Lecture 1. Definition and Examples

Mathematics is a study of patterns. Category theory is the study of patterns in mathematics.

To make the two sentences less cryptic, we start by defining what a category is. We pick one of several (mostly equivalent) definitions and proceed from there.

Definition 1.1. A category \( \mathcal{C} \) consists of the following data:

1. A collection of objects \( \mathcal{C}_0 \).
2. For each pair of objects \( a, b \in \mathcal{C}_0 \), a collection of morphisms \( \text{Hom}_\mathcal{C}(a, b) \), which may be empty if \( a \neq b \). We write \( a \xrightarrow{f} b \) or \( b \xleftarrow{f} a \) if \( f \in \text{Hom}_\mathcal{C}(a, b) \).
   We require that \( \text{Hom}_\mathcal{C}(a, b) \cap \text{Hom}_\mathcal{C}(c, d) = \emptyset \) if \( a \neq c \) and \( b \neq d \).
3. For each object \( a \in \mathcal{C}_0 \), a distinguished identity morphism \( \text{id}_a \in \text{Hom}_\mathcal{C}(a, a) \);
4. For each triple of object \( a, b, c \in \mathcal{C}_0 \), a composition map

\[
\circ : \text{Hom}_\mathcal{C}(b, c) \times \text{Hom}_\mathcal{C}(a, b) \to \text{Hom}_\mathcal{C}(a, c), \quad (c \xleftarrow{g} b, b \xrightarrow{f} a) \mapsto (c \xleftarrow{gf} a)
\]

which is subject to the following conditions:
   (i) for all \( a, b \in \mathcal{C}_0 \) and for all \( f \in \text{Hom}_\mathcal{C}(a, b) \), \( f \circ \text{id}_a = f = \text{id}_b \circ f \);
   (ii) The composition \( \circ \) is associative: for all \( d \xrightarrow{h} c, c \xrightarrow{g} b, b \xrightarrow{f} a \), \( h \circ (g \circ f) = (h \circ g) \circ f \).

We denote the collection of all morphisms of the category \( \mathcal{C} \) by \( \mathcal{C}_1 \). Thus \( \mathcal{C}_1 = \bigcup_{a,b \in \mathcal{C}_0} \text{Hom}_\mathcal{C}(a, b) \).

Remark 1.2. Given an object \( a \) of a category \( \mathcal{C} \) we write \( a \in \mathcal{C} \). (Writing \( a \in \mathcal{C}_0 \) is more precise and more pedantic.)

Remark 1.3. Given a category \( \mathcal{C} \) and two object \( a, b \in \mathcal{C} \), the collection of morphisms \( \text{Hom}_\mathcal{C}(a, b) \) from \( a \) to \( b \) may also be denoted by \( \text{Hom}(a, b) \) when the category \( \mathcal{C} \) is understood. Some authors write \( \mathcal{C}(a, b) \) for \( \text{Hom}_\mathcal{C}(a, b) \). We will not use \( \mathcal{C}(a, b) \).
Definition 1.4. Given a category \( C \) and a morphism \( f \in \text{Hom}_C(a, b) \), we call \( a \) the source of \( f \) and \( b \) the target of \( f \). We write \( t(f) = a \) and \( s(f) = a \). In other word, we have two maps \( s, t : C_1 \to C_0 \) from morphisms to objects. Note that had we not required that \( \text{Hom}_C(a, b) \cap \text{Hom}_C(c, d) = \emptyset \) for \((a, b) \neq (c, d)\), the source and target maps would not be well-defined.

Remark 1.5. We also have a unit map \( u : C_0 \to C_1, a \mapsto \text{id}_a \) that assigns to each object of a category \( C \) the identity morphism on that object.

Examples

Example 1.6. Sets and functions form a category \( \text{Set} \). The composition in the category \( \text{Set} \) is the composition of functions.

Example 1.7. Groups and group homomorphisms form a category \( \text{Group} \). \( \text{Group} \) is an example of a category whose objects are sets with some structure and whose morphisms are structure preserving maps. Not all categories are like this, but many are.

Example 1.8. There is the empty category with the empty set of objects and the empty set of morphisms, denoted by \( \emptyset \).

Example 1.9. There is a category, often denoted by \( 1 \), with one objects \( * \) and one morphism \( \text{id}_* : * \to * \). This is the smallest nonempty category. Exercise: what’s the composition map?

Definition 1.10. A category \( C \) is small if both the collection of objects \( C_0 \) and the collection of morphisms \( C_1 \) are sets (rather than some bigger collections of objects).

Example 1.11. The categories \( \emptyset \) and \( 1 \) are small. The category \( \text{Set} \) of all sets and functions is not small thanks to Russell’s paradox. We’ll come back to this point.

Definition 1.12. A preorder is a category with at most one morphism between any two objects.

Remark 1.13. A small preorder \( C \) is the same thing as set \( C_0 \) equipped with a binary relation \( \leq \) which is reflexive and transitive:

1. for all \( a \in C_0 \), \( a \leq a \) and
2. if \( a \leq b \) and \( b \leq c \), then \( a \leq c \) for all \( a, b, c \in C_0 \).

Proof. Let \( C \) be a preorder and a small category. Define a relation \( \leq \) on the set of objects \( C_0 \) by

\[
    a \leq b \iff \text{there is a morphism (which is necessarily unique) } f_{ba} : a \to b.
\]

Since for every object \( a \) in \( C \) there is the identity morphism \( \text{id}_a : a \to a \), the relation \( \leq \) is reflexive. Given \( a \leq b \) and \( b \leq c \) there are morphisms \( f_{ba} : a \to b \) and \( f_{cb} : b \to a \). The composite \( f_{cb} \circ f_{ba} : a \to c \) is a morphism from \( a \) to \( c \). Hence \( \text{Hom}_C(a, c) \neq \emptyset \). But \( C \) is a preorder, so there is at most one morphism from \( a \) to \( c \). We conclude that \( f_{cb} \circ f_{ba} = f_{ca} \). That is,

\[
    a \leq b \text{ and } b \leq c \Rightarrow a \leq c.
\]

Conversely given a relation \( \leq \) on a set \( X \) which is reflexive and transitive define a small category \( C \) by setting its collection of objects to be \( X \). Given two elements \( x, y \in X \) we define

\[
    \text{Hom}_C(x, y) = \begin{cases} 
        \{f_{yx} : x \to y\}, & \text{if } x \leq y; \\
        \emptyset & \text{otherwise.}
    \end{cases}
\]

It is not hard to check that \( C \) so defined is in fact a category. \( \square \)

Remark 1.14. Preorders occur naturally in mathematics. Recall that for a set \( Y \) the powerset \( \mathcal{P}(Y) \) is the set of all subsets of \( Y \):

\[
    \mathcal{P}(Y) := \{A \mid A \subseteq Y\}
\]
The subset relation $\subseteq$ on $\mathcal{P}(Y)$ is reflexive and transitive. Hence $\mathcal{P}(Y)$ is a preorder. Note that for a given pair $A, B \subseteq Y$ it may well happen that neither $A \subseteq B$ nor $B \subseteq A$, i.e., the corresponding set $\text{Hom}(A, B)$ in the preorder $\mathcal{P}(Y)$ may well be empty.

You have seen binary relations on a given set. One can also talk about relations between two distinct sets. The following terminology is slightly nonstandard:

**Definition 1.15.** A relation from a set $X$ to a set $Y$ is a subset $R$ of $X \times Y$.

Note that relations can be composed: if $R \subseteq X \times Y$ and $S \subseteq Y \times Z$, then define the composition to be

$$S \circ R = \{(x, z) \in X \times Z \mid \exists y \in Y, (x, y) \in R, (y, z) \in Z\}.$$ 

It is not hard to show that

1. Composition of relations is associative.
2. Denote the diagonal $\Delta_X := \{(x, y) \in X \times X \mid x = y\}$ in the product of a set set $X$ with itself by $\text{id}_X$. Then for a relation $R \subseteq X \times Y$, $\text{id}_Y \circ R = R$ and $R \circ \text{id}_X = R$.

Therefore, sets and relations form a category.

**Notation 1.16.** We denote the category of sets and relations by $\mathbf{Rel}$. In particular for any two sets $X$ and $Y$, $\text{Hom}_{\mathbf{Rel}}(X, Y) = \mathcal{P}(X \times Y)$.

**Remark 1.17.** The category $\mathbf{Rel}$ of relations is an example of a category whose morphisms are not functions.

**Example 1.18.** The collection of real vector spaces and linear maps form a category $\mathbf{Vect}_\mathbb{R}$. The composition is the composition of linear maps. This is yet another example of a category of sets with structure and structure preserving functions.

The next category is similar to Example 1.18 but is not a category of sets with structure and structure preserving maps.

**Example 1.19.** The category $\mathbf{Mat}$ of real matrices whose objects are natural numbers (including 0). A morphism from $n \in \mathbb{N}$ to $m \in \mathbb{N}$ (for $n, m > 0$) is a real $m \times n$ matrix. The composition of morphisms in $\mathbf{Mat}$ is matrix multiplication. Of course each $n \in \mathbb{N}$ is secretly $\mathbb{R}^n$. So $0 \in \mathbb{N}$ is secretly $\mathbb{R}^0 = \{0\}$, the zero-dimensional vector space. For this reason we define

$$\text{Hom}_{\mathbf{Mat}}(m, 0) = \{O_{0,m}\}$$
$$\text{Hom}_{\mathbf{Mat}}(0, n) = \{O_{n,0}\}$$

such that for all $A \in \text{Hom}_{\mathbf{Mat}}(n, m)$, $O_{0,m} \circ A = O_{n,0}$, and $O_{n,0} \circ A = O_{0,m}$. These data make $\mathbf{Mat}$ into a category.

