INTRODUCTION TO CATEGORY THEORY

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CONTENTS

Lecture 1. Definition and Examples 2
Lecture 2. Opposite category. Initial and terminal objects 4
Lecture 3. Functors. Universal mapping property 7
Lecture 4. Contravariant functors. Locally small categories 9
Lecture 5. Hom functors. Binary products. CAT. Products of categories 12
Lecture 6. Products as terminal objects. Coproducts 16
Lecture 7. Monos, epis, fully faithful functors 19
Lecture 8. Metric and topological space 23
Lecture 9. Continuity, closed sets, subspace topology, and products 26
Lecture 10. Products and coproducts in Top. General products 29
Lecture 11. More examples of products. Subcategories 32
Lecture 13. Russel’s paradox, Freyd’s theorem, universes 37
Lecture 14. Limits, diagrams, equalizers 39
Lecture 15. Fiber products. Completeness 43
Lecture 16. Completeness of categories with products and equalizers 47
Lecture 17. Colimits and coequalizers 51
Lecture 18. Cocompleteness of Set and of other categories 53
Lecture 19. Natural transformations and functor categories 55
Lecture 20. Graphs. Isomorphisms and equivalences of categories 59
Lecture 21. Equivalences of categories and fully faithful and essentially surjective functors 62
Lecture 22. Equivalences of categories. Concrete categories 65
Lecture 23. Yoneda lemma 69
Lecture 24. Yoneda embeddings. Representable functors 72
Lecture 25. Representation of a functor. Universal elements 76
Lecture 26. Universal arrows. Comma categories 78
Lecture 27. Horizontal composition of natural transformations. Adjunctions 82
Lecture 28. Examples of adjunctions 86
Lecture 29. Adjunctions from the universal arrows 89
Lecture 30. Colimits and limits as adjoints to the diagonal functor ∆ 92
Lecture 31. Universal arrows; units and counits. Uniqueness of left adjoints 95
Lecture 32. Triangle identities 98
Lecture 33. Adjunctions compose. RAFL 101
Lecture 34. Proof that Right Adjoints Preserve Limits. Adjoint equivalences 104
Lecture 35. Representable functors preserve limits. Monads 107
Lecture 36. Monads from adjunctions 111
Lecture 37. The Kleisli category 114

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Lecture 1. Definition and Examples

Mathematics is a study of patterns. Category theory is the study of patterns in mathematics.

To make the two sentences less cryptic, we start by defining what a category is. We pick one of several (mostly equivalent) definitions and proceed from there.

**Definition 1.1.** A category $C$ consists of the following data:

1. A collection of objects $C_0$.
2. For each pair of objects $a, b \in C_0$, a collection of morphisms $\text{Hom}_C(a, b)$, which may be empty if $a \neq b$. We write $a \xrightarrow{f} b$ or $b \xleftarrow{f} a$ if $f \in \text{Hom}_C(a, b)$.
3. For each object $a \in C_0$, a distinguished identity morphism $\text{id}_a \in \text{Hom}_C(a, a)$;
4. For each triple of objects $a, b, c \in C_0$, a composition map

   $\circ : \text{Hom}_C(b, c) \times \text{Hom}_C(a, b) \to \text{Hom}_C(a, c), \quad \left( c \xleftarrow{g} b, b \xrightarrow{f} a \right) \mapsto \left( c \xleftarrow{g \circ f} a \right)$

   which is subject to the following conditions:
   
   (i) for all $a, b \in C_0$ and for all $f \in \text{Hom}_C(a, b)$, $f \circ \text{id}_a = f = \text{id}_b \circ f$;
   
   (ii) The composition $\circ$ is associative: for all $d \xleftarrow{h} c, c \xleftarrow{g} b, b \xrightarrow{f} a, (h \circ (g \circ f) = (h \circ g) \circ f)$.

We denote the collection of all morphisms of the category $C$ by $C_1$. Thus $C_1 = \bigcup_{a, b \in C_0} \text{Hom}_C(a, b)$.

**Remark 1.2.** Given an object $a$ of a category $C$ we write $a \in C$. (Writing $a \in C_0$ is more precise and more pedantic.)

**Remark 1.3.** Given a category $C$ and two objects $a, b \in C$, the collection of morphisms $\text{Hom}_C(a, b)$ from $a$ to $b$ may also be denoted by $\text{Hom}(a, b)$ when the category $C$ is understood. Some authors write $C(a, b)$ for $\text{Hom}_C(a, b)$. We will not use $C(a, b)$.

**Definition 1.4.** Given a category $C$ and a morphism $f \in \text{Hom}_C(a, b)$, we call $a$ the source of $f$ and $b$ the target of $f$. We write $t(f) = a$ and $s(f) = a$. In other word, we have two maps $s, t : C_1 \to C_0$ from morphisms to objects. Note that had we not required that $\text{Hom}_C(a, b) \cap \text{Hom}_C(c, d) = \emptyset$ for $(a, b) \neq (c, d)$, the source and target maps would not be well-defined.

**Remark 1.5.** We also have a unit map $u : C_0 \to C_1$, $a \mapsto \text{id}_a$ that assigns to each object of a category $C$ the identity morphism on that object.

**Examples**

**Example 1.6.** Sets and functions form a category $\text{Set}$. The composition in the category $\text{Set}$ is the composition of functions.

**Example 1.7.** Groups and group homomorphisms form a category $\text{Group}$. Group is an example of a category whose objects are sets with some structure and whose morphisms are structure preserving maps. Not all categories are like this, but many are.

**Example 1.8.** There is the empty category with the empty set of objects and the empty set of morphism, denoted by $\emptyset$.

**Example 1.9.** There is a category, often denoted by $\mathbf{1}$, with one objects $\ast$ and one morphism $\text{id}_\ast : \ast \to \ast$. This is the smallest nonempty category. Exercise: what’s the composition map?
Definition 1.10. A category $\mathcal{C}$ is small if both the collection of objects $\mathcal{C}_0$ and the collection of morphisms $\mathcal{C}_1$ are sets (rather than some bigger collections of objects).

Example 1.11. The categories $\emptyset$ and $\mathbb{1}$ are small. The category $\text{Set}$ of all sets and functions is not small thanks to Russell’s paradox. We’ll come back to this point.

Definition 1.12. A preorder is a category with at most one morphism between any two objects.

Remark 1.13. A small preorder $\mathcal{C}$ is the same thing as set $\mathcal{C}_0$ equipped with a binary relation $\leq$ which is reflexive and transitive:

1. For all $a \in \mathcal{C}_0$, $a \leq a$ and
2. If $a \leq b$ and $b \leq c$, then $a \leq c$ for all $a, b, c \in \mathcal{C}_0$.

Proof. Let $\mathcal{C}$ be a preorder and a small category. Define a relation $\leq$ on the set of objects $\mathcal{C}_0$ by

$$a \leq b \iff \text{there is a morphism (which is necessarily unique) } f_{ba} : a \to b.$$ 

Since for every object $a$ in $\mathcal{C}$ there is the identity morphism $\text{id}_a : a \to a$, the relation $\leq$ is reflexive. Given $a \leq b$ and $b \leq c$ there are morphisms $f_{ba} : a \to b$ and $f_{cb} : b \to a$. The composite $f_{cb} \circ f_{ba} : a \to c$ is a morphism from $a$ to $c$. Hence $\text{Hom}_C(a, c) \neq \emptyset$. But $\mathcal{C}$ is a preorder, so there is at most one morphism from $a$ to $c$. We conclude that $f_{cb} \circ f_{ba} = f_{ca}$. That is,

$$a \leq b \text{ and } b \leq c \implies a \leq c.$$

Conversely given a relation $\leq$ on a set $X$ which is reflexive and transitive define a small category $\mathcal{C}$ by setting its collection of objects to be $X$. Given two elements $x, y \in X$ we define

$$\text{Hom}_C(x, y) = \begin{cases} \{f_{yx} : x \to y\}, & \text{if } x \leq y; \\ \emptyset, & \text{otherwise.} \end{cases}$$

It is not hard to check that $\mathcal{C}$ so defined is in fact a category. \qed

Remark 1.14. Preorders occur naturally in mathematics. Recall that for a set $Y$ the powerset $\mathcal{P}(Y)$ is the set of all subsets of $Y$:

$$\mathcal{P}(Y) := \{A \mid A \subseteq Y\}$$

The the subset relation $\subseteq$ on $\mathcal{P}(Y)$ is reflexive and transitive. Hence $\mathcal{P}(Y)$ is a preorder. Note that for a given pair $A, B \subseteq Y$ it may well happen that neither $A \subseteq B$ nor $B \subseteq A$, i.e., the corresponding set $\text{Hom}(A, B)$ in the preorder $\mathcal{P}(Y)$ may well be empty.

You have seen binary relations on a given set. One can also talk about relations between two distinct sets. The following terminology is slightly nonstandard:

Definition 1.15. A relation from a set $X$ to a set $Y$ is a subset $R$ of $X \times Y$.

Note that relations can be composed: if $R \subseteq X \times Y$ and $S \subseteq Y \times Z$, then define the composition to be

$$S \circ R = \{(x, z) \in X \times Z \mid \exists y \in Y, \text{ so that } (x, y) \in R, (y, z) \in Z\}.$$ 

It is not hard to show that

1. Composition of relations is associative.
2. Denote the diagonal $\Delta_X := \{(x, y) \in X \times X \mid x = y\}$ in the product of a set $X$ with itself by $\text{id}_X$. Then for a relation $R \subseteq X \times Y$, $\text{id}_Y \circ R = R$ and $R \circ \text{id}_X = R$.

Therefore, sets and relations form a category.

Notation 1.16. We denote the category of sets and relations by $\text{Rel}$. In particular for any two sets $X$ and $Y$, $\text{Hom}_{\text{Rel}}(X, Y) = \mathcal{P}(X \times Y)$.

Remark 1.17. The category $\text{Rel}$ of relations is an example of a category whose morphisms are not functions.
Example 1.18. The collection of real vector spaces and linear maps form a category $\text{Vect}_\mathbb{R}$. The composition is the composition of linear maps. This is yet another example of a category of sets with structure and structure preserving functions.

The next category is similar to Example 1.18 but is not a category of sets with structure and structure preserving maps.

Example 1.19. The category $\text{Mat}$ of real matrices whose objects are natural numbers (including 0). A morphism from $n \in \mathbb{N}$ to $m \in \mathbb{N}$ (for $n,m > 0$) is a real $m \times n$ matrix. The composition of morphisms in $\text{Mat}$ is matrix multiplication. Of course each $n \in \mathbb{N}$ is secretly $\mathbb{R}^n$. So $0 \in \mathbb{N}$ is secretly $\mathbb{R}^0 = \{0\}$, the zero-dimensional vector space. For this reason we define

$$\text{Hom}_{\text{Mat}}(m,0) = \{O_{0,m}\}$$

$$\text{Hom}_{\text{Mat}}(0,n) = \{O_{n,0}\}$$

such that for all $A \in \text{Hom}_{\text{Mat}}(n,m)$, $O_{0,m} \circ A = O_{n,0}$, and $O_{n,0} \circ A = O_{0,m}$. These data make $\text{Mat}$ into a category.

Remark 1.20. We will see later that the categories $\text{Mat}$ “is” in an appropriate sense the category $\text{FDVect}_\mathbb{R}$ of finite dimensional vector spaces. Technically the two categories are equivalent.

We end the lecture with a notion of a monoid. A monoid is an algebraic structure like a group. For some reason monoids hardly ever show up in undergraduate abstract algebra courses. They are ubiquitous in computer science. Roughly speaking a monoid is a group without inverses. More precisely we have the following definition.

Definition 1.21. A monoid is a set $M$ together with a distinguished element $e$ and a “multiplication”

$$m : M \times M \to M \quad (a,b) \mapsto m(a,b) \equiv ab,$$

such that

1. for all $a \in M$, $m(a,e) = a = m(e,a)$ (i.e., $ae = a = ea$) and
2. for all $a,b,c \in M$, $m(m(a,b),c) = m(m(a,b),c)$ (i.e., $a(bc) = (ab)c$).

Example 1.22. Any group is a monoid.

Example 1.23. $(\mathbb{N},+)$ is a monoid with $+$ as the “multiplication” and $0$ the identity element.

Example 1.24. A monoid $(M,m,e)$ gives rise to a one object category $B M$: the collection of objects of $B M$ is a singleton $B M_0 = \{\ast\}$ and $\text{Hom}_{BM}(\ast,\ast) = M$. The composition is just the multiplication in the monoid.

