Introduction to Category Theory

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Lecture 1. Definition and Examples

Mathematics is a study of patterns. Category theory is the study of patterns in mathematics.

To make the two sentences less cryptic, we start by defining what a category is. We pick one of several (mostly equivalent) definitions and proceed from there.

Definition 1.1. A category \( \mathcal{C} \) consists of the following data:

1. A collection of objects \( \mathcal{C}_0 \).
2. For each pair of objects \( a, b \in \mathcal{C}_0 \), a collection of morphisms \( \text{Hom}_{\mathcal{C}}(a, b) \), which may be empty if \( a \neq b \). We write \( a \xrightarrow{f} b \) or \( b \xleftarrow{f} a \) if \( f \in \text{Hom}_{\mathcal{C}}(a, b) \).
   We require that \( \text{Hom}_{\mathcal{C}}(a, b) \cap \text{Hom}_{\mathcal{C}}(c, d) = \emptyset \) if \( a \neq c \) and \( b \neq d \).
3. For each object \( a \in \mathcal{C}_0 \), a distinguished identity morphism \( \text{id}_a \in \text{Hom}_{\mathcal{C}}(a, a) \);
4. For each triple of object \( a, b, c \in \mathcal{C}_0 \), a composition map
   \[ \circ : \text{Hom}_{\mathcal{C}}(b, c) \times \text{Hom}_{\mathcal{C}}(a, b) \rightarrow \text{Hom}_{\mathcal{C}}(a, c), \quad (c \xleftarrow{g} b, b \xrightarrow{f} a) \mapsto (c \xleftarrow{g \circ f} a) \]
   which is subject to the following conditions:
   (i) for all \( a, b \in \mathcal{C}_0 \) and for all \( f \in \text{Hom}_{\mathcal{C}}(a, b) \), \( f \circ \text{id}_a = f = \text{id}_b \circ f \);
   (ii) The composition \( \circ \) is associative: for all \( d \xleftarrow{c} b, b \xrightarrow{f} a, h \xrightarrow{(g \circ f)} (h \circ g) \circ (g \circ f) \).

We denote the collection of all morphisms of the category \( \mathcal{C} \) by \( \mathcal{C}_1 \). Thus \( \mathcal{C}_1 = \bigcup_{a, b \in \mathcal{C}_0} \text{Hom}_{\mathcal{C}}(a, b) \).

Remark 1.2. Given an object \( a \) of a category \( \mathcal{C} \) we write \( a \in \mathcal{C} \). (Writing \( a \in \mathcal{C}_0 \) is more precise and more pedantic.)

Remark 1.3. Given a category \( \mathcal{C} \) and two object \( a, b \in \mathcal{C} \), the collection of morphisms \( \text{Hom}_{\mathcal{C}}(a, b) \) from \( a \) to \( b \) may also be denoted by \( \text{Hom}(a, b) \) when the category \( \mathcal{C} \) is understood. Some authors write \( \mathcal{C}(a, b) \) for \( \text{Hom}_{\mathcal{C}}(a, b) \). We will not use \( \mathcal{C}(a, b) \).

Definition 1.4. Given a category \( \mathcal{C} \) and a morphism \( f \in \text{Hom}_{\mathcal{C}}(a, b) \), we call \( a \) the source of \( f \) and \( b \) the target of \( f \). We write \( t(f) = a \) and \( s(f) = a \). In other word, we have two maps \( s, t : \mathcal{C}_1 \rightarrow \mathcal{C}_0 \) from morphisms to objects. Note that had we not required that \( \text{Hom}_{\mathcal{C}}(a, b) \cap \text{Hom}_{\mathcal{C}}(c, d) = \emptyset \) for \( (a, b) \neq (c, d) \), the source and target maps would not be well-defined.
Remark 1.5. We also have a unit map \( u : C_0 \to C_1, a \mapsto \text{id}_a \) that assigns to each object of a category \( C \) the identity morphism on that object.

**Examples**

**Example 1.6.** Sets and functions form a category \( \text{Set} \). The composition in the category \( \text{Set} \) is the composition of functions.

**Example 1.7.** Groups and group homomorphisms form a category \( \text{Group} \). \( \text{Group} \) is an example of a category whose objects are sets with some structure and whose morphisms are structure preserving maps. Not all categories are like this, but many are.

**Example 1.8.** There is the empty category with the empty set of objects and the empty set of morphism, denoted by \( \emptyset \).

**Example 1.9.** There is a category, often denoted by \( 1 \), with one objects \( * \) and one morphism \( \text{id}_* : * \to * \). This is the smallest nonempty category. Exercise: what’s the composition map?

**Definition 1.10.** A category \( C \) is small if both the collection of objects \( C_0 \) and the collection of morphisms \( C_1 \) are sets (rather than some bigger collections of objects).

**Example 1.11.** The categories \( \emptyset \) and \( 1 \) are small. The category \( \text{Set} \) of all sets and functions is not small thanks to Russell’s paradox. We’ll come back to this point.