**Remark 1.20.** We will see later that the categories $\mathbf{Mat}$ “is” in an appropriate sense the category $\mathbf{FDVect}_\mathbb{R}$ of finite dimensional vector spaces. Technically the two categories are equivalent.

We end the lecture with a notion of a **monoid**. A monoid is an algebraic structure like a group. For some reason monoids hardly ever show up in undergraduate abstract algebra courses. They are ubiquitous in computer science. Roughly speaking a monoid is a group without inverses. More precisely we have the following definition.

**Definition 1.21.** A monoid is a set $M$ together with a distinguished element $e$ and a “multiplication”

$$m : M \times M \to M \quad (a, b) \mapsto m(a, b) \equiv ab,$$

such that

1. for all $a \in M$, $m(a, e) = a = m(e, a)$ (i.e., $ae = a = ea$) and
(2) for all \( a, b, c \in M \), \( m(a, m(b, c)) = m(m(a, b), c) \) (i.e., \( a(bc) = (ab)c \)).

**Example 1.22.** Any group is a monoid.

**Example 1.23.** \( (\mathbb{N}, +, 0) \) is a monoid with + as the “multiplication” and 0 the identity element.

**Example 1.24.** A monoid \( (M, m, e) \) gives rise to a one object category \( \text{BM} \): the collection of objects of \( \text{BM} \) is a singleton \( \text{BM}_0 = \{ \ast \} \) and \( \text{Hom}_{\text{BM}}(\ast, \ast) = M \). The composition is just the multiplication in the monoid.

**Remark 1.25.** If \( C \) is a small category, then for any object \( a \in C_0 \), \( \text{Hom}_C(a, a) \) is a monoid: the identity morphism \( \text{id}_a \in \text{Hom}_C(a, a) \) is the distinguished element of the monoid and the composition

\[ \circ : \text{Hom}_C(a, a) \times \text{Hom}_C(a, a) \to \text{Hom}_C(a, a) \]

is multiplication in the monoid.

**Lecture 2. Opposite category. Initial and terminal objects**

**Last time:** Definition of a category. Examples including the notion of a preorder and the categories \( \text{Mat} \) and \( \text{Rel} \). Definition of a monoid.

**Definition 2.1.** Let \( C \) be a category, a morphism \( f : a \to b \) in \( C \) is an isomorphism if there exists a morphism \( g : b \to a \) such that \( f \circ g = \text{id}_a \) and \( g \circ f = \text{id}_b \).

**Example 2.2.** In the category \( \text{Set} \), isomorphisms are bijection between sets. In the category \( \text{Vect}_R \), isomorphisms are invertible linear maps. In the category \( \text{Group} \), isomorphisms are isomorphism of groups.

**Remark 2.3.** If \( f : a \to b \) is an isomorphism in a category \( C \) then there is only one morphism \( g : b \to a \) so that \( f \circ g = \text{id}_a \) and \( g \circ f = \text{id}_b \). Here is a quick proof: suppose \( h : b \to a \) is another morphism with \( \text{id}_a = f \circ h \) and \( \text{id}_b = h \circ f \). Then, since the composition \( \circ \) is associative,

\[ g = g \circ \text{id}_a = g \circ (f \circ h) = (g \circ f) \circ h = \text{id}_b \circ h = h. \]

**Definition 2.4.** Let \( f : a \to b \) be an isomorphism in a category \( C \). We call the unique morphism \( g : b \to a \) so that \( f \circ g = \text{id}_a \) and \( g \circ f = \text{id}_b \) the inverse of \( f \) and denote it by \( f^{-1} \).

We now introduce the opposite category, that is, the category opposite to a given category \( C \).

**Definition 2.5.** Let \( C \) be a category. The **opposite category** \( C^{\text{op}} \) is defined by reversing all the morphisms of \( C \). Formally, we defined the collection of objects \( (C^{\text{op}})_0 \) of \( C^{\text{op}} \) to be the collection of objects of \( C \). For every two objects \( a, b \in C^{\text{op}}_0 = C_0 \) we define

\[ \text{Hom}_{C^{\text{op}}}(b, a) := \text{Hom}_C(a, b). \]

That is, for every \( f : b \to a \) in \( C \), we have \( b \xrightarrow{f^{\text{op}}} a \) in \( C^{\text{op}} \).

Given two composable morphisms \( a \xrightarrow{f} b \xrightarrow{g} c \) in \( C \), we have \( c \xrightarrow{g^{\text{op}}} b \xrightarrow{f^{\text{op}}} a \) in \( C^{\text{op}} \). We then define the composition in \( C^{\text{op}} \) by

\[ f^{\text{op}} \circ g^{\text{op}} := (g \circ f)^{\text{op}}. \]

**Exercise 2.6.** Check that the definition of the opposite category \( C^{\text{op}} \) makes sense: if \( C \) is a category, then \( C^{\text{op}} \) as defined above actually is a category.

**Exercise 2.7.** Check that if \( \text{id}_a : a \to a \) is an identity morphism in a category \( C \) then \( (\text{id}_a)^{\text{op}} : a \to a \) is an identity morphism in \( C^{\text{op}} \).

Hint: what do you need to check?
Remark 2.8. There are several reasons for defining the opposite category. One has to do with contravariant functors which we will define later. The main reason is the principle of duality: every categorical concept, theorem, and proof that holds in an arbitrary category \( C \) also holds in the opposite category \( C^{op} \). This gives us dual concepts, theorems and proofs that are obtained by reversing the direction of all the morphisms. The principle saves a lot of work, but takes time getting used to.

The concept of initial and terminal objects serves as our first illustration of the principle of duality. We first define terminal objects in a category.

**Definition 2.9.** An object \( t \) in a category \( C \) is terminal if for any object \( a \) in \( C \), there is exactly one morphism \( a \to t \), i.e. \( \text{Hom}_C(a,t) \) is a one-element set for any object \( a \).

**Example 2.10.** In the category \( \text{Set} \), the one-element set \( \{\ast\} \) is terminal: for any set \( X \), there is exactly one function \( f: X \to \{\ast\} \). Namely, \( f \) is the constant function \( f(x) = \ast \) for all \( x \in X \).

**Example 2.11.** In the category \( \text{Group} \), the trivial group \( \{e\} \) is terminal: for any group \( G \), there is exactly one homomorphism \( \varphi: G \to \{e\} \): \( \varphi(g) = e \) for all \( g \in G \).

Not all categories has terminal objects. One simple class examples comes from partially ordered sets (posets) that are ubiquitous in mathematics.

**Definition 2.12.** A preorder \( (X, \leq) \) is a poset if whenever \( a \leq b \) and \( b \leq a \) we must have \( a = b \). Equivalently a small category \( C \) is a poset if there is at most one morphisms between any two objects and any two isomorphic objects are equal.

**Remark 2.13.** Not all preorders are posets: Let \( C \) be a category with two objects \( a, b \) and exactly two non-identity morphisms \( a \xrightarrow{f} b \), \( b \xrightarrow{g} a \). Then \( g \circ f = \text{id}_a \) and \( f \circ g = \text{id}_b \) because there is no other choice for what \( g \circ f \) and \( f \circ g \) can be — there are no other morphisms:

\[
\begin{align*}
\text{id}_a & \xrightarrow{f} b \\
& \xrightarrow{g} \text{id}_b
\end{align*}
\]

Then in particular, \( C \) is a preorder. Notice that \( a \) and \( b \) are isomorphic in \( C \) but \( a \neq b \).

We are not in position to give examples of categories with no terminal objects. Let \( (X, \leq) \) be a poset. Observe that \( t \in X \) is terminal if for any \( a \in X \), \( a \leq t \), i.e. \( t \) is a maximal element of \( X \).

The set of integers with the standard ordering has no maximal element. So \( (\mathbb{Z}, \leq) \) considered as a category has no terminal objects.

**Proposition 2.14.** Any two terminal objects \( t_1 \) and \( t_2 \) in a category \( C \) are uniquely isomorphic.

**Proof.** Since \( t_1 \) is terminal, there exists a unique morphism \( t_2 \xrightarrow{g} t_1 \). Since \( t_2 \) is terminal, there exists a unique morphism \( t_1 \xrightarrow{f} t_2 \). Since \( t_1 \) is terminal, there is only one morphism in \( \text{Hom}_C(t_1, t_1) \) which must be \( \text{id}_{t_1} \). Since \( g \circ f \in \text{Hom}_C(t_1, t_1) \), \( g \circ f = \text{id}_{t_1} \). Similarly, \( f \circ g = \text{id}_{t_2} \). Therefore, \( t_1 \) and \( t_2 \) are isomorphic, and \( g, h \) are the unique isomorphisms. \( \square \)

Now let’s see what duality gives us.

**Definition 2.15.** An object \( i \) in a category \( C \) is initial if for any object \( a \) in \( C \), there is exactly one morphism \( i \to a \). Equivalently \( \text{Hom}_C(i, a) \) is a one-element set for any object \( a \).

**Example 2.16.** In the category \( \text{Set} \) of sets and functions the empty set \( \emptyset \) is initial: for any set \( X \), there is exactly one function \( f: \emptyset \to X \), namely, the empty function.

**Example 2.17.** In the category \( \text{Group} \) of groups and homomorphisms, the trivial group \( \{e\} \) is initial: for any group \( G \), there is exactly one homomorphism \( \varphi: \{e\} \to G \), \( e \mapsto e_G \) where \( e_G \) is the identity element in \( G \).
Initial objects in a category are unique up to a unique isomorphism. There are several ways to prove this fact. We can mimic the proof that terminal objects are unique (Proposition 2.14) or we can use the principle of duality and deduce it from Proposition 2.14. We’ll do the latter carefully to illustrate how duality works.