Remark 1.25. If $C$ is a small category, then for any object $a \in C_0$, $\text{Hom}_{C}(a,a)$ is a monoid: the identity morphism $\text{id}_a \in \text{Hom}_{C}(a,a)$ is the distinguished element of the monoid and the composition

$$\circ : \text{Hom}_{C}(a,a) \times \text{Hom}_{C}(a,a) \to \text{Hom}_{C}(a,a)$$

is multiplication in the monoid.

Lecture 2. Opposite category. Initial and terminal objects

Last time: Definition of a category. Examples including the notion of a preorder and the categories $\text{Mat}$ and $\text{Rel}$. Definition of a monoid.

Definition 2.1. Let $C$ be a category, a morphism $a \xrightarrow{f} b$ in $C$ is an isomorphism if there exists a morphism $b \xrightarrow{g} a$ such that $f \circ g = \text{id}_a$ and $g \circ f = \text{id}_b$. 

4
Example 2.2. In the category $\text{Set}$, isomorphisms are bijection between sets. In the category $\text{Vect}_\mathbb{R}$, isomorphisms are invertible linear maps. In the category $\text{Group}$, isomorphisms are isomorphism of groups.

Remark 2.3. If $f : a \to b$ is an isomorphism in a category $\mathcal{C}$ then there is only one morphism $g : b \to a$ so that $f \circ g = \text{id}_a$ and $g \circ f = \text{id}_b$. Here is a quick proof: suppose $h : b \to a$ is another morphism with $\text{id}_a = f \circ h$ and $\text{id}_b = h \circ f$. Then, since the composition $\circ$ is associative,

$$g = g \circ \text{id}_a = g \circ (f \circ h) = (g \circ f) \circ h = \text{id}_b \circ h = h.$$

Definition 2.4. Let $f : a \to b$ be an isomorphism in a category $\mathcal{C}$. We call the unique morphism $g : b \to a$ so that $f \circ g = \text{id}_a$ and $g \circ f = \text{id}_b$ the inverse of $f$ and denote it by $f^{-1}$.

We now introduce the opposite category, that is, the category opposite to a given category $\mathcal{C}$.

Definition 2.5. Let $\mathcal{C}$ be a category. The opposite category $\mathcal{C}^{\text{op}}$ is defined by reversing all the morphisms of $\mathcal{C}$. Formally, we defined the collection of objects $(\mathcal{C}^{\text{op}})_0$ of $\mathcal{C}^{\text{op}}$ to be the collection of objects of $\mathcal{C}$. For every two objects $a, b \in \mathcal{C}_0 = \mathcal{C}_0^{\text{op}}$ we define

$$\text{Hom}_{\mathcal{C}^{\text{op}}}(b, a) := \text{Hom}_\mathcal{C}(a, b).$$

That is, for every $f : a \to b$ in $\mathcal{C}$, we have $b \xrightarrow{f^{\text{op}}} a$ in $\mathcal{C}^{\text{op}}$.

Given two composable morphisms $a \xrightarrow{f} b \xrightarrow{g} c$ in $\mathcal{C}$, we have $c \xrightarrow{g^{\text{op}}} b \xrightarrow{f^{\text{op}}} a$ in $\mathcal{C}^{\text{op}}$. We then define the composition in $\mathcal{C}^{\text{op}}$ by

$$f^{\text{op}} \circ g^{\text{op}} := (g \circ f)^{\text{op}}.$$

Exercise 2.6. Check that the definition of the opposite category $\mathcal{C}^{\text{op}}$ makes sense: if $\mathcal{C}$ is a category, then $\mathcal{C}^{\text{op}}$ as defined above actually is a category.

Exercise 2.7. Check that if $\text{id}_a : a \to a$ is an identity morphism in a category $\mathcal{C}$ then $(\text{id}_a)^{\text{op}} : a \to a$ is an identity morphism in $\mathcal{C}^{\text{op}}$.

Hint: what do you need to check?

Remark 2.8. There are several reasons for defining the opposite category. One has to do with contravariant functors which we will define later. The main reason is the principle of duality: every categorical concept, theorem, and proof that holds in an arbitrary category $\mathcal{C}$ also holds in the opposite category $\mathcal{C}^{\text{op}}$. This give us dual concepts, theorems and proofs that are obtained by reversing the direction of all the morphisms. The principle saves a lot of work, but takes time getting used to.

The concept of initial and terminal objects serves as our first illustration of the principle of duality. We first define terminal objects in a category.

Definition 2.9. An object $t$ in a category $\mathcal{C}$ is terminal if for any object $a$ in $\mathcal{C}$, there is exactly one morphism $a \to t$, i.e. $\text{Hom}_\mathcal{C}(a, t)$ is a one-element set for any object $a$.

Example 2.10. In the category $\text{Set}$, the one-element set $\{\ast\}$ is terminal: for any set $X$, there is exactly one function $f : X \to \{\ast\}$. Namely, $f$ is the constant function $f(x) = \ast$ for all $x \in X$.

Example 2.11. In the category $\text{Group}$, the trivial group $\{e\}$ is terminal: for any group $G$, there is exactly one homomorphism $\varphi : G \to \{e\} : \varphi(g) = e$ for all $g \in G$.

Not all categories has terminal objects. One simple class examples comes from partially ordered sets (posets) that are ubiquitous in mathematics.

Definition 2.12. A preorder $(X, \leq)$ is a poset if whenever $a \leq b$ and $b \leq a$ we must have $a = b$. Equivalently a small category $\mathcal{C}$ is a poset if there is at most one morphisms between any two objects and any two isomorphic objects are equal.
Remark 2.13. Not all preorders are posets: Let \( C \) be a category with two objects \( a, b \) and exactly two non-identity morphisms \( a \xrightarrow{f} b, b \xrightarrow{g} a \). Then \( g \circ f = \text{id}_a \) and \( f \circ g = \text{id}_b \) because there is no other choice for what \( g \circ f \) and \( f \circ g \) can be — there are no other morphisms:

\[
\text{id}_a \xrightarrow{f} \xrightarrow{g} \text{id}_b
\]  

Then in particular, \( C \) is a preorder. Notice that \( a \) and \( b \) are isomorphic in \( C \) but \( a \neq b \).

We are not in position to give examples of categories with no terminal objects. Let \((X, \leq)\) be a poset. Observe that \( t \in X \) is terminal if for any \( a \in X, a \leq t \), i.e. \( t \) is a maximal element of \( X \).

The set of integers with the standard ordering has no maximal element. So \((\mathbb{Z}, \leq)\) considered as a category has no terminal objects.

**Proposition 2.14.** Any two terminal objects \( t_1 \) and \( t_2 \) in a category \( C \) are uniquely isomorphic.

**Proof.** Since \( t_1 \) is terminal, there exists a unique morphism \( t_2 \xrightarrow{g} t_1 \). Since \( t_2 \) is terminal, there exists a unique morphism \( t_1 \xrightarrow{f} t_2 \). Since \( t_1 \) is terminal, there exists only one morphism in \( \text{Hom}_C(t_1, t_1) \) which must be \( \text{id}_{t_1} \). Since \( g \circ f \in \text{Hom}_C(t_1, t_1) \), \( g \circ f = \text{id}_{t_1} \). Similarly, \( f \circ g = \text{id}_{t_2} \). Therefore, \( t_1 \) and \( t_2 \) are isomorphic, and \( g, h \) are the unique isomorphisms. \( \square \)

Now let’s see what duality gives us.

**Definition 2.15.** An object \( i \) in a category \( C \) is initial if for any object \( a \) in \( C \), there is exactly one morphism \( i \to a \). Equivalently \( \text{Hom}_C(i, a) \) is a one-element set for any object \( a \).

**Example 2.16.** In the category \( \text{Set} \) of sets and functions the empty set \( \emptyset \) is initial: for any set \( X \), there is exactly one function \( f : \emptyset \to X \), namely, the empty function.

**Example 2.17.** In the category \( \text{Group} \) of groups and homomorphisms, the trivial group \( \{e\} \) is initial: for any group \( G \), there is exactly one homomorphism \( \varphi : \{e\} \to G, e \mapsto e_G \) where \( e_G \) is the identity element in \( G \).

Initial objects in a category are unique up to a unique isomorphism. There are several ways to prove this fact. We can mimic the proof that terminal objects are unique (Proposition 2.14) or we can use the principle of duality and deduce it from Proposition 2.14. We’ll do the latter carefully to illustrate how duality works.

**Proposition 2.18.** Two initial objects \( i_1 \) and \( i_2 \) in a category \( C \) are uniquely isomorphic.

**Proof.** Suppose \( i_1, i_2 \in C_0 \) are initial, then \( i_1, i_2 \) are terminal in \( C^{\text{op}} \). By Proposition 2.14 there exist unique isomorphisms \( i_1 \xrightarrow{f} i_2 \) and \( i_2 \xrightarrow{g} i_1 \) in \( C^{\text{op}} \) such that \( f^{\text{op}} \circ g^{\text{op}} = (\text{id}_{i_2})_{C^{\text{op}}} \) and \( g^{\text{op}} \circ f^{\text{op}} = (\text{id}_{i_1})_{C^{\text{op}}} \). But for every object \( a \) in \( C^{\text{op}} \), the identity morphism \( (\text{id}_a)_{C^{\text{op}}} \) in \( C^{\text{op}} \) is the opposite of the identity morphism \( (\text{id}_a)_C \) in \( C \) by Exercise 2.7. Then

\[
(2.2) \quad (\text{id}_{i_2})_{C^{\text{op}}} = (\text{id}_{i_2})_C^{\text{op}} = f^{\text{op}} \circ g^{\text{op}} = (g \circ f)^{\text{op}}
\]

and

\[
(2.3) \quad (\text{id}_{i_1})_{C^{\text{op}}} = (\text{id}_{i_1})_C^{\text{op}} = g^{\text{op}} \circ f^{\text{op}} = (f \circ g)^{\text{op}}
\]

Therefore, \( (\text{id}_{i_2})_C = g \circ f \) and \( (\text{id}_{i_1})_C = f \circ g \). Hence \( i_2 \xrightarrow{f} i_1 \) and \( i_1 \xrightarrow{g} i_2 \) are the desired unique isomorphisms.

**Remark 2.19.** It follows from the proof of Proposition 2.18 that if \( f \xrightarrow{\sim} \) is an isomorphism in a category \( C \), then \( b \xrightarrow{\text{op}} \) is an isomorphism in the opposite category \( C^{\text{op}} \).
Lecture 3. Functors. The free functor $F : \text{Set} \to \text{Vect}$.

**Last time:**
(1) defined isomorphism in a category;
(2) defined initial and terminal objects;
(3) introduced posets (partially ordered sets);
(4) introduced opposite categories and duality.

Just as social life of objects in a category depends on morphisms, the social life of categories themselves depends on functors.

**Definition 3.1.** A functor $F$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ (we write $F : \mathcal{C} \to \mathcal{D}$) is a pair of functions $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$ and $F_1 : \mathcal{C}_1 \to \mathcal{D}_1$ on objects and morphisms so that

1. for all morphisms $f \in \text{Hom}_\mathcal{C}(a, b)$, $F_1(f) \in \text{Hom}_\mathcal{D}(F_0(a), F_0(b))$
2. for all objects $a \in \mathcal{C}$, $F_1(\text{id}_a) = \text{id}_{F_0(a)}$
   (i.e., $F$ preserves identities)
3. for all composable morphisms $f \in \text{Hom}_\mathcal{C}(a, b)$, $g \in \text{Hom}_\mathcal{C}(b, c)$, $F_1(g \circ f) = F_1(g) \circ F_1(f)$
   (i.e., $F$ preserves composition)

**Remark 3.2.**
1. We will often drop the subscript 0 and 1. For example, we would write $F_0(a) \xrightarrow{F_1(f)} F_0(b)$ as $F(a) \xrightarrow{F(f)} F(b)$.
2. The first condition in Definition 3.1 says that the maps $F_0$ and $F_1$ are compatible with the source and target $s, t : \mathcal{C}_1 \to \mathcal{C}_0$, $s, t : \mathcal{D}_1 \to \mathcal{C}_0$ in the following sense: for any morphism $f$ in $\mathcal{C}$
   $$s(F(f)) = F(s(f)) \quad \text{and} \quad t(F(f)) = F(t(f))$$
   (strictly speaking we should write $s_c$ instead of just $s$ etc, but this clutters the notation).
3. The second condition in Definition 3.1 says that $F_0$ and $F_1$ are compatible with the unit maps $u : \mathcal{C}_0 \to \mathcal{C}_1$ and $u : \mathcal{D}_0 \to \mathcal{D}_1$: for any object $a$ in $\mathcal{C}$
   $$u(F_0(a)) = F_1(u(a)).$$

**Example 3.3.** We have the forgetful functor $U$ from $\text{Group}$ to $\text{Set}$ that forgets the group structure: for every group $G$, $U(G)$ is the set of elements of $G$ (we’ll call it the underlying set of the group $G$). For every homomorphism $\varphi : G \to H$, $U(\varphi)$ is the corresponding function on the underlying sets.