**Definition 1.12.** A preorder is a category with at most one morphism between any two objects.

Remark 1.13. A small preorder \( C \) is the same thing as set \( C_0 \) equipped with a binary relation \( \leq \) which is reflexive and transitive:

1. For all \( a \in C_0 \), \( a \leq a \) and
2. If \( a \leq b \) and \( b \leq c \), then \( a \leq c \) for all \( a, b, c \in C_0 \).

**Proof.** Let \( C \) be a preorder and a small category. Define a relation \( \leq \) on the set of objects \( C_0 \) by

\[
a \leq b \iff \text{there is a morphism (which is necessarily unique) } f_{ba} : a \to b.
\]

Since for every object \( a \) in \( C \) there is the identity morphism \( \text{id}_a : a \to a \), the relation \( \leq \) is reflexive. Given \( a \leq b \) and \( b \leq c \) there are morphisms \( f_{ba} : a \to b \) and \( f_{cb} : b \to a \). The composite \( f_{cb} \circ f_{ba} : a \to c \) is a morphism from \( a \) to \( c \). Hence \( \text{Hom}_C(a, c) \neq \emptyset \). But \( C \) is a preorder, so there is at most one morphism from \( a \) to \( c \). We conclude that \( f_{cb} \circ f_{ba} = f_{ca} \). That is,

\[
a \leq b \text{ and } b \leq c \implies a \leq c.
\]

Conversely given a relation \( \leq \) on a set \( X \) which is reflexive and transitive define a small category \( C \) by setting its collection of objects to be \( X \). Given two elements \( x, y \in X \) we define

\[
\text{Hom}_C(x, y) = \begin{cases} \{ f_{yx} : x \to y \}, & \text{if } x \leq y; \\ \emptyset, & \text{otherwise.} \end{cases}
\]

It is not hard to check that \( C \) so defined is in fact a category. \( \square \)

Remark 1.14. Preorders occur naturally in mathematics. Recall that for a set \( Y \) the powerset \( \mathcal{P}(Y) \) is the set of all subsets of \( Y \):

\[
\mathcal{P}(Y) := \{ A \mid A \subseteq Y \}
\]

The the subset relation \( \subseteq \) on \( \mathcal{P}(Y) \) is reflexive and transitive. Hence \( \mathcal{P}(Y) \) is a preorder. Note that for a given pair \( A, B \subseteq Y \) it may well happen that neither \( A \subseteq B \) nor \( B \subseteq A \), i.e., the corresponding set \( \text{Hom}(A, B) \) in the preorder \( \mathcal{P}(Y) \) may well be empty.

You have seen binary relations on a given set. One can also talk about relations between two distinct sets. The following terminology is slightly nonstandard:
We denote the category of sets and relations by \( \text{Rel} \).

Note that relations can be composed: if \( R \subseteq X \times Y \) and \( S \subseteq Y \times Z \), then define the composition to be
\[
S \circ R = \{(x, z) \in X \times Z \mid \exists y \in Y, \text{ so that } (x, y) \in R, (y, z) \in S\}.
\]
It is not hard to show that
\( \text{Rel} \) is a category.

Definition 1.16. A relation from a set \( X \) to a set \( Y \) is a subset \( R \) of \( X \times Y \).

Example 1.18. The collection of real vector spaces and linear maps form a category \( \text{Vect}_R \). The composition is the composition of linear maps. This is yet another example of a category of sets with structure and structure preserving functions.

The next category is similar to Example 1.18 but is not a category of sets with structure and structure preserving maps.

Example 1.19. The category \( \text{Mat} \) of real matrices whose objects are natural numbers (including \( 0 \)). A morphism from \( n \in \mathbb{N} \) to \( m \in \mathbb{N} \) (for \( n, m > 0 \)) is a real \( m \times n \) matrix. The composition of morphisms in \( \text{Mat} \) is matrix multiplication. Of course each \( n \in \mathbb{N} \) is secretly \( \mathbb{R}^n \). So \( 0 \in \mathbb{N} \) is secretly \( \mathbb{R}^0 = \{0\} \), the zero-dimensional vector space. For this reason we define
\[
\text{Hom}_{\text{Mat}}(m, 0) = \{O_{0,m}\} \\
\text{Hom}_{\text{Mat}}(0, n) = \{O_{n,0}\}
\]
such that for all \( A \in \text{Hom}_{\text{Mat}}(n, m) \), \( O_{0,m} \circ A = O_{n,0} \), and \( O_{n,0} \circ A = O_{0,m} \). These data make \( \text{Mat} \) into a category.

Remark 1.20. We will see later that the categories \( \text{Mat} \) “is” in an appropriate sense the category \( \text{FDVect}_R \) of finite dimensional vector spaces. Technically the two categories are equivalent.

We end the lecture with a notion of a monoid. A monoid is an algebraic structure like a group. For some reason monoids hardly ever show up in undergraduate abstract algebra courses. They are ubiquitous in computer science. Roughly speaking a monoid is a group without inverses. More precisely we have the following definition.