**Proposition 2.18.** Two initial objects \( i_1 \) and \( i_2 \) in a category \( C \) are uniquely isomorphic.

**Proof.** Suppose \( i_1, i_2 \in C_0 \) are initial, then \( i_1, i_2 \) are terminal in \( C^{\text{op}} \). By Proposition 2.14 there exist unique isomorphisms \( i_1 \overset{f^{\text{op}}}{\longrightarrow} i_2 \) and \( i_2 \overset{g^{\text{op}}}{\longrightarrow} i_1 \) in \( C^{\text{op}} \) such that \( f^{\text{op}} \circ g^{\text{op}} = (id_{i_1})_{C^{\text{op}}} \) and \( g^{\text{op}} \circ f^{\text{op}} = (id_{i_2})_{C^{\text{op}}} \). But for every object \( a \) in \( C^{\text{op}} \), the identity morphism \((id_a)_{C^{\text{op}}} \) in \( C^{\text{op}} \) is the opposite of the identity morphism \((id_a)_{C} \) in \( C \) by Exercise 2.7. Then

\[
(2.2) \quad (id_{i_2})_{C^{\text{op}}} = ((id_{i_2})_{C})^{\text{op}} = f^{\text{op}} \circ g^{\text{op}} = (g \circ f)^{\text{op}}
\]

and

\[
(2.3) \quad (id_{i_1})_{C^{\text{op}}} = ((id_{i_1})_{C})^{\text{op}} = g^{\text{op}} \circ f^{\text{op}} = (f \circ g)^{\text{op}}
\]

Therefore, \((id_{i_2})_{C} = g \circ f \) and \((id_{i_1})_{C} = f \circ g \). Hence \( i_2 \overset{f}{\longrightarrow} i_1 \) and \( i_1 \overset{g}{\longrightarrow} i_2 \) are the desired unique isomorphisms. \( \square \)

**Remark 2.19.** It follows from the proof of Proposition 2.18 that if \( a \overset{f}{\longrightarrow} b \) is an isomorphism in a category \( C \), then \( b \overset{f^{\text{op}}}{\longrightarrow} a \) is an isomorphism in the opposite category \( C^{\text{op}} \).

**Lecture 3. Functors.** The free functor \( F : \text{Set} \to \text{Vect} \).

**Last time:**

(1) defined isomorphism in a category;
(2) defined initial and terminal objects;
(3) introduced posets (partially ordered sets);
(4) introduced opposite categories and duality.

Just as social life of objects in a category depends on morphisms, the social life of categories themselves depends on functors.

**Definition 3.1.** A functor \( F \) from a category \( C \) to a category \( D \) (we write \( F : C \to D \)) is a pair of functions \( F_0 : C_0 \to D_0 \) and \( F_1 : C_1 \to D_1 \) on objects and morphisms so that

(1) for all morphisms \( f \in \text{Hom}_C(a,b) \), \( F_1(f) \in \text{Hom}_D(F_0(a),F_0(b)) \)
(2) for all objects \( a \in C \), \( F_1(id_a) = id_{F_0(a)} \)
(i.e., \( F \) preserves identities)
(3) for all composable morphisms \( f \in \text{Hom}_C(a,b), g \in \text{Hom}_C(b,c) \), \( F_1(g \circ f) = F_1(g) \circ F_1(f) \)
(i.e., \( F \) preserves composition)

**Remark 3.2.**

(1) We will often drop the subscript 0 and 1. For example, we would write \( F_0(a) \overset{F_1(f)}{\longrightarrow} F_0(b) \) as \( F(a) \overset{F(f)}{\longrightarrow} F(b) \).
(2) The first condition in Definition 3.1 says that the maps \( F_0 \) and \( F_1 \) are compatible with the source and target \( s,t : C_1 \to C_0 \), \( s,t : D_1 \to C_0 \) in the following sense: for any morphism \( f \) in \( C \)

\[
s(F(f)) = F(s(f)) \quad \text{and} \quad t(F(f)) = F(t(f))
\]

(strictly speaking we should write \( s_C \) instead of just \( s \) etc, but this clutters the notation).
(3) The second condition in Definition 3.1 says that \( F_0 \) and \( F_1 \) are compatible with the unit maps \( u : C_0 \to C_1 \) and \( u : D_0 \to D_1 \): for any object \( a \) in \( C \)
\[
u(F_0(a)) = F_1(u(a)).
\]

**Example 3.3.** We have the forgetful functor \( U \) from \( \text{Group} \) to \( \text{Set} \) that forgets the group structure: for every group \( G \), \( U(G) \) is the set of elements of \( G \) (we’ll call it the underlying set of the group \( G \)). For every homomorphism \( \phi : G \to H \), \( U(\phi) \) is the corresponding function on the underlying sets.

It is not hard to check that \( U : \text{Group} \to \text{Set} \) is actually a functor (do it).

**Example 3.4.** Recall that \( \text{Rel} \) denotes the category of sets and relations. There is a functor \( F : \text{Set} \to \text{Rel} \): \( F \) does nothing on objects and sends functions to their graphs: \( F(f) = \{(x, y) \in X \times Y | y = f(x)\} \).

Again it’s not hard to check that \( F \) is a functor. You should make sure to check that \( F \) preserves composition.

**Definition 3.5.** The category \( \text{Ab} \) of abelian groups is defined as follows: the objects of \( \text{Ab} \) are abelian groups. The morphisms in \( \text{Ab} \) are homomorphisms of groups.

**Example 3.6.** There is the inclusion functor \( i : \text{Ab} \to \text{Group} \) that is the identity on both objects and morphisms. This functor is practically invisible.

There is a forgetful functor \( U : \text{Vect}_R \to \text{Ab} \) that forgets the scalar multiplication and only remembers addition. Note that since every real vector space is an abelian group under addition and since linear maps preserve addition of vectors the definition makes sense.

The functor \( U \) allows us to say that a real vector space is an abelian group with extra structure.

**Remark 3.7.** Given two categories \( C \) and \( D \) it may be impossible to extend a given map \( F : C_0 \to D_0 \) on objects to a functor. Here is an example. Let \( C = \text{Group}, D = \text{Ab} \) and \( Z : \text{Group}_0 \to \text{Ab}_0 \) be defined by \( Z(G) = \{z \in G | zg = ga \text{ for all } g \in G\} \), the center of \( G \). It turns out that the map \( Z \) cannot be extended to a functor from \( \text{Group} \) to \( \text{Ab} \).

**Example 3.8.** There is a functor \( F : \text{Mat} \to \text{FDVect}_R \) from the category of real matrices (Example 1.19) to the category of (real) finite dimensional vector spaces. On objects, \( F(n) = \mathbb{R}^n \) for all \( n \in \mathbb{N} \). On morphism, \( F \) sends an \( m \times n \) matrix \( A \) to the linear map \( F(A) : \mathbb{R}^n \to \mathbb{R}^m \):
\[
F(A) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.
\]
Matrix multiplication is defined in such a way as to make \( F \) preserve composition. Notice that \( F \) is not surjective on objects (since not every finite dimensional vector space is \( \mathbb{R}^n \)). However, for all \( n, m \in \mathbb{N} \), the map
\[
F : \text{Hom}_{\text{Mat}}(n, m) \to \text{Hom}_{\text{FDVect}_R}(\mathbb{R}^n, \mathbb{R}^m)
\]
is a bijection. This is one of the reasons why matrices are so useful in linear algebra.

There is a forgetful functor \( U : \text{Vect}_R \to \text{Set} \) (same name as in Example 3.3 but not the same functor) that forget the vector space structure and remembers only the underlying set. Interestingly enough there is also a functor \( F : \text{Set} \to \text{Vect}_R \) that starts with a set \( X \) and assigns to it a vector spaces \( F(X) \) that has elements of \( X \) as a basis. You have seen \( F \) in linear algebra classes for finite sets.

The pair of functors
\[
F : \text{Set} \quad \Rightarrow \quad \text{Vect}_R : U
\]
will be an important example of a pair of adjoint functors in the second half of the course.
Theorem 3.9. There is a functor $F : \text{Set} \to \text{Vect}_\mathbb{R}$ that assigns to each set $X$ a vector space $F(X)$ with the following universal property:

(i) for any set $X$, there is a function $\eta_X : X \to U(F(X))$ where $U(F(X))$ is the set underlying the vector space $F(X)$;

(ii) given any vector space $W$ and any function (morphism in $\text{Set}$) $h : X \to U(W)$, there is a unique linear map $\tilde{h} : F(X) \to W$ so that $U(\tilde{h}) \circ \eta_X = h$.

Remark 3.10. Condition (ii) can be restated in terms of diagrams: given a function $h : X \to U(W)$

$$F(X) \xrightarrow{\eta_X} U(F(X))$$

$$\exists \tilde{h}$$

so that $h \circ \eta_X$ commutes.

$$W \xrightarrow{U(\tilde{h})} U(W)$$

Proof. We construct the functor $F : \text{Set} \to \text{Vect}_\mathbb{R}$ with the desired properties.

For any set $X$, let $\mathbb{R}^X$ denote the set of functions from a set $X$ to $\mathbb{R}$,

$$\mathbb{R}^X = \{f : X \to \mathbb{R}\}.$$ The set $\mathbb{R}^X$ is a real vector space under the point-wise addition and scalar multiplication: for any $\lambda \in \mathbb{R}$ and any $f, g \in \mathbb{R}^X$, we define $(\lambda f)(x) := \lambda f(x)$ and $(f + g)(x) := f(x) + g(x)$ for all $x \in X$.

Now define

$$F(X) = \{f \in \mathbb{R}^X \mid f(x) = 0\; \text{for all but finitely many} \; x \in X\}.$$ Note that if $X$ is finite then $F(X) = \mathbb{R}^X$ and otherwise $F(X) \subset \mathbb{R}^X$. It’s not hard to check that $F(X)$ is a vector subspace of $\mathbb{R}^X$.