It is not hard to check that $U : \text{Group} \to \text{Set}$ is actually a functor (do it).

**Example 3.4.** Recall that $\text{Rel}$ denotes the category of sets and relations. There is a functor $F : \text{Set} \to \text{Rel}$: $F$ does nothing on objects and sends functions to their graphs: $F(f) = \{(x, y) \in X \times Y \mid y = f(x)\}$.

Again it’s not hard to check that $F$ is a functor. You should make sure to check that $F$ preserves composition.

**Definition 3.5.** The category $\text{Ab}$ of abelian groups is defined as follows: the objects of $\text{Ab}$ are abelian groups. The morphisms in $\text{Ab}$ are homomorphisms of groups.

**Example 3.6.** There is the inclusion functor $i : \text{Ab} \to \text{Group}$ that is the identity on both objects and morphisms. This functor is practically invisible.

There is a forgetful functor $U : \text{Vect}_\mathbb{R} \to \text{Ab}$ that forgets the scalar multiplication and only remembers addition. Note that since every real vector space is an abelian group under addition and since linear maps preserve addition of vectors the definition makes sense.
The functor $U$ allows us to say that a real vector space is an abelian group with extra structure.

**Remark 3.7.** Given two categories $\mathcal{C}$ and $\mathcal{D}$ it may be impossible to extend a given map $F : \mathcal{C}_0 \to \mathcal{D}_0$ on objects to a functor. Here is an example. Let $\mathcal{C} = \text{Group}$, $\mathcal{D} = \text{Ab}$ and $Z : \text{Group}_0 \to \text{Ab}_0$ be defined by $Z(G) = \{z \in G \mid zg = ga \text{ for all } g \in G\}$, the center of $G$. It turns out that the map $Z$ cannot be extended to a functor from $\text{Group}$ to $\text{Ab}$.

**Example 3.8.** There is a functor $F : \text{Mat} \to \text{FDVect}_\mathbb{R}$ from the category of real matrices (Example 1.19) to the category of (real) finite dimensional vector spaces. On objects, $F(n) = \mathbb{R}^n$ for all $n \in \mathbb{N}$. On morphism, $F$ sends an $m \times n$ matrix $A$ to the linear map $F(A) : \mathbb{R}^n \to \mathbb{R}^m$:

$$F(A) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$  

Matrix multiplication is defined in such a way as to make $F$ preserve composition. Notice that $F$ is not surjective on objects (since not every finite dimensional vector space is $\mathbb{R}^n$). However, for all $n, m \in \mathbb{N}$, the map

$$F : \text{Hom}_{\text{Mat}}(n, m) \to \text{Hom}_{\text{FDVect}_\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^m)$$

is a bijection. This is one of the reasons why matrices are so useful in linear algebra.

There is a forgetful functor $U : \text{Vect}_\mathbb{R} \to \text{Set}$ (same name as in Example 3.3 but not the same functor) that forget the vector space structure and remembers only the underlying set. Interestingly enough there is also a functor $F : \text{Set} \to \text{Vect}_\mathbb{R}$ that starts with a set $X$ and assigns to it a vector spaces $F(X)$ that has elements of $X$ as a basis. You have seen $F$ in linear algebra classes for finite sets.

The pair of functors

$$F : \text{Set} \iff \text{Vect}_\mathbb{R} : U$$

will be an important example of a pair of adjoint functors in the second half of the course.

**Theorem 3.9.** There is a functor $F : \text{Set} \to \text{Vect}_\mathbb{R}$ that assigns to each set $X$ a vector space $F(X)$ with the following universal property:

(i) for any set $X$, there is a function $\eta_X : X \to U(F(X))$ where $U(F(X))$ is the set underlying the vector space $F(X)$;

(ii) given any vector space $W$ and any function (morphism in $\text{Set}$) $h : X \to U(W)$, there is a unique linear map $\tilde{h} : F(X) \to W$ so that $U(\tilde{h}) \circ \eta_X = h$.

**Remark 3.10.** Condition (ii) can be restated in terms of diagrams: given a function $h : X \to U(W)$

$$F(X) \xrightarrow{\eta_X} U(F(X)) \xleftarrow{h} U(W)$$

so that

$$W \xrightarrow{\exists \tilde{h}} X \xleftarrow{U(\tilde{h})} U(W)$$

commutes.

**Proof.** We construct the functor $F : \text{Set} \to \text{Vect}_\mathbb{R}$ with the desired properties.

For any set $X$, let $\mathbb{R}^X$ denote the set of functions from a set $X$ to $\mathbb{R}$,

$$\mathbb{R}^X = \{f : X \to \mathbb{R}\}.$$  

The set $\mathbb{R}^X$ is a real vector space under the point-wise addition and scalar multiplication: for any $\lambda \in \mathbb{R}$ and any $f, g \in \mathbb{R}^X$, we define $(\lambda f)(x) := \lambda f(x)$ and $(f + g)(x) := f(x) + g(x)$ for all $x \in X$. 

8
Now define
\[ F(X) = \{ f \in \mathbb{R}^X \mid f(x) = 0 \text{ for all but finitely many } x \in X \}. \]
Note that if \( X \) is finite then \( F(X) = \mathbb{R}^X \) and otherwise \( F(X) \subseteq \mathbb{R}^X \). It’s not hard to check that \( F(X) \) is a vector subspace of \( \mathbb{R}^X \).

For every \( x \in X \), we have a function \( \eta_X^x : X \to \mathbb{R} \) defined by
\[
(3.4) \\
\eta_X^x(y) = \begin{cases} 
1 & \text{if } x = y \\
0 & \text{if } x \neq y
\end{cases}
\]
Clearly, \( \eta_X^x \in F(X) \). We then have a map
\[ \eta_X : X \to F(X), \quad x \mapsto \eta_X^x. \]
Note that \( X \) is a set and \( F(X) \) is a vector space, so what we really have is a function \( \eta_X : X \to U(F(X)) \). This may seem pedantic, but it is useful when thinking about functors and categories.

**Lecture 4. The free functor \( F : \text{Vect}_\mathbb{R} \to \text{Set} \). Contravariant functors. Locally small categories.**

**Last time:** Defined functors and looked at some examples. There are forgetful functors \( U : \text{Group} \to \text{Set} \) (which forgets the group structure), \( U : \text{Vect}_\mathbb{R} \to \text{Ab} \) (which forgets scalar multiplication). There is the inclusion functor \( i : \text{Ab} \to \text{Group} \) (any abelian group is a group and a homomorphism of abelian groups is a homomorphism of groups).

**Remark 4.1.** Functors can be composed to obtain a new functor. Suppose \( F : C \to D \) and \( G : D \to E \) are two functors. Their composite \( G \circ F : C \to E \) is defined as follows:

- **On objects:** \( (G \circ F)(a) = G(F(a)) \) for any \( a \in C \)
- **On morphisms:** \( (G \circ F)(a \xrightarrow{f} b) := G(F(a)) \xrightarrow{G(f)} G(F(b)) \).

**Exercise 4.2.** Check that \( G \circ F \) so defined is a functor. Hint: what do you actually need to check?

**Example 4.3.** Composing
\[ \text{Vect}_\mathbb{R} \xrightarrow{U} \text{Ab} \xrightarrow{i} \text{Group} \xrightarrow{U} \text{Set} \]
we get a forgetful functor
\[ \text{Vect}_\mathbb{R} \to \text{Set} \]
that assigns to a vector space the underlying set. I will call it \( U \) again. (So now we have three functors called \( U \) for “underlying”. I hope this is not too confusing.)

Last time we also started constructing a functor \( F : \text{Set} \to \text{Vect} \equiv \text{Vect}_\mathbb{R} \) with the following universal property.

(i) for any set \( X \), there is a function \( \eta_X : X \to U(F(X)) \) where \( U(F(X)) \) is the set underlying the vector space \( F(X) \);
(ii) given any vector space \( W \) and any function \( h : X \to U(W) \), there is a unique linear map \( \tilde{h} : F(X) \to W \) so that \( U(\tilde{h}) \circ \eta_X = h \).

On objects we defined \( F \) by
\[ F(X) = \{ f : X \to \mathbb{R} \mid f(x) = 0 \text{ for all but finitely many } x \in X \}. \]
For every set \( X \) we also defined a function
\[ \eta_X : X \to U(F(X)), \quad \eta_X(x) := \eta_X^x \]
where
\[ \eta^x_X(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases} \]

We now argue that the function \( \eta_X : X \to U(F(X)) \) has the desired universal property.

**Claim** The set \( \{ \eta^x_X \}_{x \in X} \) is linearly independent and spans \( F(X) \) hence forms a basis of the vector space \( F(X) \).

**Proof of claim.** Suppose that \( \{ \eta^x_X \}_{x \in X} \) is linearly dependent. This means that there is \( n > 0 \), \( x_1, \ldots, x_n \in X \) and \( c_1, \ldots, c_n \in \mathbb{R} \) not all 0 so that
\[ c_1 \eta^x_{x_1} + \cdots + c_n \eta^x_{x_n} = 0. \]
But then for any \( j, 1 \leq j \leq n \)
\[ \sum_{i=1}^{n} c_i \eta^x_{x_i}(x_j) = c_j = 0 \]
Contradiction. Therefore \( \{ \eta^x_X \}_{x \in X} \) is linearly independent.

Given \( f \in F(X) \) there is \( n \geq 0 \) and \( x_1, \ldots, x_n \in X \) so that \( f(x_i) \neq 0 \) and \( f(y) = 0 \) for \( y \notin \{ x_1, \ldots, x_n \} \). Then
\[ f(z) = \sum_{i=1}^{n} f(x_i) \eta^x_{x_i}(z) \]
for all \( z \in X \). Hence
\[ f = \sum_{i=1}^{n} f(x_i) \eta^x_{x_i} \]
and consequently
\[ f = \sum_{x \in X} f(x) \eta^x_X. \]
Thus the set \( \{ \eta^x_X \}_{x \in X} \) spans \( F(X) \). \( \square \)

Now suppose we have a vector space \( W \) and a function \( h : X \to U(W) \). Since \( \{ \eta^x_X \}_{x \in X} \) is a basis of the vector space \( F(X) \) there is a unique linear map \( \tilde{h} : F(X) \to W \) with \( \tilde{h}(\eta_X(x)) = h(x) \).

Explicitly, for all \( f \in F(X) \)
\[ \tilde{h}(f) := \tilde{h} \left( \sum_{x \in X} f(x) \eta_X(x) \right) = \sum_{x \in X} f(x)h(x) \]
Note that \( f(x) \) is a real number and \( h(x) \) is a vector in \( W \), so \( f(x)h(x) \) makes sense. Moreover the sum \( \sum_{x \in X} f(x)h(x) \) is actually finite since since \( f(x) \) is zero for all but finitely many \( x \in X \).

We now construct the desired functor \( F : \text{Set} \to \text{Vect} \). Given a function \( \varphi : X \to Y \) between two sets we need to construct a linear map \( F(\varphi) : F(X) \to F(Y) \). Consider the function
\[ h : X \to U(F(Y)), \quad h := \eta_Y \circ \varphi \]
The function \( h \) is defined to make the diagram
\[ \begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \downarrow{h} & & \downarrow{\eta_Y} \\ Y & \xrightarrow{\eta_Y} & U(F(Y)) \end{array} \]
commute. By the universal property of the function \( X \xrightarrow{\eta_X} U(F(X)) \) there is a unique linear map
\[ \tilde{h} : F(X) \to F(Y) \]
so that
\[ U(\tilde{h}) \circ \eta_X = h, \]
i.e., the diagram
\[
\begin{array}{c}
X \quad \xrightarrow{\eta_X} \quad U(F(X)) \\
\uparrow \quad h \\
Y \quad \xrightarrow{\eta_Y} \quad U(F(Y)) \\
\varphi \\
\end{array}
\]
commutes. We define \( F(\varphi) \) to be this \( \tilde{h} \):
\begin{equation}
F(\varphi) := \tilde{h}. 
\end{equation}

**Exercise 4.4.** Show that \( F : \text{Set} \to \text{Vect} \), \( F(X \xrightarrow{\varphi} Y) = F(X) \xrightarrow{\tilde{h}} F(Y) \), where \( \tilde{h} \) is the linear map constructed above, is in fact a functor.