Definition 1.21. A monoid is a set \( M \) together with a distinguished element \( e \) and a “multiplication”
\[
m : M \times M \rightarrow M \quad (a, b) \mapsto m(a, b) \equiv ab,
\]
such that
\begin{enumerate}
\item for all \( a \in M \), \( m(a, e) = a = m(e, a) \) (i.e., \( ae = a = ea \)) and
\item for all \( a, b, c \in M \), \( m(a, m(b, c)) = m(m(a, b), c) \) (i.e., \( abc = (ab)c \)).
\end{enumerate}

Example 1.22. Any group is a monoid.

Example 1.23. \( (\mathbb{N}, +, 0) \) is a monoid with \( + \) as the “multiplication” and \( 0 \) the identity element.

Example 1.24. A monoid \( (M, m, e) \) gives rise to a one object category \( \text{BM} \): the collection of objects of \( \text{BM} \) is a singleton \( \text{BM}_0 = \{\ast\} \) and \( \text{Hom}_{\text{BM}}(\ast, \ast) = M \). The composition is just the multiplication in the monoid.
Remark 1.25. If \( C \) is a small category, then for any object \( a \in C_0 \), \( \text{Hom}_C(a, a) \) is a monoid: the identity morphism \( \text{id}_a \in \text{Hom}_C(a, a) \) is the distinguished element of the monoid and the composition

\[ \circ : \text{Hom}_C(a, a) \times \text{Hom}_C(a, a) \to \text{Hom}_C(a, a) \]

is multiplication in the monoid.

Lecture 2. Opposite category. Initial and terminal objects

Last time: Definition of a category. Examples including the notion of a preorder and the categories \( \text{Mat} \) and \( \text{Rel} \). Definition of a monoid.

Definition 2.1. Let \( C \) be a category, a morphism \( a \xrightarrow{f} b \) in \( C \) is an isomorphism if there exists a morphism \( b \xrightarrow{g} a \) such that \( f \circ g = \text{id}_a \) and \( g \circ f = \text{id}_b \).

Example 2.2. In the category \( \text{Set} \), isomorphisms are bijection between sets. In the category \( \text{Vect}_\mathbb{R} \), isomorphisms are isometric linear maps. In the category \( \text{Group} \), isomorphisms are isomorphism of groups.

Remark 2.3. If \( f : a \to b \) is an isomorphism in a category \( C \) then there is only one morphism \( g : b \to a \) so that \( f \circ g = \text{id}_a \) and \( g \circ f = \text{id}_b \). Here is a quick proof: suppose \( h : b \to a \) is another morphism with \( \text{id}_a = f \circ h \) and \( \text{id}_b = h \circ f \). Then, since the composition \( \circ \) is associative,

\[ g = g \circ \text{id}_a = g \circ (f \circ h) = (g \circ f) \circ h = \text{id}_b \circ h = h. \]

Definition 2.4. Let \( f : a \to b \) be an isomorphism in a category \( C \). We call the unique morphism \( g : b \to a \) so that \( f \circ g = \text{id}_a \) and \( g \circ f = \text{id}_b \) the inverse of \( f \) and denote it by \( f^{-1} \).

We now introduce the opposite category, that is, the category opposite to a given category \( C \).

Definition 2.5. Let \( C \) be a category. The opposite category \( C^{\text{op}} \) is defined by reversing all the morphisms of \( C \). Formally, we defined the collection of objects \( (C^{\text{op}})_0 \) of \( C^{\text{op}} \) to be the collection of objects of \( C \). For every two objects \( a, b \in C^{\text{op}}_0 = C_0 \) we define

\[ \text{Hom}_{C^{\text{op}}}(b, a) := \text{Hom}_C(a, b). \]

That is, for every \( a \xrightarrow{f} b \) in \( C \), we have \( b \xrightarrow{f^{\text{op}}} a \) in \( C^{\text{op}} \).

Given two composable morphisms \( a \xrightarrow{f} b \xrightarrow{g} c \) in \( C \), we have \( c \xrightarrow{g^{\text{op}}} b \xrightarrow{f^{\text{op}}} a \) in \( C^{\text{op}} \). We then define the composition in \( C^{\text{op}} \) by

\[ f^{\text{op}} \circ g^{\text{op}} := (g \circ f)^{\text{op}}. \]

Exercise 2.6. Check that the definition of the opposite category \( C^{\text{op}} \) makes sense: if \( C \) is a category, then \( C^{\text{op}} \) as defined above actually is a category.

Exercise 2.7. Check that if \( \text{id}_a : a \to a \) is an identity morphism in a category \( C \) then \( (\text{id}_a)^{\text{op}} : a \to a \) is an identity morphism in \( C^{\text{op}} \).

Hint: what do you need to check?

Remark 2.8. There are several reasons for defining the opposite category. One has to do with contravariant functors which we will define later. The main reason is the principle of duality: every categorical concept, theorem, and proof that holds in an arbitrary category \( C \) also holds in the opposite category \( C^{\text{op}} \). This give us dual concepts, theorems and proofs that are obtained by reversing the direction of all the morphisms. The principle saves a lot of work, but takes time getting used to.