For every $x \in X$, we have a function $\eta_X : X \to \mathbb{R}$ defined by

$$(3.4) \quad \eta_X^x(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

Clearly, $\eta_X^x \in F(X)$. We then have a map

$$\eta_X : X \to F(X), \quad x \mapsto \eta_X^x.$$ Note that $X$ is a set and $F(X)$ is a vector space, so what we really have is a function $\eta_X : X \to U(F(X))$. This may seem pedantic, but it is useful when thinking about functors and categories.

Lecture 4. The free functor $F : \text{Vect}_\mathbb{R} \to \text{Set}$. Contravariant functors. Locally small categories.

Last time: Defined functors and looked at some examples. There are forgetful functors $U : \text{Group} \to \text{Set}$ (which forgets the group structure), $U : \text{Vect}_\mathbb{R} \to \text{Ab}$ (which forgets scalar multiplication). There is the inclusion functor $i : \text{Ab} \to \text{Group}$ (any abelian group is a group and a homomorphism of abelian groups is a homomorphism of groups).

Remark 4.1. Functors can be composed to obtain a new functor. Suppose $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ are two functors. Their composite $G \circ F : \mathcal{C} \to \mathcal{E}$ is defined as follows:

On objects: $(G \circ F)(a) = G(F(a))$ for any $a \in \mathcal{C}$

On morphisms: $(G \circ F) \left( \begin{array}{c} a \\ \downarrow f \end{array} \right) := G(F(a)) \xrightarrow{G(F(f))} G(F(b))$.

Exercise 4.2. Check that $G \circ F$ so defined is a functor. Hint: what do you actually need to check?
Example 4.3. Composing
\[ \text{Vect}_\mathbb{R} \xrightarrow{U} \text{Ab} \xrightarrow{i} \text{Group} \xrightarrow{U} \text{Set} \]
we get a forgetful functor
\[ \text{Vect}_\mathbb{R} \rightarrow \text{Set} \]
that assigns to a vector space the underlying set. I will call it \( U \) again. (So now we have three functors called \( U \) for “underlying”. I hope this is not too confusing.)

Last time we also started constructing a functor \( F : \text{Set} \rightarrow \text{Vect} \equiv \text{Vect}_\mathbb{R} \) with the following universal property.

(i) for any set \( X \), there is a function \( \eta_X : X \rightarrow U(F(X)) \) where \( U(F(X)) \) is the set underlying the vector space \( F(X) \);
(ii) given any vector space \( W \) and any function \( h : X \rightarrow U(W) \), there is a unique linear map \( \tilde{h} : F(X) \rightarrow W \) so that \( U(\tilde{h}) \circ \eta_X = h \).

On objects we defined \( F \) by
\[ F(X) = \{ f : X \rightarrow \mathbb{R} \mid f(x) = 0 \text{ for all but finitely many } x \in X \}. \]

For every set \( X \) we also defined a function
\[ \eta_X : X \rightarrow U(F(X)), \quad \eta_X(x) := \eta^x_X \]
where
\[ \eta^x_X(y) = \begin{cases} 
1 & \text{if } x = y \\
0 & \text{if } x \neq y 
\end{cases} \]

We now argue that the function \( \eta_X : X \rightarrow U(F(X)) \) has the desired universal property.

Claim The set \( \{ \eta^x_X \}_{x \in X} \) is linearly independent and spans \( F(X) \) hence forms a basis of the vector space \( F(X) \).

Proof of claim. Suppose that \( \{ \eta^x_X \}_{x \in X} \) is linearly dependent. This means that there is \( n > 0 \), \( x_1, \ldots, x_n \in X \) and \( c_1, \ldots, c_n \in \mathbb{R} \) not all \( 0 \) so that
\[ c_1 \eta^x_{x_1} + \cdots + c_n \eta^x_{x_n} = 0. \]

But then for any \( j, 1 \leq j \leq n \)
\[ \sum_{i=1}^n c_i \eta^x_{x_i}(x_j) = c_j = 0 \]
Contradiction. Therefore \( \{ \eta^x_X \}_{x \in X} \) is linearly independent.

Given \( f \in F(X) \) there is \( n \geq 0 \) and \( x_1, \ldots, x_n \in X \) so that \( f(x_i) \neq 0 \) and \( f(y) = 0 \) for \( y \not\in \{ x_1, \ldots, x_n \} \). Then
\[ f(z) = \sum_{i=1}^n f(x_i) \eta^x_{x_i}(z) \]
for all \( z \in X \). Hence
\[ f = \sum_{i=1}^n f(x_i) \eta^x_{x_i} \]
and consequently
\[ f = \sum_{x \in X} f(x) \eta^x_X. \]

Thus the set \( \{ \eta^x_X \}_{x \in X} \) spans \( F(X) \). \( \square \)
Now suppose we have a vector space $W$ and a function $h : X \to U(W)$. Since $\{\eta_X^x\}_{x \in X}$ is a basis of the vector space $F(X)$ there is a unique linear map
\[ \tilde{h} : F(X) \to W \quad \text{with} \quad \tilde{h}(\eta_X(x)) = h(x). \]
Explicitly, for all $f \in F(X)$
\[ \tilde{h}(f) := \tilde{h} \left( \sum_{x \in X} f(x)\eta_X(x) \right) = \sum_{x \in X} f(x)h(x) \]
Note that $f(x)$ is a real number and $h(x)$ is a vector in $W$, so $f(x)h(x)$ makes sense. Moreover the sum $\sum_{x \in X} f(x)h(x)$ is actually finite since since $f(x)$ is zero for all but finitely many $x \in X$.

We now construct the desired functor $F : \text{Set} \to \text{Vect}$. Given a function $\varphi : X \to Y$ between two sets we need to construct a linear map $F(\varphi) : F(X) \to F(Y)$. Consider the function
\[ h : X \to U(F(Y)), \quad h := \eta_Y \circ \varphi \]
The function $h$ is defined to make the diagram
\[ \begin{array}{ccc}
X & \xrightarrow{\eta_X} & U(F(X)) \\
\downarrow{\varphi} & & \downarrow{U(h)} \\
Y & \xrightarrow{\eta_Y} & U(F(Y))
\end{array} \]
commute. By the universal property of the function $X \overset{\eta_X}{\to} U(F(X))$ there is a unique linear map
\[ \tilde{h} : F(X) \to F(Y) \]
so that
\[ U(\tilde{h}) \circ \eta_X = h, \]
i.e., the diagram
\[ \begin{array}{ccc}
X & \overset{\eta_X}{\longrightarrow} & U(F(X)) \\
\downarrow{\varphi} & & \downarrow{U(\tilde{h})} \\
Y & \overset{\eta_Y}{\longrightarrow} & U(F(Y))
\end{array} \]
commutes. We define $F(\varphi)$ to be this $\tilde{h}$:
\begin{equation}
F(\varphi) := \tilde{h}.
\end{equation}

**Exercise 4.4.** Show that $F : \text{Set} \to \text{Vect}$, $F(X \overset{\varphi}{\to} Y) = F(X) \overset{\tilde{h}}{\to} F(Y)$, where $\tilde{h}$ is the linear map constructed above, is in fact a functor.

Hints: Use the universal properties of the collection of functions $\{\eta_X : X \to U(F(X))\}_{X \in \text{Set}}$.

If you have trouble getting the universal properties approach to work show that $F(\varphi)$ is given by
\[ F(\varphi)f = \sum_{x \in X} f(x)\eta_Y^\varphi(x) \]
for all $f \in F(X) \subset \mathbb{R}^X$.

**Remark 4.5.** The functors that we have considered so far are called **covariant functors** in older literature.
Definition 4.6. A contravariant functor $F$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a pair of functions $F_0 : \mathcal{C}_0 \rightarrow \mathcal{D}_0$ and $F_1 : \mathcal{C}_1 \rightarrow \mathcal{D}_1$ on objects and morphisms so that

- for all morphisms $f \in \text{Hom}_{\mathcal{C}}(a, b)$,
  $$F_0(b) \xrightarrow{F_1(f)} F_0(a)$$

  [Note the direction of the morphism $F_1(f)$!]
- for all objects $a \in \mathcal{C}$, $F_1(\text{id}_a) = \text{id}_{F_0(a)}$
- for all pairs of composable morphisms $a \xrightarrow{f} b \xrightarrow{g} c$
  $$F_1(g \circ f) = F_1(f) \circ F_1(g) : F(a) \xrightarrow{F(f)} F(b) \xrightarrow{F(g)} F(c)$$

Example 4.7. Consider the category $\text{Vect}_\mathbb{R}$. For any vector $V$ space over $\mathbb{R}$, we have the dual vector space

$$V^* := \{ \ell : V \rightarrow \mathbb{R} \mid \ell \text{ is linear} \}.$$

For any linear map $T : V \rightarrow W$, we have the pullback (dual map)

$$T^* : W^* \rightarrow V^*, \quad T^*(\ell) = \ell \circ T$$

for all $\ell : W \rightarrow \mathbb{R}$. Note that $\text{id}_V^*(\ell) = \ell \circ \text{id}_V = \ell$. Hence

$$(\text{id}_V)^* = \text{id}_{V^*}$$

for all vector spaces $V$.

Given a pair of composable linear maps $V \xrightarrow{T} W \xrightarrow{S} U$ and $\ell \in U^*$

$$(S \circ T)^*(\ell) = \ell \circ S \circ T = T^*(\ell \circ S) = T^*(S^*(\ell))$$

It follows that

$$*: \text{Vect} \rightarrow \text{Vect}, \quad (V \xrightarrow{T} W) \mapsto W^* \xrightarrow{T^*} V^*$$

is a contravariant functor.