Hints: Use the universal properties of the collection of functions \( \{ \eta_X : X \to U(F(X)) \}_{X \in \text{Set}} \).

If you have trouble getting the universal properties approach to work show that \( F(\varphi) \) is given by
\[ F(\varphi)f = \sum_{x \in X} f(x)\eta_Y^{\varphi(x)} \]
for all \( f \in F(X) \subset \mathbb{R}^X \).

---

**Remark 4.5.** The functors that we have considered so far are called **covariant functors** in older literature.

**Definition 4.6.** A **contravariant functor** \( F \) from a category \( \mathcal{C} \) to a category \( \mathcal{D} \) is a pair of functions \( F_0 : \mathcal{C}_0 \to \mathcal{D}_0 \) and \( F_1 : \mathcal{C}_1 \to \mathcal{D}_1 \) on objects and morphisms so that
- for all morphisms \( f \in \text{Hom}_\mathcal{C}(a, b) \),
  \[ F_0(b) \xrightarrow{F_1(f)} F_0(a) \]
  [Note the direction of the morphism \( F_1(f) \)!]
- for all objects \( a \in \mathcal{C} \), \( F_1(\text{id}_a) = \text{id}_{F_0(a)} \)
- for all pairs of composable morphisms \( a \xrightarrow{f} b \xrightarrow{g} c \)
  \[ F_1(g \circ f) = F_1(f) \circ F_1(g) : \]

**Example 4.7.** Consider the category \( \text{Vect}_\mathbb{R} \). For any vector \( V \) space over \( \mathbb{R} \), we have the dual vector space
\[ V^* := \{ \ell : V \to \mathbb{R} \mid \ell \text{ is linear} \} . \]
For any linear map \( T : V \to W \), we have the pullback (dual map)
\[ T^* : W^* \to V^*, \quad T^*(\ell) = \ell \circ T \]
for all $\ell : W \to \mathbb{R}$. Note that $\id^*_V(\ell) = \ell \circ \id_V = \ell$. Hence

$$(\id_V)^* = \id_V^*$$

for all vector spaces $V$.

Given a pair of composable linear maps $V \xrightarrow{T} W \xrightarrow{S} U$ and $\ell \in U^*$

$$(S \circ T)^*(\ell) = \ell \circ S \circ T = T^*(\ell \circ S) = T^*(S^*(\ell))$$

It follows that

$$\ast : \text{Vect} \to \text{Vect}, \quad (V \xrightarrow{T} W) \mapsto W^* \xrightarrow{T^*} V^*$$

is a contravariant functor.

Remark 4.8. Any contravariant functor $F$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a (covariant) functor $F : \mathcal{C} \to \mathcal{D}^{\text{op}}$ and a (covariant) functor $F : \mathcal{C}^{\text{op}} \to \mathcal{D}$. For this reason in contemporary literature one simply writes

$$F : \mathcal{C}^{\text{op}} \to \mathcal{D}$$

when $F$ is a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$.

We end the lecture with a definition.

Definition 4.9. A category $\mathcal{C}$ is locally small if for any $a, b \in \mathcal{C}$, $\text{Hom}_{\mathcal{C}}(a, b)$ is a set.

Example 4.10. The categories $\text{Set}$, $\text{Group}$, $\text{Vect}_\mathbb{R}$ and $\text{Rel}$ are all locally small but not small. Any small category is, of course, locally small.

Lecture 5. Hom functors. Binary products. CAT.

Last time:

- Constructed a functor $F : \text{Set} \to \text{Vect}$ by constructing for every set $X$ a vector space $F(X)$ and an injective map $\eta_X : X \to U(F(X))$ so that $\eta_X(X)$ is a basis of $F(X)$.
- Defined contravariant functors from a category $\mathcal{C}$ to a category $\mathcal{D}$. They are ordinary (covariant) functors from $\mathcal{C}^{\text{op}}$ to $\mathcal{D}$.
- The dual vector space functor $\ast : \text{Vect}^{\text{op}} \to \text{Vect}$, $V \mapsto V^* = \text{Hom}_{\text{Vect}}(V, \mathbb{R})$ is an example of a contravariant functor.
- Defined locally small categories: these are categories $\mathcal{C}$ so that $\text{Hom}_{\mathcal{C}}(a, b)$ is a set for any pair of objects $a, b$.

The functor $\text{Hom}_{\mathcal{C}}(-, c) : \mathcal{C}^{\text{op}} \to \text{Set}$

Fix a locally small category $\mathcal{C}$ and an object $c \in \mathcal{C}$. Define the functor

$$\text{Hom}_{\mathcal{C}}(-, c) : \mathcal{C}^{\text{op}} \to \text{Set}$$

as follows: for any object $a \in \mathcal{C}$ set

$$(\text{Hom}_{\mathcal{C}}(-, c))(a) = \text{Hom}_{\mathcal{C}}(a, c).$$

For a morphism $a \xrightarrow{f} b$ in $\mathcal{C}$, define

$$f^* \equiv \text{Hom}_{\mathcal{C}}(-, c)(f) : \text{Hom}_{\mathcal{C}}(b, c) \to \text{Hom}_{\mathcal{C}}(a, c)$$

by

$$f^*(b \xrightarrow{g} c) := a \xrightarrow{\gamma \circ f} c.$$

Then for any pair of composable morphisms $a \xrightarrow{f} b \xrightarrow{g} d$ in $\mathcal{C}$ and for any $d \xrightarrow{\tau} e \in \text{Hom}_{\mathcal{C}}(d, c)$

$$(g \circ f)^*(\tau) = \tau \circ (g \circ f) = (\tau \circ g) \circ f = f^*(g^* \tau).$$
Note also that for any object $a$ of $C$, $\text{id}_a^* = \text{id}_{\text{Hom}_C(a,a)}$ since for any $\tau: a \to a$, $\text{id}_a^* \tau = \tau \circ \text{id}_a = \tau$. Consequently $\text{Hom}_C(-, c)$ is a contravariant functor.

**Remark 5.1.** If $C = \text{Vect}_R$ and $c = R$ then $\text{Hom}_{\text{Vect}_R}(-, R)$ is (almost!) the dual vector space functor $^*: \text{Vect}_R^{\text{op}} \to \text{Vect}_R$.

It just happens to be that for any pair of vector spaces $V$ and $W$, the set $\text{Hom}_{\text{Vect}_R}(V, W)$ is not just a set but also a vector space over $R$.

**The functor** $\text{Hom}_C(c, -): C \to \text{Set}$

Fix a locally small category $C$ and an object $c \in C$. Define the functor $\text{Hom}_C(c, -): C^{\text{op}} \to \text{Set}$ by

$$\text{Hom}_C(c, -)(a \xrightarrow{f} b) := \left( \text{Hom}_C(c, a) \xrightarrow{f_*} \text{Hom}_C(c, b) \right)$$

where

$$f_*(c \xrightarrow{\tau} a) = c \xrightarrow{f \circ \tau} b.$$ 

**Exercise 5.2.** Check that $\text{Hom}_C(c, -)$ is a functor.

**Products**

Recall that given two sets $X$ and $Y$ their **Cartesian product** is the set of ordered pairs

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}.$$ 

(Note that the product is empty if one of $X$ and $Y$ is empty.) The Cartesian product comes with two canonical functions

$$p_X : X \times Y \to X, \quad (x, y) \mapsto x$$

$$p_Y : X \times Y \to Y, \quad (x, y) \mapsto y.$$ 

The triple $(X \times Y, p_X, p_Y)$ possesses the following universal property:

For any set $Z$ and for any pair of functions $f_X: Z \to X$ and $f_Y: Z \to Y$, there exists a unique function $f: Z \to X \times Y$ such that $f_X = p_X \circ f$ and $f_Y = p_Y \circ f$. It is defined by

$$f(z) := (f_X(z), f_Y(z)).$$ 

for all $z \in Z$. One also writes $(f_X, f_Y)$ for the function $f$.

Diagrammatically we have:

\[
\begin{array}{c}
Z \\
\downarrow f_X \quad \exists f \\
X \xleftarrow{p_X} X \times Y \xrightarrow{p_Y} Y \\
\downarrow f_Y \\
\end{array}
\]

Similarly given two groups $G$ and $H$ their **product** is the group with the underlying set $G \times H$ and multiplication defined coordinate-wise:

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, h_1 h_2)$$

for all $(g_1, h_1), (g_2, h_2) \in G \times H$. 

13
Note that the projections
\[ \pi_G : G \times H \to G, \quad (g, h) \mapsto g \]
\[ \pi_H : G \times H \to H, \quad (g, h) \mapsto h \]
are group homomorphisms, i.e., morphisms in the category Group. The triple \((G \times H, \pi_G, \pi_H)\) has the same universal property as the Cartesian product of two sets: for any group \(K\) and any pair of homomorphisms of groups \(\varphi_G : K \to G, \varphi_H : K \to H\), there is a unique homomorphism \(\varphi : K \to G \times H\) such that \(\varphi_G = \pi_G \circ \varphi\) and \(\varphi_H = \pi_H \circ \varphi\). Namely we define \(\varphi\) by
\[ \varphi(k) = (\varphi_G(k), \varphi_H(k)) \]
for all \(k \in K\).

We now generalize:

**Definition 5.3.** Let \(\mathcal{C}\) be a category. A (binary categorical) product of two objects \(a, b \in \mathcal{C}\) is a triple \((c, c \xrightarrow{p_a} a, c \xrightarrow{p_b} b)\) (where \(c\) is an object in \(\mathcal{C}\) and \(p_a, p_b\) are morphisms in \(\mathcal{C}\)) with the following universal property: for any object \(d \in \mathcal{C}\) and any pair of morphisms \(d \xrightarrow{f_a} a, d \xrightarrow{f_b} b\), there exists a unique morphism \(d \xrightarrow{f} c\) such that \(f_a = p_a \circ f\) and \(f_b = p_b \circ f\). In other words the following diagram commutes

\[ \begin{array}{ccc}
    d & \xrightarrow{f} & c \\
    f_a \downarrow & & \downarrow p_a \\
    a & & b \\
    f_b \downarrow & & \downarrow p_b \\
    b & & \end{array} \]

**Example 5.4.** In the category \(\text{Set}\), the product is just the Cartesian product.

**Example 5.5.** In the category \(\text{Group}\), the product is just the product of groups.

**Example 5.6.** Let \((X, \leq)\) be a poset (considered as a category), \(a, b \in X\). Recall that in this category there is a morphism \(f_{cd} : c \to d\) if and only if \(c \leq d\).

The product of \(a\) and \(b\), if it exists, is \(c \in X\) such that
1. \(c \leq a\) and \(c \leq b\);
2. if \(d \leq a\) and \(d \leq b\), then \(d \leq c\).

Thus the product of \(a\) and \(b\) is the greatest lower bound of \(\{a, b\}\), which may or may not exist.

For example suppose \((X, \leq)\) be a poset with four elements \(a, b, c\) and \(d\) so that \(a, b \leq c\) and \(a, b \leq d\). Then the product of \(a\) and \(b\) doesn’t exist. (Why not?)

Products in a category are not unique on the nose, but just like terminal objects they are unique up to a unique isomorphism.

**Lemma 5.7.** Let \(\mathcal{C}\) be a category, \(a, b \in \mathcal{C}\). Any two products \((c, c \xrightarrow{p_a} a, c \xrightarrow{p_b} b)\) and \((d, d \xrightarrow{q_a} a, d \xrightarrow{q_b} b)\) of \(a, b\) are uniquely isomorphic.

**Proof.** Since \(a \xrightarrow{p_a} d \xrightarrow{q_b} b\) is a product of \(a\) and \(b\), and \(c \xrightarrow{p_a} a, c \xrightarrow{p_b} b\) are morphisms in \(\mathcal{C}\), there exists a unique morphism
\[ \varphi : c \to d \quad \text{so that} \quad p_a = q_a \circ \varphi \text{ and } p_b = q_b \circ \varphi. \]

Similarly, there exists a unique morphism
\[ \psi : d \to c \quad \text{so that} \quad q_a = p_a \circ \psi \text{ and } q_b = p_b \circ \psi. \]
Now consider the composites $\psi \circ \varphi : c \to c \quad \varphi \circ \psi : d \to d$.