The concept of initial and terminal objects serves as our first illustration of the principle of duality. We first define terminal objects in a category.
**Definition 2.9.** An object $t$ in a category $\mathcal{C}$ is **terminal** if for any object $a$ in $\mathcal{C}$, there is exactly one morphism $a \to t$, i.e. $\text{Hom}_\mathcal{C}(a,t)$ is a one-element set for any object $a$.

**Example 2.10.** In the category $\text{Set}$, the one-element set $\{\ast\}$ is terminal: for any set $X$, there is exactly one function $f : X \to \{\ast\}$. Namely, $f$ is the constant function $f(x) = \ast$ for all $x \in X$.

**Example 2.11.** In the category $\text{Group}$, the trivial group $\{e\}$ is terminal: for any group $G$, there is exactly one homomorphism $\varphi : G \to \{e\} : \varphi(g) = e$ for all $g \in G$.

Not all categories has terminal objects. One simple class examples comes from partially ordered sets (posets) that are ubiquitous in mathematics.

**Definition 2.12.** A preorder $(X, \leq)$ is a **poset** if whenever $a \leq b$ and $b \leq a$ we must have $a = b$. Equivalently a small category $\mathcal{C}$ is a poset if there is at most one morphisms between any two objects and any two isomorphic objects are equal.

**Remark 2.13.** Not all preorders are posets: Let $\mathcal{C}$ be a category with two objects $a,b$ and exactly two non-identity morphisms $a \xrightarrow{f} b$, $b \xrightarrow{g} a$. Then $g \circ f = \text{id}_a$ and $f \circ g = \text{id}_b$ because there is no other choice for what $g \circ f$ and $f \circ g$ can be — there are no other morphisms:

$$
\begin{array}{c}
\text{id}_a
\end{array}
\xrightarrow{\text{id}_b}
\begin{array}{c}
a
\xrightarrow{f}
\end{array}
\xrightarrow{g}
\begin{array}{c}
b
\xrightarrow{\text{id}_a}
\end{array}
$$

Then in particular, $\mathcal{C}$ is a preorder. Notice that $a$ and $b$ are isomorphic in $\mathcal{C}$ but $a \neq b$.

We are not in position to give examples of categories with no terminal objects. Let $(X, \leq)$ be a poset. Observe that $t \in X$ is terminal if for any $a \in X$, $a \leq t$, i.e. $t$ is a **maximal element** of $X$.

The set of integers with the standard ordering has no maximal element. So $(\mathbb{Z}, \leq)$ considered as a category has no terminal objects.

**Proposition 2.14.** Any two terminal objects $t_1$ and $t_2$ in a category $\mathcal{C}$ are uniquely isomorphic.

**Proof.** Since $t_1$ is terminal, there exists a unique morphism $t_2 \xrightarrow{t_1} t_1$. Since $t_2$ is terminal, there exists a unique morphism $t_1 \xrightarrow{t_2} t_2$. Since $t_1$ is terminal, there is only one morphism in $\text{Hom}_\mathcal{C}(t_1, t_1)$ which must be $\text{id}_{t_1}$. Since $g \circ f \in \text{Hom}_\mathcal{C}(t_1, t_1)$, $g \circ f = \text{id}_{t_1}$. Similarly, $f \circ g = \text{id}_{t_2}$. Therefore, $t_1$ and $t_2$ are isomorphic, and $g$, $h$ are the unique isomorphisms.

Now let’s see what duality gives us.

**Definition 2.15.** An object $i$ in a category $\mathcal{C}$ is **initial** if for any object $a$ in $\mathcal{C}$, there is exactly one morphism $i \to a$. Equivalently $\text{Hom}_\mathcal{C}(i,a)$ is a one-element set for any object $a$.

**Example 2.16.** In the category $\text{Set}$ of sets and functions the empty set $\emptyset$ is initial: for any set $X$, there is exactly one function $f : \emptyset \to X$, namely, the empty function.

**Example 2.17.** In the category $\text{Group}$ of groups and homomorphisms, the trivial group $\{e\}$ is initial: for any group $G$, there is exactly one homomorphism $\varphi : \{e\} \to G$; $e \mapsto e_G$ where $e_G$ is the identity element in $G$.

Initial objects in a category are unique up to a unique isomorphism. There are several ways to prove this fact. We can mimic the proof that terminal objects are unique (Proposition 2.14) or we can use the principle of duality and deduce it from Proposition 2.14. We’ll do the latter carefully to illustrate how duality works.