Remark 4.8. Any contravariant functor $F$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a (covariant) functor $F : \mathcal{C} \rightarrow \mathcal{D}^{\text{op}}$ and a (covariant) functor $F : \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$. For this reason in contemporary literature one simply writes

$$F : \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$$

when $F$ is a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$.

We end the lecture with a definition.

Definition 4.9. A category $\mathcal{C}$ is locally small if for any $a, b \in \mathcal{C}$, $\text{Hom}_\mathcal{C}(a, b)$ is a set.

Example 4.10. The categories $\text{Set}$, $\text{Group}$, $\text{Vect}_\mathbb{R}$ and $\text{Rel}$ are all locally small but not small. Any small category is, of course, locally small.
Lecture 5. Hom functors. Binary products. CAT.

Last time:

• Constructed a functor $F : \text{Set} \to \text{Vect}$ by constructing for every set $X$ a vector space $F(X)$ and an injective map $\eta_X : X \to U(F(X))$ so that $\eta_X(X)$ is a basis of $F(X)$.
• Defined contravariant functors from a category $\mathcal{C}$ to a category $\mathcal{D}$. They are ordinary (covariant) functors from $\mathcal{C}^{\text{op}}$ to $\mathcal{D}$.
• The dual vector space functor $\ast : \text{Vect}^{\text{op}} \to \text{Vect}$, $V \mapsto V^\ast = \text{Hom}_{\text{Vect}}(V, \mathbb{R})$ is an example of a contravariant functor.
• Defined locally small categories: these are categories $\mathcal{C}$ so that $\text{Hom}_\mathcal{C}(a, b)$ is a set for any pair of objects $a, b$.

The functor $\text{Hom}_\mathcal{C}(-, c) : \mathcal{C}^{\text{op}} \to \text{Set}$

Fix a locally small category $\mathcal{C}$ and an object $c \in \mathcal{C}$. Define the functor

$$\text{Hom}_\mathcal{C}(-, c) : \mathcal{C}^{\text{op}} \to \text{Set}$$

as follows: for any object $a \in \mathcal{C}$ set

$$(\text{Hom}_\mathcal{C}(-, c))(a) = \text{Hom}_\mathcal{C}(a, c).$$

For a morphism $f : a \to b$ in $\mathcal{C}$, define

$$f^\ast \equiv \text{Hom}_\mathcal{C}(-, c)(f) : \text{Hom}_\mathcal{C}(b, c) \to \text{Hom}_\mathcal{C}(a, c)$$

by

$$f^\ast(b \xrightarrow{\tau} c) := a \xrightarrow{\tau \circ f} c.$$  

Then for any pair of composable morphisms $f : a \to b \xrightarrow{g} d$ in $\mathcal{C}$ and for any $c \in \text{Hom}_\mathcal{C}(d, c)$

$$(g \circ f)^\ast(c) = \tau \circ (g \circ f) = (\tau \circ g) \circ f = f^\ast(g^\ast \tau).$$

Note also that for any object $a$ of $\mathcal{C}$, $\text{id}_a^\ast \equiv \text{id}_{\text{Hom}_\mathcal{C}(a, a)}$ since for any $\tau : a \to a$, $\text{id}_a^\ast \tau = \tau \circ \text{id}_a = \tau$. Consequently $\text{Hom}_\mathcal{C}(-, c)$ is a contravariant functor.

Remark 5.1. If $\mathcal{C} = \text{Vect}_\mathbb{R}$ and $c = \mathbb{R}$ then $\text{Hom}_{\text{Vect}_\mathbb{R}}(-, \mathbb{R})$ is (almost!) the dual vector space functor

$$\ast : \text{Vect}_\mathbb{R}^{\text{op}} \to \text{Vect}_\mathbb{R}.$$  

It just happens to be that for any pair of vector spaces $V$ and $W$, the set $\text{Hom}_{\text{Vect}_\mathbb{R}}(V, W)$ is not just a set but also a vector space over $\mathbb{R}$.

The functor $\text{Hom}_\mathcal{C}(c, -) : \mathcal{C} \to \text{Set}$

Fix a locally small category $\mathcal{C}$ and an object $c \in \mathcal{C}$. Define the functor

$$\text{Hom}_\mathcal{C}(c, -) : \mathcal{C}^{\text{op}} \to \text{Set}$$

by

$$\text{Hom}_\mathcal{C}(c, -)(a \xrightarrow{f} b) := \left( \text{Hom}_\mathcal{C}(c, a) \xrightarrow{f^\ast} \text{Hom}_\mathcal{C}(c, b) \right)$$

where

$$f^\ast(c \xrightarrow{\tau} a) = c \xrightarrow{\tau \circ f^\ast} b.$$

Exercise 5.2. Check that $\text{Hom}_\mathcal{C}(c, -)$ is a functor.
**Products**

Recall that given two sets $X$ and $Y$ their **Cartesian product** is the set of ordered pairs

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}.$$  

(Note that the product is empty if one of $X$ and $Y$ is empty.) The Cartesian product comes with two canonical functions

$$p_X : X \times Y \to X, \quad (x, y) \mapsto x$$

$$p_Y : X \times Y \to Y, \quad (x, y) \mapsto y.$$  

The triple $(X \times Y, p_X, p_Y)$ possesses the following universal property:

For any set $Z$ and for any pair of functions $f_X : Z \to X$ and $f_Y : Z \to Y$, there exists a unique function $f : Z \to X \times Y$ such that $f_X = p_X \circ f$ and $f_Y = p_Y \circ f$. It is defined by

$$f(z) := (f_X(z), f_Y(z)),$$

for all $z \in Z$. One also writes $(f_X, f_Y)$ for the function $f$.

Diagrammatically we have:

$$\begin{array}{ccc}
Z & \xrightarrow{f_X} & X \\
\parallel & f \downarrow & \parallel \\
X & \xleftarrow{p_X} & X \times Y \\
\parallel & p_Y \downarrow & \parallel \\
& Y & \\
\end{array}$$

Similarly given two groups $G$ and $H$ their **product** is the group with the underlying set $G \times H$ and multiplication defined coordinate-wise:

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1g_2, h_1h_2)$$

for all $(g_1, h_1), (g_2, h_2) \in G \times H$.

Note that the projections

$$\pi_G : G \times H \to G, \quad (g, h) \mapsto g$$

$$\pi_H : G \times H \to H, \quad (g, h) \mapsto h$$

are group homomorphisms, i.e., morphisms in the category **Group**. The triple $(G \times H, \pi_G, \pi_H)$ has the same universal property as the Cartesian product of two sets: for any group $K$ and any pair of homomorphisms of groups $\varphi_G : K \to G$, $\varphi_H : K \to H$, there is a unique homomorphism $\varphi : K \to G \times H$ such that $\varphi_G = \pi_G \circ \varphi$ and $\varphi_H = \pi_H \circ \varphi$. Namely we define $\varphi$ by

$$\varphi(k) = (\varphi_G(k), \varphi_H(k))$$

for all $k \in K$.

We now generalize:

**Definition 5.3.** Let $\mathcal{C}$ be a category. A (binary categorical) **product** of two objects $a, b \in \mathcal{C}$ is a triple $(c, c \xrightarrow{p_a} a, c \xrightarrow{p_b} b)$ (where $c$ is an object in $\mathcal{C}$ and $p_a$, $p_b$ are morphisms in $\mathcal{C}$) with the following universal property: for any object $d \in \mathcal{C}$ and any pair of morphisms $d \xrightarrow{f_a} a, d \xrightarrow{f_b} b$, there exists a unique morphism $d \xrightarrow{f} c$ such that $f_a = p_a \circ f$ and $f_b = p_b \circ f$. In other words the following
Example 5.4. In the category Set, the product is just the Cartesian product.

Example 5.5. In the category Group, the product is just the product of groups.

Example 5.6. Let \((X, \leq)\) be a poset (considered as a category), \(a, b \in X\). Recall that in this category there is a morphism \(f_{cd} : c \to d\) if and only if \(c \leq d\).

The product of \(a\) and \(b\), if it exists, is \(c \in X\) such that
1. \(c \leq a\) and \(c \leq b\);
2. if \(d \leq a\) and \(d \leq b\), then \(d \leq c\).

Thus the product of \(a\) and \(b\) is the greatest lower bound of \(\{a, b\}\), which may or may not exist.

For example suppose \((X, \leq)\) be a poset with four elements \(a, b, c\) and \(d\) so that \(a, b \leq c\) and \(a, b \leq d\). Then the product of \(a\) and \(b\) doesn’t exist. (Why not?)

Products in a category are not unique on the nose, but just like terminal objects they are unique up to a unique isomorphism.

Lemma 5.7. Let \(C\) be a category, \(a, b \in C\). Any two products \((c, c \xrightarrow{p_a} a, c \xrightarrow{p_b} b)\) and \((d, d \xrightarrow{q_a} a, d \xrightarrow{q_b} b)\) of \(a, b\) are uniquely isomorphic.

Proof. Since \(a \xleftarrow{q_a} d \xrightarrow{q_b} b\) is a product of \(a\) and \(b\), and \(c \xrightarrow{p_a} a, c \xrightarrow{p_b} b\) are morphisms in \(C\), there exists a unique morphism
\[
\varphi : c \to d \quad \text{so that} \quad p_a = q_a \circ \varphi \quad \text{and} \quad p_b = q_b \circ \varphi.
\]

Similarly, there exists a unique morphism
\[
\psi : d \to c \quad \text{so that} \quad q_a = p_a \circ \psi \quad \text{and} \quad q_b = p_b \circ \psi.
\]

Now consider the composites
\[
\psi \circ \varphi : c \to c \quad \varphi \circ \psi : d \to d.
\]

Since the diagrams
\[
\begin{array}{ccc}
  c & \xrightarrow{\varphi} & d \\
  \downarrow{p_a} & & \downarrow{\psi} \\
  a & \xrightarrow{p_i} & c
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
  c & \xrightarrow{\psi} & d \\
  \downarrow{p_b} & & \downarrow{\varphi} \\
  b & \xrightarrow{p_i} & c
\end{array}
\]

commute, the diagram \(\begin{array}{ccc}
  c & \xrightarrow{\psi \circ \varphi} & c \\
  \downarrow{p_i} & & \downarrow{p_i} \\
  i & \xrightarrow{i} & i
\end{array}\) commutes for \(i = a, b\). But \(\begin{array}{ccc}
  c & \xrightarrow{id_c} & c \\
  \downarrow{p_i} & & \downarrow{p_i} \\
  i & \xrightarrow{i} & i
\end{array}\) also commutes for \(i = a, b\). By the universal property of the product \(a \xrightarrow{p_a} c \xrightarrow{p_b} b\), we must have \(\psi \circ \varphi = \text{id}_c\).