Since the diagrams

\[
\begin{array}{ccc}
 c & \xrightarrow{\varphi} & d \\
\downarrow{p_a} & & \downarrow{q_b} \\
 a & \xrightarrow{p_i} & b \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
 c & \xrightarrow{\psi \circ \varphi} & c \\
\downarrow{p_b} & & \downarrow{p_b} \\
 d & \xrightarrow{\varphi} & c \\
\end{array}
\]

commute, the diagram

\[
\begin{array}{ccc}
 c & \xrightarrow{\psi \circ \varphi} & c \\
\downarrow{i} & & \downarrow{i} \\
 d & \xrightarrow{\varphi} & c \\
\end{array}
\]

commutes for $i = a, b$. But

\[
\begin{array}{ccc}
 c & \xrightarrow{id_c} & c \\
\downarrow{i} & & \downarrow{i} \\
 d & \xrightarrow{\varphi} & c \\
\end{array}
\]

also commutes for $i = a, b$. By the universal property of the product $a \xleftarrow{p_a} c \xrightarrow{p_b} b$, we must have $\psi \circ \varphi = id_c$.

Similarly $\varphi \circ \psi = id_d$. \qed

We next turn the collection of all locally small categories into a category. This will allow us to define the product of two locally small categories.

**Definition 5.8.** The category $\text{CAT}$ of all locally small categories is defined as follows.

- The objects in $\text{CAT}$ are locally small categories.
- The morphisms in $\text{CAT}$ are functors

The composition of morphisms in $\text{CAT}$ is the composition of functors. For any locally small category $\mathcal{C}$, there is the identity functor $id_C$ defined to be the identity map on objects and morphisms:

$$id_C(a \twoheadrightarrow b) := a \twoheadrightarrow b$$

for all $a \twoheadrightarrow b \in \mathcal{C}$.

The category $\text{CAT}$ has products. Here is a construction: let $\mathcal{A}, \mathcal{B}$ be two categories. We define the collection of objects of $\mathcal{A} \times \mathcal{B}$ to be the collection

$$(\mathcal{A} \times \mathcal{B})_0 = \{(a, b) \mid a \in \mathcal{A}, b \in \mathcal{B}\}.$$

A morphism in $\mathcal{A} \times \mathcal{B}$ from $(a, b)$ to $(a', b')$ is a pair of morphisms $(a \xrightarrow{f} a', b \xrightarrow{g} b') \in A_1 \times B_1$.

The composition is defined coordinate-wise:

$$\left((a', b') \xrightarrow{(f', g')} (a'', b'')\right) \circ \left((a, b) \xrightarrow{(f, g)} (a', b')\right) := (a, b) \xrightarrow{(f' \circ f', g' \circ g)} (a'', b'')$$

and

$$id_{(a, b)} = (id_a, id_b).$$

The projections $P_A : \mathcal{A} \times \mathcal{B} \to \mathcal{A}$, $P_B : \mathcal{A} \times \mathcal{B} \to \mathcal{B}$ are defined by

$$P_A \left((a, b) \xrightarrow{(f, g)} (a', b')\right) = a \xrightarrow{f} a'$$

$$P_B \left((a, b) \xrightarrow{(f, g)} (a', b')\right) = b \xrightarrow{g} b'$$

It is not hard to check that $P_A$ and $P_B$ are functors and that the triple $(\mathcal{A} \times \mathcal{B}, P_A, P_B)$ satisfies the universal property of products.
Example 5.9. Let $\mathcal{A}$ be the category with two objects and exactly one non-identity morphism. That is,

$$\mathcal{A} = \xymatrix{ a & b \ar[l]_f \ar[r]^{-id_a, id_b} \ar[dl]^{id_a} & \ar[d]_{id_b} a \ar[r]^{f} & b \ar[l]_{id_b} }.$$  

Then

$$(id_a, id_b) \xymatrix{ (a, b) \ar[r]^{(f, id_b)} & (b, b) \ar[l]_{(id_b, id_a)} } =$$

$$(id_a, f) \xymatrix{ (a, a) \ar[r]^{f, id_a} & (b, a) \ar[l]_{(id_b, f)} } =$$

$$(id_b, f) \xymatrix{ (a, b) \ar[r]^{id_a} & (b, b) \ar[l]_{id_b} } =$$

$$(id_b, id_b) \xymatrix{ (a, b) \ar[r]^{(f, id_a)} & (b, a) \ar[l]_{(id_b, id_b)} }.$$


Last time:

- Defined locally small categories. Locally small categories and functors form a category $\text{CAT}$.
- If $\mathcal{C}$ is locally small then for every object $c \in \mathcal{C}$ we have functors

$$\text{Hom}_\mathcal{C}(-, c) : \mathcal{C}^{\text{op}} \to \text{Set}, \quad (a \xrightarrow{f} b) \mapsto f^* : \text{Hom}_\mathcal{C}(b, c) \to \text{Hom}_\mathcal{C}(a, c)$$

$$f^*(b \xrightarrow{\gamma} c) := a \xrightarrow{\gamma \circ f} c,$$

$$\text{Hom}_\mathcal{C}(c, -) : \mathcal{C}^{\text{op}} \to \text{Set}, \quad (a \xrightarrow{f} b) \mapsto f_* : \text{Hom}_\mathcal{C}(c, a) \to \text{Hom}_\mathcal{C}(c, b)$$

$$f_*(c \xrightarrow{\gamma} a) := a \xrightarrow{f \circ \gamma} c.$$

- Defined (categorical) products.
- Observed that the category $\text{CAT}$ has products.

Recall the definition of a product of two objects $a, b$ in a category $\mathcal{C}$:

A product of $a$ and $b$ is a triple $(c, c \xrightarrow{p_a} a, c \xrightarrow{p_b} b)$ (where $c$ is an object in $\mathcal{C}$ and $p_a, p_b$ are morphisms in $\mathcal{C}$) with the following universal property: for any object $d \in \mathcal{C}$ and any pair of morphisms $d \xrightarrow{f_a} a$, $d \xrightarrow{f_b} b$, there exists a unique morphism $d \xrightarrow{f} c$ such that the following diagram commutes

$$
\begin{array}{c}
\begin{tikzcd}
& d \\
\downarrow{f_a} & \downarrow{f} & \downarrow{f_b} \\
\vdots & \downarrow{f} & \downarrow{f_b} \\
a & c & b
\end{tikzcd}
\end{array}
$$

We proved that for any two products $(c, c \xrightarrow{p_a} a, c \xrightarrow{p_b} b)$, $(d, q_a : d \rightarrow a, q_b : d \rightarrow b)$ of $a$ and $b$ there is a unique isomorphism $\varphi : c \rightarrow d$ so that the diagrams

$$
\begin{array}{c}
\begin{tikzcd}
& c \\
\downarrow{p_0} & \downarrow{q_a} & \downarrow{q_b} \\
a & c & b
\end{tikzcd}
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\begin{tikzcd}
& c \\
\downarrow{p_0} & \downarrow{q_b} \\
a & b
\end{tikzcd}
\end{array}
$$

commute. For this reason we may talk about “the” product of $a$ and $b$ and write $(a \times b, p_a : a \times b \rightarrow a, p_b : a \times b \rightarrow b)$.

In particular, given two locally small categories $\mathcal{A}, \mathcal{B}$ we denote their product by $(\mathcal{A} \times \mathcal{B}, p_A : \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{A}, p_B : \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{B})$.  

16
**Remark 6.1.** If $\mathcal{A}$ and $\mathcal{B}$ are not necessarily locally small categories, their product $\mathcal{A} \times \mathcal{B}$ still makes sense and is defined exactly the same way. The only difference is that $\mathcal{A}$ and $\mathcal{B}$ need not be objects of CAT.

**Remark 6.2.** Suppose $\mathcal{C}$ is a locally small category. Then the two types of Hom functors can be put together into one functor

$$\text{Hom}_\mathcal{C}(-, \cdot) : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \text{Set}$$

as follows: a morphism $(a, b) \xrightarrow{(f, g)} (a', b') \in \mathcal{C}^{\text{op}} \times \mathcal{C}$ a a pair of morphisms $\xrightarrow{f} \, a \xrightarrow{g} \, b$ in $\mathcal{C}$. We define

$$\text{Hom}_\mathcal{C} \left( (a, b) \xrightarrow{(f, g)} (a', b') \right) : \text{Hom}_\mathcal{C}(a, b) \to \text{Hom}_\mathcal{C}(a', b')$$

by

$$\text{Hom}_\mathcal{C}(f, g)(b \xleftarrow{\gamma} a) := b' \xleftarrow{\gamma'} b \xleftarrow{f} a \equiv g \circ \gamma \circ f \equiv (g_* \circ f^*)(\gamma).$$

That is,

$$\text{Hom}_\mathcal{C}(f, g) = g_* \circ f^*.$$

**Remark 6.3.** Let $\mathcal{C}$ be a category. A product $(a \times b, p_a : a \times b \to a, p_b : a \times b \to b)$ in $\mathcal{C}$ is a terminal object in another category $\mathcal{D}$ which we now construct.

The objects of $\mathcal{D}$ are triples $(d, d' \xrightarrow{q_a} a, d' \xrightarrow{q_b} b)$, where $a, b, d$ are objects in $\mathcal{C}$ and $q_a, q_b$ are morphisms in $\mathcal{C}$:

$$\mathcal{D}_0 := \left\{ \begin{array}{c} \begin{array}{ccc} q_a & a & \Rightarrow & d' \xrightarrow{q_b} b \\ - & - & - & - \\ d & \in & \mathcal{C}_0 \end{array} \end{array} \right\}$$

Morphisms in $\mathcal{D}$ are defined by

$$\text{Hom}_\mathcal{D}((d, q_a, q_b), (d', q_{a}', q_{b}')) = \left\{ d \xrightarrow{\varphi} d' \text{ in } \mathcal{C} \mid \begin{array}{c} d \xrightarrow{\varphi} d' \text{ and } d \xrightarrow{\varphi} d' \text{ commute} \\ q_a \xleftarrow{\phi} d \xrightarrow{\varphi} d' \text{ and } q_b \xleftarrow{\phi} d \xrightarrow{\varphi} d' \text{ commute} \end{array} \right\}.$$

Note that the identity morphism in $\mathcal{C}$ on a triple $(d, q_a, q_b)$ is just $\text{id}_d$ in $\mathcal{C}$.

Given morphisms $(d, q_a, q_b) \xrightarrow{\varphi} (d', q_{a}', q_{b}') \xrightarrow{\psi} (d'', q_{a}'', q_{b}'')$ in $\mathcal{D}$, the composite $\psi \circ \varphi$ in $\mathcal{C}$ is a morphism in $\mathcal{D}$ since the diagram

$$\begin{array}{c} d \xrightarrow{\varphi} d' \xrightarrow{\psi} d'' \\ q_i \downarrow \quad \downarrow q_i' \\ i \end{array}$$

commutes for $i = a, b$. We define the composition in $\mathcal{D}$ by

$$\left((d'', q_{a}'', q_{b}'') \xrightarrow{\psi} (d', q_{a}', q_{b}')) \circ (d', q_{a}', q_{b}') \xrightarrow{\varphi} (d, q_a, q_b)\right) := \left((d'', q_{a}'', q_{b}'') \xrightarrow{\psi \circ \varphi} (d, q_a, q_b)\right).$$

It is not hard to check that $\mathcal{D}$ is a category.

Moreover an object $(c, p_a, p_b)$ is terminal in $\mathcal{D} \iff$ for all objects $(d, q_a, q_b)$ in $\mathcal{D}$ there exists a unique morphism $(d, q_a, q_b) \xrightarrow{\varphi} (c, p_a, p_b)$ in $\mathcal{D} \iff$
for all objects \((d, q_a, q_b)\) in \(D\) there exists a unique morphism \(d \xrightarrow{\varphi} c\) in \(C\) such that
\[
\begin{array}{c}
\xymatrix{d & c \\
q_a \ar[ur]^i \ar@{.>}[dr]_{p_a} & }
\end{array}
\]
commutes for \(i = a, b \iff (c, p_a, p_b)\) is a product of \(a\) and \(b\) in \(C\).

Since terminal objects are unique up to a unique isomorphism, products are unique up to a unique isomorphism. We obtain an alternative proof of Lemma 5.7.

**Coproducts**

By the principle of duality, dual to the notion of a product we have the dual notion of a coproduct.

**Definition 6.4.** Let \(C\) be a category. A **coproduct** of two objects \(a, b \in C\) (if it exists) is their product in the opposite category \(C^{op}\) (recall that the categories \(C\) and \(C^{op}\) have the same objects).

Explicitly a coproduct of \(a\) and \(b\) is a triple \((c, i_a : a \to c, i_b : b \to c)\) so that for any object \(e \in C\) and any pair of morphisms \(f_a : e \to a, f_b : e \to b\) there is a unique morphism \(f : e \to c\) so that the diagram
\[
\begin{array}{c}
\xymatrix{ & c \\
 & e \\
a \ar[ur]^{i_a} \ar[dr]_{f_a} & \\
b \ar[ur]^{i_b} \ar[dr]_{f_b} & }
\end{array}
\]
commutes.