**Proposition 2.18.** Two initial objects $i_1$ and $i_2$ in a category $\mathcal{C}$ are uniquely isomorphic.
Proof. Suppose $i_1, i_2 \in C_0$ are initial, then $i_1, i_2$ are terminal in $C^{op}$. By Proposition 2.14, there exist unique isomorphisms $i_1 \xrightarrow{\phi_{i_1}} i_2$ and $i_2 \xrightarrow{\phi_{i_2}} i_1$ in $C^{op}$ such that $\phi_{i_2} \circ \phi_{i_1} = (\text{id}_{i_1})_{C^{op}}$ and $\phi_{i_1} \circ \phi_{i_2} = (\text{id}_{i_2})_{C^{op}}$. But for every object $a$ in $C^{op}$, the identity morphism $(\text{id}_a)_{C^{op}}$ in $C^{op}$ is the opposite of the identity morphism $(\text{id}_a)_{C}$ by Exercise 2.7. Then

$$(2.2) \quad (\text{id}_{i_1})_{C^{op}} = ((\text{id}_{i_1})_{C})^{op} = f^{op} \circ g^{op} = (g \circ f)^{op}$$

and

$$(2.3) \quad (\text{id}_{i_2})_{C^{op}} = ((\text{id}_{i_2})_{C})^{op} = g^{op} \circ f^{op} = (f \circ g)^{op}$$

Therefore, $(\text{id}_{i_2})_{C} = g \circ f$ and $(\text{id}_{i_1})_{C} = f \circ g$. Hence $i_2 \xrightarrow{\phi_{i_1}} i_1$ and $i_1 \xrightarrow{\phi_{i_2}} i_2$ are the desired unique isomorphisms. \qed

Remark 2.19. It follows from the proof of Proposition 2.18 that if $a \xrightarrow{j} b$ is an isomorphism in a category $C$, then $b \xrightarrow{op} a$ is an isomorphism in the opposite category $C^{op}$.

Lecture 3. Functors. The free functor $F : \text{Set} \to \text{Vect}$.

Last time:

1. defined isomorphism in a category;
2. defined initial and terminal objects;
3. introduced posets (partially ordered sets);
4. introduced opposite categories and duality.

Just as social life of objects in a category depends on morphisms, the social life of categories themselves depends on functors.

Definition 3.1. A functor $F$ from a category $C$ to a category $D$ (we write $F : C \to D$) is a pair of functions $F_0 : C_0 \to D_0$ and $F_1 : C_1 \to D_1$ on objects and morphisms so that

1. for all morphisms $f \in \text{Hom}_C(a,b)$, $F_1(f) \in \text{Hom}_D(F_0(a), F_0(b))$
2. for all objects $a \in C$, $F_1(\text{id}_a) = \text{id}_{F_0(a)}$ (i.e., $F$ preserves identities)
3. for all composable morphisms $f \in \text{Hom}_C(a,b)$, $g \in \text{Hom}_C(b,c)$, $F_1(g \circ f) = F_1(g) \circ F_1(f)$ (i.e., $F$ preserves composition)

Remark 3.2.

1. We will often drop the subscript 0 and 1. For example, we would write $F_0(a) \xrightarrow{F_1(f)} F_0(b)$ as $F(a) \xrightarrow{F(f)} F(b)$.
2. The first condition in Definition 3.1 says that the maps $F_0$ and $F_1$ are compatible with the source and target $s, t : C_1 \to C_0$, $s, t : D_1 \to D_0$ in the following sense: for any morphism $f$ in $C$

\[ s(F(f)) = F(s(f)) \quad \text{and} \quad t(F(f)) = F(t(f)) \]

(strictly speaking we should write $s\circ \text{id}_C$ instead of just $s$ etc, but this clutters the notation).
3. The second condition in Definition 3.1 says that $F_0$ and $F_1$ are compatible with the unit maps $u : C_0 \to C_1$ and $u : D_0 \to D_1$: for any object $a$ in $C$

\[ u(F_0(a)) = F_1(u(a)) \]

Example 3.3. We have the forgetful functor $U$ from Group to Set that forgets the group structure: for every group $G$, $U(G)$ is the set of elements of $G$ (we’ll call it the underlying set of the group $G$). For every homomorphism $\varphi : G \to H$, $U(\varphi)$ is the corresponding function on the underlying sets.

It is not hard to check that $U : \text{Group} \to \text{Set}$ is actually a functor (do it).
**Example 3.4.** Recall that Rel denotes the category of sets and relations. There is a functor $F : \text{Set} \to \text{Rel}$: $F$ does nothing on objects and sends functions to their graphs: $F(f) = \{(x, y) \in X \times Y \mid y = f(x)\}$.

Again it’s not hard to check that $F$ is a functor. You should make sure to check that $F$ preserves composition.

**Definition 3.5.** The category Ab of abelian groups is defined as follows: the objects of Ab are abelian groups. The morphisms in Ab are homomorphisms of groups.

**Example 3.6.** There is the inclusion functor $i : \text{Ab} \to \text{Group}$ that is the identity on both objects and morphisms. This functor is practically invisible.

There is a forgetful functor $U : \text{Vect}_\mathbb{R} \to \text{Ab}$ that forgets the scalar multiplication and only remembers addition. Note that since every real vector space is an abelian group under addition and since linear maps preserve addition of vectors the definition makes sense.

The functor $U$ allows us to say that a real vector space is an abelian group with extra structure.