Similarly \(\varphi \circ \psi = \text{id}_d\). \qed

We next turn the collection of all locally small categories into a category. This will allow us to define the product of two locally small categories.
**Definition 5.8.** The category $\text{CAT}$ of all locally small categories is defined as follows.

- The objects in $\text{CAT}$ are locally small categories.
- The morphisms in $\text{CAT}$ are functors

The composition of morphisms in $\text{CAT}$ is the composition of functors. For any locally small category $\mathcal{C}$, there is the identity functor $\text{id}_\mathcal{C}$ defined to be the identity map on objects and morphisms:

$$\text{id}_\mathcal{C}(a \to b) := a \to b$$

for all $a \to b \in \mathcal{C}$.

The category $\text{CAT}$ has products. Here is a construction: let $\mathcal{A}, \mathcal{B}$ be two categories. We define the collection of objects of $\mathcal{A} \times \mathcal{B}$ to be the collection $(\mathcal{A} \times \mathcal{B})^0 = \{(a, b) \mid a \in \mathcal{A}, b \in \mathcal{B}\}$.

A morphism in $\mathcal{A} \times \mathcal{B}$ from $(a, b)$ to $(a', b')$ is a pair of morphisms $(a \to a', b \to b') \in \mathcal{A}_1 \times \mathcal{B}_1$.

The composition is defined coordinate-wise:

$$\left( (a', b') \to (a'', b'') \right) \circ \left( (a, b) \to (a', b') \right) := (a, b) \to (a'', b'')$$

and

$$\text{id}_{(a,b)} = (\text{id}_a, \text{id}_b).$$

The projections $P_\mathcal{A} : \mathcal{A} \times \mathcal{B} \to \mathcal{A}, P_\mathcal{B} : \mathcal{A} \times \mathcal{B} \to \mathcal{B}$ are defined by

$$P_\mathcal{A} \left( (a, b) \to (a', b') \right) = a \to a'$$

$$P_\mathcal{B} \left( (a, b) \to (a', b') \right) = b \to b'$$

It is not hard to check that $P_\mathcal{A}$ and $P_\mathcal{B}$ are functors and that the triple $(\mathcal{A} \times \mathcal{B}, P_\mathcal{A}, P_\mathcal{B})$ satisfies the universal property of products.

**Example 5.9.** Let $\mathcal{A}$ be the category with two objects and exactly one non-identity morphism. That is,

$$\mathcal{A} = \xymatrix{ a \ar[r]^f & b \ar[l]^{\text{id}_a} }.$$

Then

$$\mathcal{A} \times \mathcal{A} = \xymatrix{ (a, a) \ar[r]^{(f, f)} & (b, b) \ar[l]_{(\text{id}_a, \text{id}_a)} }$$


Last time:

- Defined locally small categories. Locally small categories and functors form a category $\text{CAT}$.  

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• If $\mathcal{C}$ is locally small then for every object $c \in \mathcal{C}$ we have functors
  \[
  \text{Hom}_\mathcal{C}(-, c) : \mathcal{C}^{\text{op}} \to \text{Set}, \quad (a \xrightarrow{f} b) \mapsto f^* : \text{Hom}_\mathcal{C}(b, c) \to \text{Hom}_\mathcal{C}(a, c)
  \]
  \[f^*(b \xrightarrow{\gamma} c) := a \xrightarrow{\gamma \circ f} c,
  \]
  \[
  \text{Hom}_\mathcal{C}(c, -) : \mathcal{C}^{\text{op}} \to \text{Set}, \quad (a \xrightarrow{f} b) \mapsto f^* : \text{Hom}_\mathcal{C}(c, a) \to \text{Hom}_\mathcal{C}(c, b)
  \]
  \[f^*(c \xrightarrow{\gamma} a) := a \xrightarrow{f \circ \gamma} c.
  \]

• Defined (categorical) products.
• Observed that the category $\text{CAT}$ has products.

Recall the definition of a product of two objects $a, b$ in a category $\mathcal{C}$:

A product of $a$ and $b$ is a triple $(c, c^p_a \xrightarrow{f_a} a, c^p_b \xrightarrow{f_b} b)$ (where $c$ is an object in $\mathcal{C}$ and $p_a, p_b$ are morphisms in $\mathcal{C}$) with the following universal property: for any object $d \in \mathcal{C}$ and any pair of morphisms $d \xrightarrow{f_a} a$, $d \xrightarrow{f_b} b$, there exists a unique morphism $d \xrightarrow{f_c} c$ such that the following diagram commutes.

\[
\begin{array}{ccc}
  d & \xrightarrow{f} & b \\
  \downarrow \exists f & & \downarrow \\
  c & \xrightarrow{f_a} & a \\
\end{array}
\]

We proved that for any two products $(c, c^p_a \xrightarrow{f_a} a, c^p_b \xrightarrow{f_b} b)$, $(d, q_a : d \to a, q_b : d \to b)$ of $a$ and $b$ there is a unique isomorphism $\varphi : c \to d$ so that the diagrams

\[
\begin{array}{ccc}
  c & \xrightarrow{\varphi} & d \\
  \downarrow p_a & & \downarrow q_a \\
  a & \xrightarrow{f_a} & b
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
  c & \xrightarrow{\varphi} & d \\
  \downarrow p_b & & \downarrow q_b \\
  a & \xrightarrow{f_b} & b
\end{array}
\]

commute. For this reason we may talk about “the” product of $a$ and $b$ and write $(a \times b, p_a : a \times b \to a, p_b : a \times b \to b)$.

In particular, given two locally small categories $\mathcal{A}, \mathcal{B}$ we denote their product by $(\mathcal{A} \times \mathcal{B}, p_A : \mathcal{A} \times \mathcal{B} \to \mathcal{A}, p_B : \mathcal{A} \times \mathcal{B} \to \mathcal{B})$.

**Remark 6.1.** If $\mathcal{A}$ and $\mathcal{B}$ are not necessarily locally small categories, their product $\mathcal{A} \times \mathcal{B}$ still makes sense and is defined exactly the same way. The only difference is that $\mathcal{A}$ and $\mathcal{B}$ need not be objects of $\text{CAT}$.

**Remark 6.2.** Suppose $\mathcal{C}$ is a locally small category. Then the two types of Hom functors can be put together into one functor

\[
\text{Hom}_\mathcal{C}(-, \cdot) : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \text{Set}
\]

as follows: a morphism $(a, b) \xrightarrow{(f_{\text{op}, g})} (a', b') \in \mathcal{C}^{\text{op}} \times \mathcal{C}$ a pair of morphisms $f : \xrightarrow{a'} a$, $g : \xrightarrow{b'} b$ in $\mathcal{C}$. We define

\[
\text{Hom}_\mathcal{C}((a, b) \xrightarrow{(f_{\text{op}, g})} (a', b')) : \text{Hom}_\mathcal{C}(a, b) \to \text{Hom}_\mathcal{C}(a', b')
\]

by

\[
\text{Hom}_\mathcal{C}(f_{\text{op}}, g)(b \xrightarrow{\gamma} a) := b' \xrightarrow{g \circ \gamma} a \xrightarrow{f} a' \equiv g \circ \gamma \circ f \equiv (g \circ f^*)(\gamma).
\]

That is,

\[
\text{Hom}_\mathcal{C}(f_{\text{op}}, g) = g \circ f^*.
\]
Remark 6.3. Let $\mathcal{C}$ be a category. A product $(a \times b, p_a : a \times b \to a, p_b : a \times b \to b)$ in $\mathcal{C}$ is a terminal object in another category $\mathcal{D}$ which we now construct.

The objects of $\mathcal{D}$ are triples $(d, d \xrightarrow{q_a} a, d \xrightarrow{q_b} b)$, where $a, b, d$ are objects in $\mathcal{C}$ and $q_a, q_b$ are morphisms in $\mathcal{C}$:

$$\mathcal{D}_0 := \left\{ \frac{q_a}{a} \right\} \quad \left| \quad d \in \mathcal{C}_0 \right\}$$

Morphisms in $\mathcal{D}$ are defined by

$$\text{Hom}_\mathcal{D}((d, q_a, q_b), (d', q_a', q_b')) = \left\{ d \xrightarrow{\varphi} d' \text{ in } \mathcal{C} \mid \begin{array}{c}
 d \xrightarrow{\varphi} d' \text{ and } q_a \xrightarrow{\varphi} q_a' \text{ commute for } d \in \mathcal{D}_0 \\
 q_b \xrightarrow{\varphi} q_b' \text{ commute for } d \in \mathcal{D}_0
\end{array} \right\}.$$  

Note that the identity morphism in $\mathcal{C}$ on a triple $(d, q_a, q_b)$ is just $\text{id}_d$ in $\mathcal{C}$.