**Example 6.5.** Given two sets \(X, Y\), their coproduct exists in \(\text{Set}\), it is the disjoint union \(X \sqcup Y\) together with the two inclusions \(i_X, i_Y\).

There are various way to construct/define the disjoint union. For example we may define
\[
X \sqcup Y := (X \times \{0\}) \cup (Y \times \{1\}),
\]
and
\[
i_X : X \to X \sqcup Y, \quad i_X(x) := (x, 0) \quad \text{for all } x \in X
\]
\[
i_Y : Y \to X \sqcup Y \quad i_Y(y) := (y, 1) \quad \text{for all } y \in Y.
\]
The universal property is easy to check: given functions \(f_X : X \to Z\) and \(f_Y : Y \to Z\), we define \(f : X \sqcup Y \to Z\) by
\[
f(w, i) = \begin{cases} f_X(w) & \text{if } i = 0 \\ f_Y(w) & \text{if } i = 1 \end{cases}.
\]

**Remark 6.6.** A coproduct \((c, i_a, i_b)\) of \(a, b \in C\) is initial in the category with objects \((d, a \xrightarrow{j_a} d, b \xrightarrow{j_b} d)\) (and appropriately defined morphisms). Since initial objects are unique up to a unique isomorphism coproducts are unique up to a unique isomorphism.

Alternatively, since a coproduct in \(C\) is a product in the opposite category \(C^{op}\), it’s unique up to a unique isomorphism in \(C^{op}\) (which is also an isomorphism in \(C\)).

**Example 6.7.** Given two vector space over the reals \(V, W\), their coproduct exists in \(\text{Vect}_\mathbb{R}\), it is the direct sum \(V \oplus W = \{(v, w) = v \oplus w \mid v \in V, w \in W\}\) with the structure maps \(i_V : V \to V \oplus W, v \mapsto (v, 0)\) and \(i_W : W \to V \oplus W, w \mapsto (0, w)\).

The universal property is easy to check: given two linear maps \(T_V : V \to U\) and \(T_W : W \to U\), there exists a unique linear map \(T = T_V \oplus T_W : V \oplus W \to U, (v, w) \mapsto T_V(v) + T_W(w)\).
**Example 6.8.** Coproducts exist in the category \textbf{Group}. They are known as free products. However, given two group \(G, H\), their coproduct is \textit{not} \(G \times H\) together with the two inclusions
\[
\begin{align*}
i_G : G &\to G \times H, \quad i_G(g) = (g, e_H) \\
i_H : H &\to G \times H, \quad i_H(h) = (e_G, h).
\end{align*}
\]

\textit{Reason.} Suppose \((G \times H, i_G, i_H)\) were a coproduct. Then for any group \(K\) and any pair of homomorphisms \(\varphi_G : G \to K, \varphi_H : H \to K\), we have a unique homomorphism \(\varphi : G \times H \to K\) such that \(\varphi(g, e_H) = \varphi_G(g)\) and \(\varphi(e_G, h) = \varphi_H(h)\) for any \(g \in G\) and \(h \in H\). Therefore we must have
\[
\varphi_G(g) \varphi_H(h) = \varphi((g, e_H)(e_G, h)) = \varphi(g, h) = \varphi(e_G, h) \varphi(g, e_H) = \varphi_H(h) \varphi_G(g)
\]
for any \(g \in G\) and \(h \in H\). But there is no reason for \(\varphi_G(g)\) and \(\varphi_H(h)\) to commute in \(K\). \(\square\)

\textbf{Lecture 7. Monos, epis, fully faithful functors.}

\textbf{Last time:}
\begin{itemize}
  \item Products in a category \(C\) are terminal objects in another (appropriate) category.
  \item Coproducts in \(C\) are products in \(C^{\text{op}}\). Coproducts are initial in an appropriate category.
\end{itemize}

We now define the category-theoretic analogue of injective and surjective functions and try to get a feel for these concepts. Recall that a function \(f : X \to Y\) between two sets is \textit{injective} if for all \(x_1, x_2 \in X\), \(f(x_1) = f(x_2) \implies x_1 = x_2\). Consequently, for any pair of functions \(g : W \to X\), \(h : W \to X\),
\[
f \circ g = f \circ h \implies g = h.
\]
This is because for any \(w \in W\), \(f(g(w)) = f(h(w))\) implies \(g(w) = h(w)\).

Similarly a function \(f : X \to Y\) between two sets is \textit{surjective} if for all \(y \in Y\), there exists \(x \in X\) such that \(f(x) = y\). Consequently, for any pair of functions \(k : Y \to Z\), \(\ell : Y \to Z\),
\[
k \circ f = \ell \circ f \implies k = \ell.
\]
This is because for any \(y \in Y\), there exists \(x \in X\) such that \(f(x) = y\). And then then \(k(y) = k(f(x)) = \ell(f(x)) = \ell(y)\) for all \(y \in Y\).

Now we generalize.

\textbf{Definition 7.1.} A morphism \(b \xrightarrow{f} c\) in a category \(C\) \textbf{is monic} (or is a \textit{monomorphism} or is a \textit{mono}) if for any \(a \in C\), and for any pair of morphisms \(a \xrightarrow{g} b\), \(a \xrightarrow{h} b\) in \(C\),
\[
f \circ g = f \circ h \implies g = h.
\]

A morphism \(b \xrightarrow{f} c\) in a category \(C\) \textbf{is epic} (or is an \textit{epimorphism} or is an \textit{epi}) if for any \(d \in C\), and for any pair of morphisms \(k, \ell : c \to d\),
\[
k \circ f = \ell \circ f \implies k = \ell.
\]

\textit{Remark 7.2.} “Epi” and “mono” are dual notions in the following sense: a morphism \(f : x \to y\) in a category \(C\) is epic \iff \(f^{\text{op}} : y \to x\) in \(C^{\text{op}}\) is monic.
Example 7.3. In Set, monos are injective functions and epis are surjective functions.

Example 7.4. In any preorder, any morphism is both a mono and an epi.

Proof. Let \( f : a \to b \) be a morphism in a preorder \( C \). Then for any pair of morphism \( g, h : c \to a \)
\[
f \circ g = f \circ h \implies g = h.
\]
because \( g \) has to equal \( h \) to begin with: in a preorder \( \text{Hom}_C(c, a) \) has at most one morphism. Hence \( f \) is monic.

A similar argument proves that \( f \) is epic. \( \square \)

Example 7.5. In the category \( \text{Group} \), any injective group homomorphism is a mono. Conversely, if \( \varphi : G \to H \) is monic in \( \text{Group} \), it is injective. Why? [This is not completely trivial.]

Warning 7.6. There are categories that are “sets with structure and structure-preserving maps” where monos need not be injective and epis need not be surjective. See Proposition 7.11 and Proposition 7.15 below.

Also, while in any category an isomorphism in a category is monic and epic (see Lemma 7.17), the converse is false. See Examples 7.4 and 7.20.

We now give an example of a category of sets with structure where epis need not be surjective. We start with a definition.

Definition 7.7. An abelian group \( G \) is is divisible if for any \( g \in G \) and any positive integer \( n \) there exists \( y \in G \) such that \( ny = g \), where \( ny := y + \cdots + y \).

Example 7.8. The abelian group \((\mathbb{Q}, +)\) is divisible, the abelian group \((\mathbb{Z}, +)\) is not divisible.

Exercise 7.9. If \( G \) is divisible and \( H \leq G \) is a subgroup then the quotient \( G/H \) is also divisible. In particular the quotient \((\mathbb{Q}/\mathbb{Z}, +)\) is a divisible group.

Notation 7.10. Divisible groups and homomorphism of divisible groups form a category which we denote by \( \text{Div} \).

Proposition 7.11. The quotient map \( \pi : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \) is a monic in \( \text{Div} \). The function \( \pi : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \) is not injective.

Proof. Suppose that \( G \) is a divisible group and \( f, g : G \to \mathbb{Q} \) are two homomorphisms such that \( \pi \circ f = \pi \circ g \). We want to show that \( f = g \). It is enough to show that \( k := f - g : G \to \mathbb{Q} \) sends everything to zero: \( k(x) = 0 \) for all \( x \in G \).

We know that
\[
\pi \circ k = \pi \circ (f - g) = \pi \circ f - \pi \circ g = 0.
\]
Hence for any \( x \in G \), \( \pi(k(x)) = 0 \) in \( \mathbb{Q}/\mathbb{Z} \). Therefore \( k(x) \in \mathbb{Z} \subset \mathbb{Q} \) for all \( x \in G \).

Now suppose there is \( x \in G \) so that \( k(x) \neq 0 \). Then \( 2|k(x)| \) is a positive integer. Since \( G \) is divisible there is \( y \in G \) so that
\[
2|k(x)|y = x.
\]
Then in \( \mathbb{Q} \)
\[
k(x) = k(y + \cdots + y) = k(y) + \cdots + k(y) = 2|k(x)|k(y).
\]
Therefore
\[
k(y) = \frac{k(x)}{2|k(x)|} = \pm \frac{1}{2}
\]
which contradicts the fact that \( k(G) \subset \mathbb{Z} \). Therefore \( k(x) = 0 \) for all \( x \in G \). Hence \( f = g \) and \( \pi : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \) is monic. \( \square \)
Remark 7.12. Since \( \pi : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \) is monic and the function \( \pi : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \) on the underlying sets is not injective, the underlying set functor \( U : \text{Div} \to \text{Set} \) does not send monos to monos. One says “\( U \) does not preserve monomorphisms.”

Definition 7.13 (The category \( \text{Mon} \) of monoids). The objects of the category \( \text{Mon} \) of monoids are monoids. A morphism \( \varphi : M \to N \) in \( \text{Mon} \) is a function that preserves identity and multiplication:

1. \( \varphi(e_M) = e_N; \)
2. for any \( m_1, m_2 \in M, \varphi(m_1m_2) = \varphi(m_1)\varphi(m_2). \)

Remark 7.14. If we view monoids as small one object categories then morphisms of monoids are just functors.

Proposition 7.15. The inclusion \( i : \mathbb{N} \to \mathbb{Z} \) is an epi in \( \text{Mon} \). The inclusion is not surjective on the underlying sets. Hence the underlying set functor \( U : \text{Mon} \to \text{Set} \) does not preserve epimorphisms.

Proof. Let \( M \) be a monoid, \( f, g : \mathbb{Z} \to M \) be two homomorphisms such that \( f \circ i = g \circ i \). We want to show \( f(n) = g(n) \) for all \( n \in \mathbb{Z} \).

If \( n \in \mathbb{Z} \) and \( n \geq 0 \), then \( i(n) = n \). Therefore, \( g(n) = g(i(n)) = f(i(n)) = f(n) \). If \( n < 0 \), then
\[
f(n) = f(n)e_M = f(n)g(0) = f(n)(g(-n)g(n)) = (f(n)f(-n))g(n) = (f(k)g(n)) = g(n).
\]

Therefore, \( f = g \). We conclude that the inclusion \( i : \mathbb{N} \to \mathbb{Z} \) is epic in \( \text{Mon} \). Clearly the underlying map of sets is not surjective.

Remark 7.16. Since the inclusion \( i : \mathbb{N} \to \mathbb{Z} \) is an injective function, it must also be monic. However, \( i \) is not an isomorphism of monoids (why not?). Consequently

| mono + epi | $\nRightarrow$ | isomorphism. |

Lemma 7.17. In any category \( C \) an isomorphism \( b \overset{f}{\to} c \) is epic and monic.

Proof. Suppose \( a \overset{g}{\to} b \) and \( a \overset{h}{\to} b \) are two morphisms in \( C \) such that \( f \circ g = f \circ h \). Then
\[
g = f^{-1} \circ (f \circ g) = f^{-1} \circ (f \circ h) = h
\]
So \( f \) is a monic. Similarly if \( k, \ell : c \to d \) are two morphisms with \( k \circ f = \ell \circ f \) then
\[
k = (k \circ f) \circ f^{-1} = (\ell \circ f) \circ f^{-1} = \ell.
\]

Lemma 7.18. Let \( F : C \to D \) be a functor. If \( a \overset{f}{\to} b \) is an isomorphism in \( C \), then \( F(a) \overset{F(f)}{\to} F(b) \) is an isomorphism in \( D \). In other words

| functors preserve isomorphisms. |

Moreover \( F(f^{-1}) = F(f)^{-1} \).

