt.$$

Therefore

$$0 = \sum_i \int_a^b \left(\frac{\partial L}{\partial x_i}(\gamma^0, \dot{\gamma}^0) - \frac{d}{dt}\left(\frac{\partial L}{\partial v_i}(\gamma^0(t), \dot{\gamma}^0(t))\right)\right) h_i(t)\, dt.$$

Since $h_i(t)$ are arbitrary, the equation above forces (13.1): see Lemma 13.4 below.

Running the computations backwards we see that if γ^0 satisfies the Euler-Lagrange equations then γ^0 is L-critical. □

Lemma 13.4. *If $f \in C^\infty([a, b])$ is a smooth function and if for any $h \in C^\infty([a, b])$ with $h(a) = h(b) = 0$ we have $\int_a^b f(t)h(t)\, dt = 0$, then $f(t) \equiv 0$.*

Proof. Exercise. □

Proposition 13.5. *Let g be a metric on $\mathbb{R}^m$ and $L(x, v) = \frac{1}{2} g_x(v, v)$ the associated Lagrangian. Then γ is L-critical if and only if it is a geodesic for the Levi-Civita connection defined by the metric g.*

Proof. We have

$$2L(x, v) = \sum_{k,l} g_{kl}(x)\, v_k v_l.$$

Therefore, for each index i,

$$2\frac{\partial L}{\partial x_i} = \sum_{k,l} \frac{\partial g_{kl}}{\partial x_i} v_k v_l$$

and

$$2\frac{\partial L}{\partial v_i} = \sum_{k,l}(g_{il}v_l + g_{ki}v_k).$$

The Euler-Lagrange equations in this case then are

$$\sum_{k,l} \frac{\partial g_{kl}}{\partial x_i}\dot\gamma_k\dot\gamma_l = \frac{d}{dt}\left(\sum_{k,l}(g_{il}\dot\gamma_l + g_{ki}\dot\gamma_k)\right).$$

Differentiating and gathering $\ddot\gamma_s$ terms on one side, we get:

$$\sum_s g_{is}\ddot\gamma_s = -\frac{1}{2}\sum_{k,l}\Big(\frac{\partial g_{ki}}{\partial x_l} + \frac{\partial g_{il}}{\partial x_k} - \frac{\partial g_{kl}}{\partial x_i}\Big)\dot\gamma_l\dot\gamma_k. \tag{13.2}$$

Here we used the fact that $\gamma_{is} = \gamma_{si}$; this is where the $\frac{1}{2}$ comes from. As before we denote the entries of the inverse of the matrix $(g_{\alpha\beta})$ by $g^{\alpha\beta}$ so that $\sum_\beta g^{\alpha\beta}g_{\beta\gamma} = \delta_{\alpha\gamma}$. Therefore if we multiply both sides of (13.2) by g^{ji} and sum on i we get

$$\ddot\gamma_j = -\frac{1}{2}\sum_{i,k,l} g^{ji}\Big(\frac{\partial g_{ki}}{\partial x_l} + \frac{\partial g_{il}}{\partial x_k} - \frac{\partial g_{kl}}{\partial x_i}\Big)\dot\gamma_l\dot\gamma_k = -\sum_{k,l}\Gamma^j_{kl}\dot\gamma_k\dot\gamma_l,$$

where Γ^i_{kl} are the Christoffel symbols for the Levi-Civita connection (cf. (12.3)). We now see that this is the geodesic equation. Thus, L-critical curves are geodesics and vice versa. $\square$

The result for Lagrangians on $\mathbb{R}^n$, Theorem 13.3, and the corresponding result for geodesics, Proposition 13.5, generalize to the manifold setting. To be precise, recall that if $(x_1, \dots, x_n) : U \to \mathbb{R}^m$ is a coordinate chart on a manifold M, then it defines an associated coordinate chart $(x_1, \dots, x_m, v_1, \dots, v_m) : TU \to \mathbb{R}^m \times \mathbb{R}^m$ on the tangent bundle of M. Namely, if $q \in U$ is a point and $w \in T_qU = T_qM$ is a vector, then there are unique numbers $v_1 = v_1(w), \dots, v_m = v_m(w)$ so that

$$w = \sum_i v_i(w)\frac{\partial}{\partial x_i}\Big|_q,$$

since $\left\{\frac{\partial}{\partial x_i}\Big|_q\right\}$ is a basis of T_qM. Of course, $v_i(w) = (dx_i)_q(w)$.

Proposition 13.6. *Let M be a manifold and $L : TM \to \mathbb{R}$ a Lagrangian. If a path $\gamma^0 : [a,b] \to M$ lies entirely inside a coordinate chart $(x_1, \dots, x_n) : U \to \mathbb{R}^m$ (i.e., $\gamma([a,b]) \subset U$), then*

$$(\gamma^0_1(t), \dots, \gamma^0_m(t), \dot\gamma^0_1(t), \dots, \dot\gamma^0_m(t)) := (x_1 \circ \gamma^0(t), \dots, x_m \circ \gamma^0(t), v_1 \circ \dot\gamma^0(t), \dots, v_m \circ \dot\gamma^0(t))$$

satisfies the Euler-Lagrange equations. Here, as above, $(x_1, \dots, x_m, v_1, \dots, v_m) : TU \to \mathbb{R}^m \times \mathbb{R}^m$ is the coordinate chart on the tangent bundle TM associated with the chart $(x_1, \dots, x_n) : U \to \mathbb{R}^m$ on the manifold M.

Proof. The only possible concern is that the image of a variation γ_s of our curve γ^0 lies outside the domain U of our coordinate chart. But we only care about γ_s for s small, and for small values of the parameter s the variation $\gamma_s(t)$ is close to $\gamma^0(t)$, hence lies in U. $\square$

From Propositions 13.5 and 13.6 we deduce:

Corollary 13.6.1. *Let M be a manifold with a Lagrangian L. A path $\gamma^0 : [a,b] \to M$ lying inside a coordinate chart on M is a geodesic for a Riemannian metric g if and only if γ^0 is critical for the energy Lagrangian $L(x,v) = \frac{1}{2}g_x(v,v)$.*

What about L-critical paths whose images cannot be covered by a single coordinate chart? Suppose $\gamma : [a,b] \to M$ is L-critical and for some time t_0 the point $\gamma(t_0)$ lies in a coordinate chart $(x_1, \dots, x_m) : U \to \mathbb{R}^m$. Then $\gamma([a',b']) \subset U$ for some subinterval $[a',b'] \subset [a,b]$ containing t_0. Any variation of $\gamma|_{[a',b']}$ is a variation of γ. Hence $\gamma|_{[a',b']}$ is also L-critical. Therefore it satisfies Euler-Lagrange equations in the chart U. In particular, if γ is critical for the energy Lagrangian, then γ is a geodesic in every coordinate chart, hence a geodesic. This proves one global direction of Theorem 13.2, as promised.

The converse is true as well, but this requires a coordinate-free description of L-critical curves which we don't have time for.