**Remark 3.7.** Given two categories $\mathcal{C}$ and $\mathcal{D}$ it may be impossible to extend a given map $F : \mathcal{C}_0 \to \mathcal{D}_0$ on objects to a functor. Here is an example. Let $\mathcal{C} = \text{Group}$, $\mathcal{D} = \text{Ab}$ and $Z : \text{Group}_0 \to \text{Ab}_0$ be defined by $Z(G) = \{z \in G \mid zg = ga \text{ for all } g \in G\}$, the center of $G$. It turns out that the map $Z$ cannot be extended to a functor from Group to Ab.

**Example 3.8.** There is a functor $F : \text{Mat} \to \text{FDVect}_\mathbb{R}$ from the category of real matrices (Example 3.3) to the category of (real) finite dimensional vector spaces. On objects, $F(n) = \mathbb{R}^n$ for all $n \in \mathbb{N}$. On morphism, $F$ sends an $m \times n$ matrix $A$ to the linear map $F(A) : \mathbb{R}^n \to \mathbb{R}^m$:

$$F(A) \begin{bmatrix} x_1 \\
\vdots \\
x_n \end{bmatrix} = A \begin{bmatrix} x_1 \\
\vdots \\
x_n \end{bmatrix}.$$  

Matrix multiplication is defined in such a way as to make $F$ preserve composition. Notice that $F$ is not surjective on objects (since not every finite dimensional vector space is $\mathbb{R}^n$). However, for all $n, m \in \mathbb{N}$, the map

$$F : \text{Hom}_{\text{Mat}}(n, m) \to \text{Hom}_{\text{FDVect}_\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^m)$$

is a bijection. This is one of the reasons why matrices are so useful in linear algebra.

There is a forgetful functor $U : \text{Vect}_\mathbb{R} \to \text{Set}$ (same name as in Example 3.3 but not the same functor) that forget the vector space structure and remembers only the underlying set. Interestingly enough there is also a functor $F : \text{Set} \to \text{Vect}_\mathbb{R}$ that starts with a set $X$ and assigns to it a vector spaces $F(X)$ that has elements of $X$ as a basis. You have seen $F$ in linear algebra classes for finite sets.

The pair of functors

$$F : \text{Set} \quad \Rightarrow \quad \text{Vect}_\mathbb{R} : U$$

will be an important example of a pair of adjoint functors in the second half of the course.

**Theorem 3.9.** There is a functor $F : \text{Set} \to \text{Vect}_\mathbb{R}$ that assigns to each set $X$ a vector space $F(X)$ with the following universal property:

(i) for any set $X$, there is a function $\eta_X : X \to U(F(X))$ where $U(F(X))$ is the set underlying the vector space $F(X)$;

(ii) given any vector space $W$ and any function (morphism in Set) $h : X \to U(W)$, there is a unique linear map $\tilde{h} : F(X) \to W$ so that $U(\tilde{h}) \circ \eta_X = h$. 

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Remark 3.10. Condition (ii) can be restated in terms of diagrams: given a function \( h : X \to U(W) \)

\[
\begin{array}{ccc}
F(X) & \xrightarrow{\eta_X} & U(F(X)) \\
\downarrow & & \downarrow U(h) \\
X & \xrightarrow{\eta_X} & U(W)
\end{array}
\]

so that \( \delta \) commutes.

Proof. We construct the functor \( F : \text{Set} \to \text{Vect}_\mathbb{R} \) with the desired properties.

For any set \( X \), let \( \mathbb{R}^X \) denote the set of functions from a set \( X \) to \( \mathbb{R} \),

\[ \mathbb{R}^X = \{ f : X \to \mathbb{R} \}. \]

The set \( \mathbb{R}^X \) is a real vector space under the point-wise addition and scalar multiplication: for any \( \lambda \in \mathbb{R} \) and any \( f, g \in \mathbb{R}^X \), we define \( (\lambda f)(x) := \lambda f(x) \) and \( (f + g)(x) := f(x) + g(x) \) for all \( x \in X \).

Now define \( F(X) = \{ f \in \mathbb{R}^X \mid f(x) = 0 \text{ for all but finitely many } x \in X \} \).

Note that if \( X \) is finite then \( F(X) = \mathbb{R}^X \) and otherwise \( F(X) \subsetneq \mathbb{R}^X \). It’s not hard to check that \( F(X) \) is a vector subspace of \( \mathbb{R}^X \).

For every \( x \in X \), we have a function \( \eta_X^x : X \to \mathbb{R} \) defined by

\[
\eta_X^x(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}
\]

(3.4)

Clearly, \( \eta_X^x \in F(X) \). We then have a map

\[ \eta_X : X \to F(X), \quad x \mapsto \eta_X^x. \]

Note that \( X \) is a set and \( F(X) \) is a vector space, so what we really have is a function \( \eta_X : X \to U(F(X)) \). This may seem pedantic, but it is useful when thinking about functors and categories.

Lecture 4. The free functor \( F : \text{Vect}_\mathbb{R} \to \text{Set} \). Contravariant functors. Locally small categories.