Proof.
\[
F(f) \circ F(f^{-1}) = F(f \circ f^{-1}) = F(id_b) = id_{F(b)}
\]
where the last equality holds since functors preserve identity morphisms. Similarly,
\[
F(f^{-1}) \circ F(f) = F(f^{-1} \circ f) = F(id_a) = id_{F(a)}.
\]
Therefore \( F(f) \) is an isomorphism in \( D \) and \( F(f^{-1}) = F(f)^{-1} \).
Definition 7.21. A functor \( F : \mathcal{A} \to \mathcal{B} \) is injective if for any pair of objects \( a, b, c \in \mathcal{A} \) the map \( F : \text{Hom}_\mathcal{A}(a,b) \to \text{Hom}_\mathcal{B}(F(a), F(b)) \) is injective.

A functor \( F : \mathcal{A} \to \mathcal{B} \) is surjective if for any pair of objects \( a, b \in \mathcal{A} \) the map \( F : \text{Hom}_\mathcal{A}(a,b) \to \text{Hom}_\mathcal{B}(F(a), F(b)) \) is surjective.

A functor \( F \) is fully faithful if \( F \) is both full and faithful.

Example 7.20. Consider the monoid \( \mathbb{N} \) and the corresponding one object category \( \mathcal{BN} \) where \( (\mathcal{BN})_0 = \{ * \} \) and \( \text{Hom}_{\mathcal{BN}}(*, *) = \mathbb{N} \). Recall that composition in \( \mathcal{BN} \) is the “multiplication” in \( \mathbb{N} \), i.e., the composition is \(+\).

For any \( a, b, c \in \text{Hom}_{\mathcal{BN}}(*, *) \),
\[
\begin{align*}
a + b = a + c & \implies b = c \quad \text{and} \\
b + a = c + a & \implies b = c.
\end{align*}
\]

Hence any morphism \( a \) in \( \mathcal{BN} \) is both monic and epic.

For \( a \) to be an isomorphism in \( \mathcal{BN} \) there needs to be a \( b \in (\mathcal{BN})_1 = \mathbb{N} \) so that
\[
a + b = 0 = b + a.
\]

Therefore the only isomorphism in \( \mathcal{BN} \) is \( 0 \).

To repeat: every morphism in \( \mathcal{BN} \) is both monic and epic, but there is only one isomorphism.

Definition 7.21. A functor \( F : \mathcal{C} \to \mathcal{D} \) is full if for any pair of objects \( a, b \in \mathcal{C} \) the map
\[
F : \text{Hom}_\mathcal{C}(a,b) \to \text{Hom}_\mathcal{D}(F(a), F(b))
\]
is surjective.

A functor \( F : \mathcal{C} \to \mathcal{D} \) is faithful if for any pair of objects \( a, b \in \mathcal{C} \) the map
\[
F : \text{Hom}_\mathcal{C}(a,b) \to \text{Hom}_\mathcal{D}(F(a), F(b))
\]
is injective.

A functor \( F \) is fully faithful if \( F \) is both full and faithful.

Example 7.22. • The inclusion functor \( i : \text{Ab} \to \text{Group} \) is fully faithful.

• The forgetful functor \( U : \text{Group} \to \text{Set} \) is faithful but not full since not all function between the underlying sets need to be homomorphisms.

• The “inclusion” functor
\[
i : \text{Set} \to \text{Rel}, \quad i(X \xrightarrow{f} Y) = X \xrightarrow{\text{graph}(f)} Y
\]
is faithful but not full since not all relations are graphs of functions.

Lemma 7.23. Let \( F : \mathcal{C} \to \mathcal{D} \) be a faithful functor and \( b \xrightarrow{f} c \) a morphism in \( \mathcal{C} \). If \( F(b) \xrightarrow{F(f)} F(c) \) is monic, then \( b \xrightarrow{f} c \) is monic. If \( F(b) \xrightarrow{F(f)} F(c) \) is epic, then \( b \xrightarrow{f} c \) is epic.

One says: “faithful functors reflect monos and epis.”

Remark 7.24. We have seen that faithful functors don’t need to send monos to monos or epis to epis, i.e., they don’t need to preserve them.

Proof of Lemma 7.23. Suppose that \( F(b) \xrightarrow{F(f)} F(c) \) is monic and \( a \xrightarrow{g} b, \ a \xrightarrow{h} b \) are morphisms in \( \mathcal{C} \). If \( f \circ g = f \circ h \), then we have \( F(f \circ g) = F(f \circ h) \). Since \( F \) is a functor, \( F(f) \circ F(g) = F(f) \circ F(h) \).

Since \( F(b) \xrightarrow{F(f)} F(c) \) is a monic, \( F(g) = F(h) \). Since \( F \) is faithful, we must have \( g = h \), therefore, \( f \) is a monic. A proof that \( F \) reflects epimorphisms is similar. \( \square \)

Remark 7.25. Alternatively, we can use the principle of duality to show that \( F \) reflects epis. A functor \( G : \mathcal{A} \to \mathcal{B} \) between two categories induces a functor \( G^{\text{op}} : \mathcal{A}^{\text{op}} \to \mathcal{B}^{\text{op}} \) between the opposite categories: \( G^{\text{op}} \) is defined by
\[
G^{\text{op}}(a \xrightarrow{\gamma} a') := G(a) \xrightarrow{G(\gamma)^{\text{op}}} G(a').
\]
Next observe that if a functor $F : C \to D$ is faithful then $F^{\text{op}} : C^{\text{op}} \to D^{\text{op}}$ is faithful as well. We then note that a morphism $f$ in a category $A$ is epic iff and only if $f^{\text{op}}$ is monic in $A^{\text{op}}$.

**Lecture 8. Metric and topological spaces**

Today we take a short break from category theory in order to define the category $\text{Top}$ of topological spaces.

Recall that the **Euclidean distance** $d(x, y)$ between two points $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in $\mathbb{R}^3$ is defined by
\[
d(x, y) := \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}.
\]
The open ball of radius $r$ center at $x \in \mathbb{R}^3$ is defined by
\[
B_r(x) = \{ y \in \mathbb{R}^3 \mid d(x, y) < r \}
\]
A subset $U$ of $\mathbb{R}^3$ is open if for every $x \in U$, there exists $r > 0$ such that $B_r(x) \subseteq U$.

The Euclidean distance function $d : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ has three important properties: for every $x, y, z \in \mathbb{R}^3$
\[
\begin{align*}
(1) \quad & d(x, y) = d(y, x) \\
(2) \quad & d(x, y) = 0 \iff x = y \\
(3) \quad & d(x, y) + d(y, z) \geq d(x, z)
\end{align*}
\]
We turn these properties into a definition.

**Definition 8.1.** A metric on a set $X$ is a function $d : X \times X \to [0, +\infty)$ such that for all $x, y, z \in X$
\[
\begin{align*}
(1) \quad & d(x, y) = d(y, x) \\
(2) \quad & d(x, y) = 0 \iff x = y \\
(3) \quad & d(x, y) + d(y, z) \geq d(x, z)
\end{align*}
\]
A metric space is a pair $(X, d)$ where $X$ is a set and $d : X \times X \to [0, \infty)$ is a metric on $X$.

**Definition 8.2.** Let $(X, d)$ be a metric space. An open ball of radius $r$ centered at $x \in X$ is the set
\[
B_r(x) = \{ y \in X \mid d(x, y) < r \}.
\]

**Example 8.3.** Let $X = \mathbb{R}$, then the function $d : \mathbb{R} \times \mathbb{R} \to [0, +\infty)$ defined by $d(x, y) = |x - y|$ is a metric. In this case, $B_r(x) = (x - r, x + r)$.

**Example 8.4.** Let $X = \mathbb{R}^n$, then $d : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$ defined by
\[
d(x, y) = \left( \sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}
\]
is a metric. Triangle identity is not obvious, it’s usually proved using the Cauchy-Schwarz inequality:
\[
|\langle x, y \rangle| \leq \sqrt{\langle x, x \rangle} \sqrt{\langle y, y \rangle}
\]
where $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$ is the usual inner product on $\mathbb{R}^n$.

**Definition 8.5.** Let $(X, d)$ be a metric space, A subset $U \subseteq X$ is open if for any $x \in U$, there exists $r > 0$ such that $B_r(x) \subseteq U$.

We expect open balls to be open sets and their are:

**Lemma 8.6.** Let $(X, d)$ be a metric space and $B_r(x)$ an open ball of radius $r$ centered at $x \in X$. Then $B_r(x)$ is an open set: for any $y \in B_r(x)$, there exist $\rho > 0$ such that $B_\rho(y) \subseteq B_r(x)$
Proof. Since \( y \in B_r(x) \), \( r > d(x,y) \). Let \( \rho = r - d(x,y) \), \( \rho \) is positive. If \( z \in B_\rho(y) \) then \( d(y,z) < \rho = r - d(x,y) \). Therefore,
\[
  r > d(x,y) + d(y,z) \geq d(x,z)
\]
by the triangle identity. Hence \( z \in B_r(x) \) and consequently \( B_\rho(x) \subseteq B_r(x) \).
\( \square \)

Remark 8.7. Let \((X,d)\) be a metric space. Then the empty set \(\emptyset\) and the whole space \(X\) are open.

Exercise 8.8. Let \((X,d)\) be a metric space.

1. If \(U, V \subseteq X\) are open subsets, then so is their intersection \(U \cap V\).
2. If \(\{U_\alpha\}_{\alpha \in A}\) is an arbitrary collection of open subsets of \(X\), then their union \(\bigcup_{\alpha \in A} U_\alpha\) is open in \(X\).

We now turn the properties of open sets in a metric space into a definition.

Definition 8.9. A \textit{topology} \(T\) on a set \(X\) is a collection of subsets of \(X\) such that

1. \(\emptyset, X \in T\);
2. if \(U, V \in T\), then \(U \cap V \in T\);
3. if \(\{U_\alpha\}_{\alpha \in A} \subseteq T\) (where \(A\) is an arbitrary indexing set), then \(\bigcup_{\alpha \in A} U_\alpha \in T\).

The elements of \(T\) are called \textit{open sets} and the pair \((X,T)\) is called a \textit{topological space}.

Example 8.10. Let \((X,d)\) be a metric space. By Exercise 8.8 the set \(T_d := \{U \subseteq X \mid \text{for any } x \in U \text{ there exists } r > 0 \text{ such that } B_r(x) \subseteq U\}\) is a topology on \(X\).

The topology \(T_d\) is called the \textit{topology defined by the metric} \(d\). A topology \(T\) on a set \(X\) so that \(T = T_d\) for some metric \(d\) on \(X\) is called a \textit{metric topology}.

Definition 8.11. The \textit{standard topology} \(T_{st}\) on \(\mathbb{R}^n\) is the topology defined by the Euclidean metric \(d; d(x,y) = \left(\sum_{i=1}^n (x_i - y_i)^2\right)^{1/2}\).

Fact 8.12. (i.e., an exercise or a theorem) Let \(X = \mathbb{R}^2\). Then

1. \(d_\infty : X \times X \to [0,\infty)\) \(d_\infty(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}\)
   is a metric on \(\mathbb{R}^2\).
2. The “open balls” with respect to \(d_\infty\) are open squares.
3. The metric \(d_\infty\) induces (defines) the same topology on \(\mathbb{R}^2\) as the Euclidean metric.

Moral: different metrics may give rise to the same topology.

Remark 8.13. For any set \(X\) we always have the following two topologies:

\[ T_{\max} = \mathcal{P}(X) = \text{ the set of all subsets of } X \]

and

\[ T_{\min} = \{\emptyset, X\}. \]

Remark 8.14. The maximal topology \(T_{\max} = \mathcal{P}(X)\) on a set \(X\) does come from a metric. Namely define \(d : X \times X \to [0,\infty)\) by

\[
  d(x,y) := \begin{cases} 
1 & \text{if } x \leq y \\
0 & \text{if } x = y.
\end{cases}
\]

It is easy to check that \(d\) is a metric (do it!). Then for any point \(x \in X\) the open ball \(B_1(x)\) of radius 1 is just the singleton \(\{x\}\). And then for any \(A \subset X\)

\[
A = \bigcup_{x \in A} \{x\}
\]
so \( A \) is open, i.e., is an element of \( \mathcal{T}_d \).

There are topologies on the real line \( \mathbb{R} \) that do not come from any metric. Here is an example. Consider
\[
\mathcal{T} = \left\{ \emptyset \right\} \cup \left\{ U \subseteq \mathbb{R} \mid U = \mathbb{R} \setminus F \text{ for some finite set } F \right\}.
\]
Let’s check that \( \mathcal{T} \) is a topology. \( \emptyset \in \mathcal{T} \) by definition and \( \mathbb{R} \in \mathcal{T} \) because \( \mathbb{R} = \mathbb{R} \setminus \emptyset \) and the empty set is finite.