Given morphisms $(d, q_a, q_b) \xrightarrow{\varphi} (d', q_a', q_b')$ and $(d', q_a', q_b') \xrightarrow{\psi} (d'', q_a'', q_b'')$ in $\mathcal{D}$, the composite $\psi \circ \varphi$ in $\mathcal{C}$ is a morphism in $\mathcal{D}$ since the diagram commutes for $i = a, b$. We define the composition in $\mathcal{D}$ by

$$\left( (d'', q_a'', q_b'') \xrightarrow{\psi} (d', q_a', q_b') \right) \circ \left( (d', q_a', q_b') \xrightarrow{\varphi} (d, q_a, q_b) \right) := \left( (d'', q_a'', q_b'') \xrightarrow{\psi \circ \varphi} (d, q_a, q_b) \right).$$

It is not hard to check that $\mathcal{D}$ is a category.

Moreover an object $(c, p_a, p_b)$ is terminal in $\mathcal{D}$ if and only if for all objects $(d, q_a, q_b)$ in $\mathcal{D}$ there exists a unique morphism $(d, q_a, q_b) \xrightarrow{\varphi} (c, p_a, p_b)$ in $\mathcal{D}$ such that

$$d \xrightarrow{\varphi} c \quad 
\left\{ 
\begin{array}{c}
 q_i \xrightarrow{\varphi} p_i \\
 q_i \xrightarrow{\varphi} p_i
\end{array} \right.$$

commutes for $i = a, b$. We obtain an alternative proof of Lemma 5.7.

**Coproducts**

By the principle of duality, dual to the notion of a product we have the dual notion of a coproduct.

**Definition 6.4.** Let $\mathcal{C}$ be a category. A coproduct of two objects $a, b \in \mathcal{C}$ (if it exists) is their product in the opposite category $\mathcal{C}^{\text{op}}$ (recall that the categories $\mathcal{C}$ and $\mathcal{C}^{\text{op}}$ have the same objects).

Explicitly a coproduct of $a$ and $b$ is a triple $(c, i_a : a \to c, i_b : b \to c)$ so that for any object $e \in \mathcal{C}$ and any pair of morphisms $f_a : e \to a, f_b : e \to b$ there is a unique morphism $f : e \to c$ so that the
Example 6.5. Given two sets $X, Y$, their coproduct exists in $\text{Set}$, it is the disjoint union $X \sqcup Y$ together with the two inclusions $i_X, i_Y$.

There are various ways to construct/define the disjoint union. For example we may define

$$X \sqcup Y := (X \times \{0\}) \cup (Y \times \{1\}).$$

and

$$i_X : X \to X \sqcup Y, \quad i_X(x) := (x, 0) \quad \text{for all } x \in X$$

$$i_Y : Y \to X \sqcup Y, \quad i_Y(y) := (y, 1) \quad \text{for all } y \in Y.$$

The universal property is easy to check: given two linear maps $f_X : X \to Z$ and $f_Y : Y \to Z$, we define $f : X \sqcup Y \to Z$ by

$$f(w, i) = \begin{cases} f_X(w) & \text{if } i = 0 \\ f_Y(w) & \text{if } i = 1 \end{cases}.$$

Remark 6.6. A coproduct $(c, i_a, i_b)$ of $a, b \in \mathcal{C}$ is initial in the category with objects $(d, a \overset{j_a}{\to} d, b \overset{j_b}{\to} d)$ (and appropriately defined morphisms). Since initial objects are unique up to a unique isomorphism coproducts are unique up to a unique isomorphism.

Alternatively, since a coproduct in $\mathcal{C}$ is a product in the opposite category $\mathcal{C}^{\text{op}}$, it’s unique up to a unique isomorphism in $\mathcal{C}^{\text{op}}$ (which is also an isomorphism in $\mathcal{C}$.)

Example 6.7. Given two vector space over the reals $V, W$, their coproduct exists in $\text{Vect}_\mathbb{R}$, it is the direct sum $V \oplus W = \{(v, w) = v + w \mid v \in V \text{ and } w \in W\}$ with the structure maps $i_V : V \to V \oplus W, v \mapsto (v, 0)$ and $i_W : W \to V \oplus W, w \mapsto (0, w)$.

The universal property is easy to check: given two linear maps $T_V : V \to U$ and $T_W : W \to U$, there exists a unique linear map $T = T_V \oplus T_W : V \oplus W \to U, (v, w) \mapsto T_V(v) + T_W(w)$.

Example 6.8. Coproducts exist in the category $\text{Group}$. They are known as free products. However, given two group $G, H$, their coproduct is not $G \times H$ together with the two inclusions

$$i_G : G \to G \times H, \quad i_G(g) = (g, e_H)$$

$$i_H : H \to G \times H, \quad i_H(h) = (e_G, h).$$

Reason. Suppose $(G \times H, i_G, i_H)$ were a coproduct. Then for any group $K$ and any pair of homomorphisms $\varphi_G : G \to K, \varphi_H : H \to K$, we have a unique homomorphism $\varphi : G \times H \to K$ such that $\varphi(g, e_H) = \varphi_G(g)$ and $\varphi(e_G, h) = \varphi_H(h)$ for any $g \in G$ and $h \in H$. Therefore we must have

$$\varphi_G(g) \varphi_H(h) = \varphi((g, e_H), (e_G, h)) = \varphi((g, e_H)(e_G, h)) = \varphi(g, h) = \varphi(e_G, h) \varphi(g, e_H) = \varphi_H(h) \varphi_G(g)$$

for any $g \in G$ and $h \in H$. But there is no reason for $\varphi_G(g)$ and $\varphi_H(h)$ to commute in $K$. $\square$
Lecture 7. Monos, epis, fully faithful functors.

Last time:

- Products in a category $C$ are terminal objects in another (appropriate) category.
- Coproducts in $C$ are products in $C^{\text{op}}$. Coproducts are initial in an appropriate category.

We now define the category-theoretic analogue of injective and surjective functions and try to get a feel for these concepts. Recall that a function $f : X \to Y$ between two sets is injective if for all $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$. Consequently, for any pair of functions $g : W \to X$, $h : W \to X$,

\[ f \circ g = f \circ h \implies g = h. \]

This is because for any $w \in W$, $f(g(w)) = f(h(w))$ implies $g(w) = h(w)$.

Similarly a function $f : X \to Y$ between two sets is surjective if for all $y \in Y$, there exists $x \in X$ such that $f(x) = y$. Consequently, for any pair of functions $k : Y \to Z$, $\ell : Y \to Z$,

\[ k \circ f = \ell \circ f \implies k = \ell. \]

This is because for any $y \in Y$, there exists $x \in X$ such that $f(x) = y$. And then then $k(y) = k(f(x)) = \ell(f(x)) = \ell(y)$ for all $y \in Y$.

Now we generalize.

**Definition 7.1.** A morphism $b \xrightarrow{f} c$ in a category $C$ is monic (or is a monomorphism or is a mono) if for any $a \in C$, and for any pair of morphisms $a \xrightarrow{g} b$, $a \xrightarrow{h} b$ in $C$,

\[ f \circ g = f \circ h \implies g = h. \]

A morphism $b \xrightarrow{f} c$ in a category $C$ is epic (or is an epimorphism or is an epi) if for any $d \in C$, and for any pair of morphisms $k, \ell : c \to d$

\[ k \circ f = \ell \circ f \implies k = \ell. \]

**Remark 7.2.** “Epi” and “mono” are dual notions in the following sense: a morphism $f : x \to y$ in a category $C$ is epic $\iff f^{\text{op}} : y \to x$ in $C^{\text{op}}$ is monic.

**Example 7.3.** In $\text{Set}$, monos are injective functions and epis are surjective functions.

**Example 7.4.** In any preorder, any morphism is both a mono and an epi.

**Proof.** Let $f : a \to b$ be a morphism in a preorder $C$. Then for any pair of morphism $g, h : c \to a$

\[ f \circ g = f \circ h \implies g = h. \]

because $g$ has to equal $h$ to begin with: in a preorder $\text{Hom}_C(c, a)$ has at most one morphism. Hence $f$ is monic.

A similar argument proves that $f$ is epic.  

**Example 7.5.** In the category $\text{Group}$, any injective group homomorphism is a mono. Conversely, if $\varphi : G \to H$ is monic in $\text{Group}$, it is injective. Why? [This is not completely trivial.]
Warning 7.6. There are categories that are “sets with structure and structure-preserving maps” where monos need not be injective and epis need not be surjective. See Proposition 7.11 and Proposition 7.15 below.

Also, while in any category an isomorphism in a category is monic and epic (see Lemma 7.17), the converse is false. See Example 7.20.

We now give an example of a category of sets with structure where epis are not surjective. We start with a definition.

Definition 7.7. An abelian group $G$ is is divisible if for any $g \in G$ and any positive integer $n$ there exists $y \in G$ such that $ny = g$, where $ny := y + \cdots + y$.

Example 7.8. The abelian group $(\mathbb{Q}, +)$ is divisible, the abelian group $(\mathbb{Z}, +)$ is not divisible.

Exercise 7.9. If $G$ is divisible and $H < G$ is a subgroup then the quotient $G/H$ is also divisible. In particular the quotient $(\mathbb{Q}/\mathbb{Z}, +)$ is a divisible group.

Notation 7.10. Divisible groups and homomorphism of divisible groups form a category which we denote by $\text{Div}$.

Proposition 7.11. The quotient map $\pi : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is a monic in $\text{Div}$. The function $\pi : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is not injective.

Proof. Suppose that $G$ is a divisible group and $f, g : G \to \mathbb{Q}$ are two homomorphisms such that $\pi \circ f = \pi \circ g$. We want to show that $f = g$. It is enough to show that $k := f - g : G \to \mathbb{Q}$ sends everything to zero: $k(x) = 0$ for all $x \in G$.

We know that

$$\pi \circ k = \pi \circ (f - g) = \pi \circ f - \pi \circ g = 0.$$ 

Hence for any $x \in G$, $\pi(k(x)) = 0$ in $\mathbb{Q}/\mathbb{Z}$. Therefore $k(x) \in \mathbb{Z} \subset \mathbb{Q}$ for all $x \in G$.