Last time: Defined functors and looked at some examples. There are forgetful functors \( U : \text{Group} \to \text{Set} \) (which forgets the group structure), \( U : \text{Vect}_\mathbb{R} \to \text{Ab} \) (which forgets scalar multiplication). There is the inclusion functor \( i : \text{Ab} \to \text{Group} \) (any abelian group is a group and a homomorphism of abelian groups is a homomorphism of groups).

Remark 4.1. Functors can be composed to obtain a new functor. Suppose \( F : \mathcal{C} \to \mathcal{D} \) and \( G : \mathcal{D} \to \mathcal{E} \) are two functors. Their composite \( G \circ F : \mathcal{C} \to \mathcal{E} \) is defined as follows:

- On objects: \( (G \circ F)(a) = G(F(a)) \) for any \( a \in \mathcal{C} \)

- On morphisms: \( (G \circ F)(a \xrightarrow{f} b) := G(F(a)) \xrightarrow{G(f)} G(F(b)) \).

Exercise 4.2. Check that \( G \circ F \) so defined is a functor. Hint: what do you actually need to check?

Example 4.3. Composing

\[
\begin{array}{ccc}
\text{Vect}_\mathbb{R} & \xrightarrow{U} & \text{Ab} & \xrightarrow{i} & \text{Group} & \xrightarrow{U} & \text{Set} \\
\end{array}
\]

we get a forgetful functor

\[
\begin{array}{ccc}
\text{Vect}_\mathbb{R} & \xrightarrow{U} & \text{Set} \\
\end{array}
\]

that assigns to a vector space the underlying set. I will call it \( U \) again. (So now we have three functors called \( U \) for “underlying”. I hope this is not too confusing.)
Last time we also started constructing a functor $F : \text{Set} \to \text{Vect} \equiv \text{Vect}_R$ with the following universal property:

(i) for any set $X$, there is a function $\eta_X : X \to U(F(X))$ where $U(F(X))$ is the set underlying the vector space $F(X)$;

(ii) given any vector space $W$ and any function $h : X \to U(W)$, there is a unique linear map $\tilde{h} : F(X) \to W$ so that $U(\tilde{h}) \circ \eta_X = h$.

On objects we defined $F$ by

$$F(X) = \{ f : X \to \mathbb{R} \mid f(x) = 0 \text{ for all but finitely many } x \in X \}.$$  

For every set $X$ we also defined a function $\eta_X : X \to U(F(X))$, $\eta_X(x) := \eta^x_X$ where

$$\eta^x_X(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}.$$  

We now argue that the function $\eta_X : X \to U(F(X))$ has the desired universal property.

**Claim** The set $\{\eta^x_X\}_{x \in X}$ is linearly independent and spans $F(X)$ hence forms a basis of the vector space $F(X)$.

**Proof of claim.** Suppose that $\{\eta^x_X\}_{x \in X}$ is linearly dependent. This means that there is $n > 0$, $x_1, \ldots, x_n \in X$ and $c_1, \ldots, c_n \in \mathbb{R}$ not all 0 so that

$$c_1 \eta^x_1 + \cdots + c_n \eta^x_n = 0.$$  

But then for any $j$, $1 \leq j \leq n$

$$\sum_{i=1}^n c_i \eta^x_i(x_j) = c_j = 0.$$  

Contradiction. Therefore $\{\eta^x_X\}_{x \in X}$ is linearly independent.

Given $f \in F(X)$ there is $n \geq 0$ and $x_1, \ldots, x_n \in X$ so that $f(x_i) \neq 0$ and $f(y) = 0$ for $y \not\in \{x_1, \ldots, x_n\}$. Then

$$f(z) = \sum_{i=1}^n f(x_i) \eta^x_i(z)$$  

for all $z \in X$. Hence

$$f = \sum_{i=1}^n f(x_i) \eta^x_i$$  

and consequently

$$f = \sum_{x \in X} f(x) \eta^x_X.$$  

Thus the set $\{\eta^x_X\}_{x \in X}$ spans $F(X)$. \hfill \square

Now suppose we have a vector space $W$ and a function $h : X \to U(W)$. Since $\{\eta^x_X\}_{x \in X}$ is a basis of the vector space $F(X)$ there is a unique linear map

$$\tilde{h} : F(X) \to W$$  

with $\tilde{h}(\eta_X(x)) = h(x)$.

Explicitly, for all $f \in F(X)$

$$\tilde{h}(f) := \tilde{h} \left( \sum_{x \in X} f(x) \eta_X(x) \right) = \sum_{x \in X} f(x) h(x)$$.
Note that \( f(x) \) is a real number and \( h(x) \) is a vector in \( W \), so \( f(x)h(x) \) makes sense. Moreover the sum \( \sum_{x \in X} f(x)h(x) \) is actually finite since \( f(x) \) is zero for all but finitely many \( x \in X \).