Now suppose there is $x \in G$ so that $k(x) \neq 0$. Then $2|k(x)|$ is a positive integer. Since $\mathbb{Q}/\mathbb{Z}$ is divisible by Exercise 7.9 there is $y \in G$ so that

$$2|k(x)|y = x.$$ 

Then in $\mathbb{Q}$

$$k(x) = k(y + \cdots + y) = k(y) + \cdots + k(y) = 2|k(x)|k(y).$$ 

Therefore

$$k(y) = \frac{k(x)}{2|k(x)|} = \pm \frac{1}{2},$$

which contradicts the fact that $k(G) \subset \mathbb{Z}$. Therefore $k(x) = 0$ for all $x \in G$ hence $f = g$ and $\pi : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is monic. \hfill $\Box$

Remark 7.12. Since $\pi : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is monic and the function $\pi : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ on the underlying sets is not injective, the underlying set functor $U : \text{Div} \to \text{Set}$ does not send monos to monos. One says “$U$ does not preserve monomorphisms.”

Definition 7.13 (The category $\text{Mon}$ of monoids). The objects of the category $\text{Mon}$ of monoids are monoids. A morphism $\varphi : M \to N$ in $\text{Mon}$ is a function that preserves identity and multiplication:

1. $\varphi(e_M) = e_N$;
2. for any $m_1, m_2 \in M$, $\varphi(m_1m_2) = \varphi(m_1)\varphi(m_2)$.

Remark 7.14. If we view monoids as small one object categories then morphisms of monoids are just functors.
Proposition 7.15. The inclusion \( i : \mathbb{N} \to \mathbb{Z} \) is an epi in \( \text{Mon} \). The inclusion is not surjective on the underlying sets. Hence the underlying set functor \( U : \text{Mon} \to \text{Set} \) does not preserve epimorphisms.

Proof. Let \( M \) be a monoid, \( f, g : \mathbb{Z} \to M \) be two homomorphisms such that \( f \circ i = g \circ i \). We want to show \( f(n) = g(n) \) for all \( n \in \mathbb{Z} \).

If \( n \in \mathbb{Z} \) and \( n \geq 0 \), then \( i(n) = n \). Therefore, \( g(n) = g(i(n)) = f(i(n)) = f(n) \). If \( n < 0 \), then

\[
f(n) = f(n)e_M = f(n)g(0) = f(n)(g(-n)g(n)) = (f(n)f(-n))g(n) = e_Mg(n) = g(n).
\]

Therefore, \( f = g \). We conclude that the inclusion \( i : \mathbb{N} \to \mathbb{Z} \) is epic in \( \text{Mon} \). Clearly the underlying map of sets is not surjective.

Remark 7.16. Since the inclusion \( i : \mathbb{N} \to \mathbb{Z} \) is an injective function, it must also be a monic. However, \( i \) is not an isomorphism of monoids (why not?). Consequently

\[
\text{mono + epi} \not\implies \text{isomorphism}.
\]

Lemma 7.17. In any category \( C \) an isomorphism \( b \xrightarrow{f} c \) is epic and monic.

Proof. Suppose \( a \xrightarrow{g} b \) and \( a \xrightarrow{h} b \) are two morphisms in \( C \) such that \( f \circ g = f \circ h \). Then

\[
g = f^{-1} \circ (f \circ g) = f^{-1} \circ (f \circ h) = h
\]

So \( f \) is a monic. Similarly if \( k, \ell : c \to d \) are two morphisms with \( k \circ f = \ell \circ f \) then

\[
k = (k \circ f) \circ f^{-1} = (\ell \circ f) \circ f^{-1} = \ell.
\]

Lemma 7.18. Let \( F : C \to D \) be a functor. If \( a \xrightarrow{f} b \) is an isomorphism in \( C \), then \( F(a) \xrightarrow{F(f)} F(b) \) is an isomorphism in \( D \). In other words

\[
\text{functors preserve isomorphisms}.
\]

Moreover \( F(f^{-1}) = F(f)^{-1} \).

Proof.

\[
F(f) \circ F(f^{-1}) = F(f \circ f^{-1}) = F(id_b) = id_{F(b)}
\]

where the last equality holds since functors preserve identity morphisms. Similarly,

\[
F(f^{-1}) \circ F(f) = F(f^{-1} \circ f) = F(id_a) = id_{F(a)}.
\]

Therefore \( F(f) \) is an isomorphism in \( D \) and \( F(f^{-1}) = F(f)^{-1} \).

Remark 7.19. We have forgetful functors \( U : \text{Vect}_R \to \text{Set} \), \( U : \text{Group} \to \text{Set} \), \( U : \text{Mon} \to \text{Set} \), and \( U : \text{Vect}_R \to \text{Set} \). Therefore, isomorphisms in \( \text{Vect}_R \), \( \text{Group} \), and \( \text{Mon} \) must be isomorphisms in \( \text{Set} \), i.e., they must be bijections.

Example 7.20. Consider the monoid \( \mathbb{N} \) and the corresponding one object category \( \mathbb{B}\mathbb{N} \) where \( (\mathbb{B}\mathbb{N})_0 = \{\ast\} \) and \( \text{Hom}_{\mathbb{B}\mathbb{N}}(\ast, \ast) = \mathbb{N} \). Recall that composition in \( \mathbb{B}\mathbb{N} \) is the “multiplication” in \( \mathbb{N} \), i.e., the composition is +.

For any \( a, b, c \in \text{Hom}_{\mathbb{B}\mathbb{N}}(\ast, \ast) \),

\[
a + b = a + c \implies b = c \quad \text{and} \quad b + a = c + a \implies b = c.
\]

Hence any morphism \( a \) in \( \mathbb{B}\mathbb{N} \) is both monic and epic.
For \( a \) to be an isomorphism in \( \mathcal{B} \mathcal{N} \) there needs to be a \( b \in (\mathcal{B} \mathcal{N})_1 = \mathbb{N} \) so that
\[
a + b = 0 = b + a.
\]
Therefore the only isomorphism in \( \mathcal{B} \mathcal{N} \) is 0.

To repeat: every morphism in \( \mathcal{B} \mathcal{N} \) is both monic and epic, but there is only one isomorphism.

**Definition 7.21.** A functor \( F : \mathcal{C} \to \mathcal{D} \) is full if for any pair of objects \( a, b \in \mathcal{C} \) the map
\[
F : \text{Hom}_\mathcal{C}(a, b) \to \text{Hom}_\mathcal{D}(F(a), F(b))
\]
is surjective.

A functor \( F : \mathcal{C} \to \mathcal{D} \) is faithful if for any pair of objects \( a, b \in \mathcal{C} \) the map
\[
F : \text{Hom}_\mathcal{C}(a, b) \to \text{Hom}_\mathcal{D}(F(a), F(b))
\]
is injective.

A functor \( F \) is fully faithful if \( F \) is both full and faithful.

**Example 7.22.**
- The inclusion functor \( i : \text{Ab} \to \text{Group} \) is fully faithful.
- The forgetful functor \( U : \text{Group} \to \text{Set} \) is faithful but not full since not all function between the underlying sets need to be homomorphisms.
- The “inclusion” functor \( i : \text{Set} \to \text{Rel} \), \( i(\mathcal{X} \xrightarrow{f} \mathcal{Y}) = \mathcal{X} \xrightarrow{\text{graph}(f)} \mathcal{Y} \) is faithful but not full since not all relations are graphs of functions.

**Lemma 7.23.** Let \( F : \mathcal{C} \to \mathcal{D} \) be a faithful functor and \( b \xrightarrow{f} c \) a morphism in \( \mathcal{C} \). If \( F(b) \xrightarrow{F(f)} F(c) \) is monic, then \( b \xrightarrow{f} c \) is monic. If \( F(b) \xrightarrow{F(f)} F(c) \) is epic, then \( b \xrightarrow{f} c \) is epic.

One says: “faithful functors reflect monos and epis.”

**Remark 7.24.** We have seen that faithful functors don’t need to send monos to monos or epis to epis, i.e., they don’t need to preserve them.

**Proof of Lemma 7.23.** Suppose that \( F(b) \xrightarrow{F(f)} F(c) \) is monic and \( a \xrightarrow{g} b, a \xrightarrow{h} b \) are morphisms in \( \mathcal{C} \). If \( f \circ g = f \circ h \), then we have \( F(f \circ g) = F(f \circ h) \). Since \( F \) is a functor, \( F(f) \circ F(g) = F(f) \circ F(h) \).

Since \( F(b) \xrightarrow{F(f)} F(c) \) is a monic, \( F(g) = F(h) \). Since \( F \) is faithful, we must have \( g = h \), therefore, \( f \) is a monic. A proof that \( F \) reflects epic is similar. \(\square\)

**Remark 7.25.** Alternatively, we can use the principle of duality to show that \( F \) reflects epis. A functor \( G : \mathcal{A} \to \mathcal{B} \) between two categories induces a functor \( G^\text{op} : \mathcal{A}^\text{op} \to \mathcal{B}^\text{op} \) between the opposite categories: \( G^\text{op} \) is defined by
\[
G^\text{op}(a \xrightarrow{\gamma^\text{op}} a') := G(a) \xrightarrow{G(\gamma)^\text{op}} G(a').
\]
Next observe that if a functor \( F : \mathcal{C} \to \mathcal{D} \) is faithful then \( F^\text{op} : \mathcal{C}^\text{op} \to \mathcal{D}^\text{op} \) is faithful as well. We then note that a morphism \( f \) in a category \( \mathcal{A} \) is epic iff and only if \( f^\text{op} \) is monic in \( \mathcal{A}^\text{op} \).