We now construct the desired functor \( F : \text{Set} \to \text{Vect} \). Given a function \( \varphi : X \to Y \) between two sets we need to construct a linear map \( F(\varphi) : F(X) \to F(Y) \). Consider the function

\[
h : X \to U(F(Y)), \quad h := \eta_Y \circ \varphi
\]

The function \( h \) is defined to make the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\varphi} & Y \\
\downarrow h & & \downarrow \eta_Y \\
U(F(Y)) & & U(F(Y))
\end{array}
\]

commute. By the universal property of the function \( \eta_X : X \to U(F(X)) \) there is a unique linear map

\[
\tilde{h} : F(X) \to F(Y)
\]

so that

\[
U(\tilde{h}) \circ \eta_X = h,
\]

i.e., the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\eta_X} & U(F(X)) \\
\downarrow \varphi & & \downarrow U(\tilde{h}) \\
Y & \xrightarrow{\eta_Y} & U(F(Y))
\end{array}
\]

commutes. We define \( F(\varphi) \) to be this \( \tilde{h} \):

\[
(4.5) \quad F(\varphi) := \tilde{h}.\]

\[\square\]

**Exercise 4.4.** Show that \( F : \text{Set} \to \text{Vect}, F(X \xrightarrow{\varphi} Y) = F(X) \xrightarrow{\tilde{h}} F(Y) \), where \( \tilde{h} \) is the linear map constructed above, is in fact a functor.

Hints: Use the universal properties of the collection of functions \( \{\eta_X : X \to U(F(X))\}_{X \in \text{Set}} \).

If you have trouble getting the universal properties approach to work show that \( F(\varphi) \) is given by

\[
F(\varphi)f = \sum_{x \in X} f(x)\eta_Y^{\varphi(x)}
\]

for all \( f \in F(X) \subset \mathbb{R}^X \).

\[\underline{Remark 4.5}.\] The functors that we have considered so far are called **covariant functors** in older literature.

**Definition 4.6.** A **contravariant functor** \( F \) from a category \( \mathcal{C} \) to a category \( \mathcal{D} \) is a pair of functions \( F_0 : \mathcal{C}_0 \to \mathcal{D}_0 \) and \( F_1 : \mathcal{C}_1 \to \mathcal{D}_1 \) on objects and morphisms so that

- for all morphisms \( f \in \text{Hom}_\mathcal{C}(a, b) \),
  \[
  F_0(b) \xrightarrow{F_1(f)} F_0(a)
  \]
  [Note the direction of the morphism \( F_1(f) \)!]
- for all objects \( a \in \mathcal{C} \), \( F_1(\text{id}_a) = \text{id}_{F_0(a)} \)
• for all pairs of composable morphisms \( a \xrightarrow{f} b \xrightarrow{g} c \)
\[
F_1(g \circ f) = F_1(f) \circ F_1(g) :
\]
\[
\begin{array}{c}
a & \xrightarrow{g \circ f} & c \\
\downarrow f & \quad & \downarrow g \\
b & \quad & a
\end{array}
\quad \begin{array}{c}
F(a) & \xrightarrow{F(f) \circ F(g)} & F(c) \\
\downarrow F(f) & \quad & \downarrow F(g) \\
F(b) & \quad & F(a)
\end{array}
\]

**Example 4.7.** Consider the category \( \text{Vect}_\mathbb{R} \). For any vector space over \( \mathbb{R} \), we have the dual vector space
\[
V^* := \{ \ell : V \to \mathbb{R} \mid \ell \text{ is linear} \}.
\]
For any linear map \( T : V \to W \), we have the pullback (dual map)
\[
T^* : W^* \to V^*, \quad T^*(\ell) = \ell \circ T
\]
for all \( \ell : W \to \mathbb{R} \). Note that \( \text{id}_V^*(\ell) = \ell \circ \text{id}_V = \ell \). Hence
\[
(\text{id}_V)^* = \text{id}_V^*.
\]
for all vector spaces \( V \).

Given a pair of composable linear maps \( V \xrightarrow{T} W \xrightarrow{S} U \) and \( \ell \in U^* \)
\[
(S \circ T)^*(\ell) = \ell \circ S \circ T = T^*(\ell \circ S) = T^*(S^*(\ell))
\]
It follows that
\[
*: \text{Vect} \to \text{Vect}, \quad (V \xrightarrow{T} W) \mapsto W^* \xrightarrow{T^*} V^*
\]
is a contravariant functor.

**Remark 4.8.** Any contravariant functor \( F \) from a category \( \mathcal{C} \) to a category \( \mathcal{D} \) is a (covariant) functor \( F : \mathcal{C} \to \mathcal{D}^\text{op} \) and a (covariant) functor \( F : \mathcal{C}^\text{op} \to \mathcal{D} \). For this reason in contemporary literature one simply writes
\[
F : \mathcal{C}^\text{op} \to \mathcal{D}
\]
when \( F \) is a contravariant functor from \( \mathcal{C} \) to \( \mathcal{D} \).

We end the lecture with a definition.

**Definition 4.9.** A category \( \mathcal{C} \) is **locally small** if for any \( a, b \in \mathcal{C} \), \( \text{Hom}_\mathcal{C}(a, b) \) is a set.

**Example 4.10.** The categories \( \text{Set} \), \( \text{Group} \), \( \text{Vect}_\mathbb{R} \) and \( \text{Rel} \) are all locally small but not small. Any small category is, of course, locally small.