Suppose \( F_1, F_2 \) are two finite sets. Then \( F_1 \cup F_2 \) is finite and therefore
\[
(\mathbb{R} \setminus F_1) \cap (\mathbb{R} \setminus F_2) = \mathbb{R} \setminus (F_1 \cup F_2) \in \mathcal{T}.
\]
For any collection \( \{F_\alpha\}_{\alpha \in A} \) of finite sets \( \bigcap_{\alpha \in A} F_\alpha \) is finite. Therefore
\[
\bigcup_{\alpha \in A} (\mathbb{R} \setminus F_\alpha) = \mathbb{R} \setminus \bigcap_{\alpha \in A} F_\alpha \in \mathcal{T}.
\]
So \( \mathcal{T} \) is indeed a topology on \( \mathbb{R} \). It is often referred to as the cofinite topology. We now check that \( \mathcal{T} \) does not come from a metric. To do that we prove a lemma.

**Lemma 8.15.** Let \((X, d)\) be a (non-empty) metric space, then for any \( x, y \in X \) such that \( x \neq y \), there are open sets \( U, V \subseteq X \) such that \( x \in U \) and \( y \in V \) and \( U \cap V = \emptyset \).

**Proof.** Since \( x \neq y \), \( d(x, y) > 0 \) by definition of metrics. Let \( r = \frac{1}{2}d(x, y) \). We argue that \( B_r(x) \cap B_r(y) = \emptyset \).

Suppose not: \( B_r(x) \cap B_r(y) \neq \emptyset \). Then for any \( z \in B_r(x) \cap B_r(y) \) we have \( d(x, z) < r \) and \( d(y, z) < r \). By the triangle identity,
\[
d(x, y) \leq d(x, z) + d(y, z) < r + r = d(x, y),
\]
which is a contradiction. Therefore \( B_r(x) \cap B_r(y) = \emptyset \). \( \square \)

We end the proof that the cofinite topology on \( \mathbb{R} \) does not come from any metric by proving

**Lemma 8.16.** For any two points \( x, y \in \mathbb{R} \) such that \( x \neq y \) and for any two open set \( U, V \in \mathcal{T} \) such that \( x \in U \) and \( y \in V \) the intersection \( U \cap V \) is nonempty.

**Proof.** Since \( U, V \) are not empty, they must both have finite complements in \( \mathbb{R} \) by definition of \( \mathcal{T} \). Then \( \mathbb{R} \setminus (U \cap V) = (\mathbb{R} \setminus U) \cup (\mathbb{R} \setminus V) \) must also be finite. Hence \( U \cap V \) must be infinite (since \( \mathbb{R} \) is infinite) and therefore nonempty. \( \square \)

To turn the collection of all topological spaces into a category we need to have a notion of a morphism. The “right” notion of a morphism between two topological spaces turns out to be that of a continuous map. The definition of a continuous map is simple and a bit criptic. I will provide a motivation next time.

**Definition 8.17.** Let \((X, \mathcal{T}_X), (Y, \mathcal{T}_Y)\) be topological spaces. A function \( f : X \to Y \) is continuous if for any \( U \in \mathcal{T}_Y \), \( f^{-1}(U) \in \mathcal{T}_X \) (i.e., preimages of open sets under \( f \) are open).

**Remark 8.18.** For any topological space \((X, \mathcal{T}_X)\) the identity map \( \text{id}_X : X \to X \) is continuous (why?).

**Lemma 8.19.** Composition of two continuous function is continuous.

**Proof.** Let \( f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y) \) and \( g : (Y, \mathcal{T}_Y) \to (Z, \mathcal{T}_Z) \) be two continuous map. We’d like to show that their composite \( g \circ f : (X, \mathcal{T}_X) \to (Z, \mathcal{T}_Z) \) is continuous.

Suppose that \( U \in \mathcal{T}_Z \) is an open set. Then
\[
(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)).
\]
Since \( g \) is continuous, \( g^{-1}(U) \in \mathcal{T}_Y \). Since \( f \) is continuous, \( f^{-1}(g^{-1}(U)) \in \mathcal{T}_X \). Hence, \( g \circ f \) is continuous. \( \square \)
**Lecture 9. Continuity, closed sets, subspace topology, and products**

**Last time:** A topology \( \mathcal{T} \) on a set \( X \) is a collection of subsets of \( X \) such that

1. \( \emptyset, X \in \mathcal{T} \);
2. if \( U, V \in \mathcal{T} \), then \( U \cap V \in \mathcal{T} \);
3. if \( \{ U_\alpha \}_{\alpha \in A} \subseteq \mathcal{T} \) (where \( A \) is an arbitrary indexing set), then \( \bigcup_{\alpha \in A} U_\alpha \in \mathcal{T} \).

The elements of \( \mathcal{T} \) are called **open sets** under \( \mathcal{T} \) are open). A function \( f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y) \) is **continuous** if for any \( U \in \mathcal{T}_Y \), \( f^{-1}(U) \in \mathcal{T}_X \) (i.e., preimages of open sets under \( f \) are open).

We proved that the composition of two continuous functions is continuous and that the identity map \( \text{id}_X : (X, \mathcal{T}_X) \to (X, \mathcal{T}_X) \) is continuous.

**Definition 9.1.** Topological spaces and continuous maps (with the usual composition) form a category that we denote by \( \text{Top} \).

**Remark 9.2.**

1. The category \( \text{Top} \) is locally small.
2. There is a forgetful/underlying set functor \( U : \text{Top} \to \text{Set} \). On objects \( U((X, \mathcal{T}_X)) = X \).

Last time I promised to motivate the definition of a continuous map. Recall that a function \( f : \mathbb{R} \to \mathbb{R} \) is **continuous at** \( x_0 \in \mathbb{R} \) if for every \( \varepsilon > 0 \) there is \( \delta > 0 \) so that for any \( x \in \mathbb{R} \)

\[ |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \varepsilon. \]

A function \( f : \mathbb{R} \to \mathbb{R} \) is **continuous** if it is continuous at every point \( x_0 \in \mathbb{R} \). This \( \varepsilon \)-\( \delta \) definition of continuity generalizes to arbitrary metric spaces.

**Definition 9.3.** Let \( (X, d_X) \) and \( (Y, d_Y) \) be metric spaces. A function \( f : X \to Y \) is **continuous at** \( x_0 \in X \) if for any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that

\[ d_X(x, x_0) < \delta \quad \Rightarrow \quad d_Y(f(x), f(x_0)) < \varepsilon. \]

A function \( f : X \to Y \) is said to be **continuous** if it is continuous at every point \( x_0 \in X \).

This definition of continuity looks rather different from the definition of a continuous map between two topological spaces. What’s the connection? We first make a technical observation.

**Lemma 9.4.** Let \( (X, d_X) \) and \( (Y, d_Y) \) be metric spaces and \( f : X \to Y \) a function. Equation (9.6) is equivalent to

\[ B_\delta(x_0) \subseteq f^{-1}(B_\varepsilon(f(x_0))) \]

where as before \( B_r(z) \) denotes an open ball of radius \( r \) centered at \( z \).

**Proof.** Recall that an open ball \( B_r(z) \) in a metric space \( (Z, d) \) is defined by

\[ B_r(z) = \{ z' \in Z \mid d(z, z') < r \} \]  

or, equivalently, \( z' \in B_r(z) \iff d(z, z') < r \). Now Equation (9.6) holds

\[ \iff (x \in B_\delta(x_0) \quad \Rightarrow \quad f(x) \in B_\varepsilon(f(x_0))) \]

\[ \iff (x \in B_\delta(x_0) \quad \Rightarrow \quad x \in f^{-1}(B_\varepsilon(f(x_0)))) \]

\[ \iff B_\delta(x_0) \subseteq f^{-1}(B_\varepsilon(f(x_0))) \]

\[ \square \]

**Lemma 9.5.** Let \( (X, d_X) \) and \( (Y, d_Y) \) be metric spaces. A function \( f : X \to Y \) is continuous in the sense of Definition 9.3 \iff for any \( U \subseteq Y \) that is open with respect to the metric \( d_Y \) the set \( f^{-1}(U) \) is open with respect to the metric \( d_X \).
Remark
Example 9.7.
Definition 9.11.
Let topological space, \( \emptyset \)
9.10
Remark
Example
Definition 9.6.
Proof. ( \( \implies \) ) Suppose that \( f \) is continuous in the sense of Definition 9.3 and \( U \subseteq Y \) is open in the metric topology \( T_{dy} \). If \( f^{-1}(U) = \emptyset \), it’s open. If \( f^{-1}(U) \) is not empty, pick any \( x_0 \in f^{-1}(U) \). Then \( f(x_0) \in U \). Since \( U \) is open, there exists \( \varepsilon > 0 \) such that \( B_{\varepsilon}(f(x_0)) \subseteq U \). Since \( f \) is continuous at \( x_0 \), there exists \( \delta > 0 \) such that
\[
d_X(x, x_0) < \delta \quad \implies \quad d_Y(f(x), f(x_0)) < \varepsilon.
\]
By Lemma 9.4
\[
B_\delta(x_0) \subseteq f^{-1}(B_\varepsilon(f(x_0))) \quad (\subseteq f^{-1}(U)).
\]
Therefore for any \( x_0 \in f^{-1}(U) \) there is \( \delta > 0 \) so that \( B_\delta(x_0) \subseteq f^{-1}(U) \). Since the choice of \( x_0 \) is arbitrary, \( f^{-1}(U) \) is open in \( X \) with respect to \( d_X \).
( \( \iff \) ) We now suppose that if \( U \subseteq Y \) is open then \( f^{-1}(U) \) is open in \( X \). Let \( x_0 \in X \) be a point and let \( \varepsilon > 0 \). Recall that open balls are open. In particular \( B_\varepsilon(f(x_0)) \subseteq Y \) is open. By assumption on \( f \) the set \( f^{-1}(B_\varepsilon(f(x_0))) \subseteq X \) is open in \( X \). Since \( f(x_0) \in B_\varepsilon(f(x_0)) \) we have \( x_0 \in f^{-1}(B_\varepsilon(f(x_0))) \).
Since \( f^{-1}(B_\varepsilon(f(x_0))) \subseteq X \) is open there exists \( \delta > 0 \) such that
\[
B_\delta(x_0) \subseteq f^{-1}(B_\varepsilon(f(x_0))).
\]
By Lemma 9.4 this inclusion is equivalent to
\[
d_X(x, x_0) < \delta \quad \implies \quad d_Y(f(x), f(x_0)) < \varepsilon.
\]
Hence the function \( f \) is continuous at \( x_0 \) in the sense of Definition 9.3 Since \( x_0 \) is arbitrary, \( f \) is continuous.
□

Moral: Definition 8.17 of continuity of maps between two topological spaces is a generalization of the \( \varepsilon\)-\( \delta \) definition of continuity.

Definition 9.6. Let \( (X, \mathcal{T}) \) be a topological space. A subset \( C \subseteq X \) is closed if the complement \( X \setminus C \) is open, i.e. \( (X \setminus C) \in \mathcal{T} \).

Example 9.7. Consider the real line \( \mathbb{R} \) with the standard topology \( T_{st} \). Then any closed interval \([a, b]\) is closed in \( \mathbb{R} \). Why?

Consider the real line \( \mathbb{R} \) with the cofinite topology. Then finite sets are closed.

Exercise 9.8. Show that if \( (X, \mathcal{T}) \) is a topological space, then
1. \( \emptyset, X \) are closed;
2. if \( \{C_a\}_{a \in A} \) is an arbitrary indexed collection of closed sets, then \( \bigcap_{a \in A} C_a \) is closed in \( X \);
3. if \( C, D \) are closed sets in \( X \), then \( U \cup V \) is closed in \( X \).

Remark 9.9. Arbitrary union of closed sets may not be closed. Consider the \( \bigcup_{n \in \mathbb{Z}^+} [\frac{1}{n}, 1 - \frac{1}{n}] = (0, 1) \), which is not closed in the standard topology.

Remark 9.10. Being open and closed are not mutually exclusive. For example, if \( (X, \mathcal{T}) \) is a topological space, \( \emptyset \) and \( X \) are both open and closed.

For any set \( X \) any subset in a topological space \( (X, \mathcal{P}(X)) \) is both open and closed.

Consider \( \mathbb{R} \) with the standard topology. Intervals of the form \([a, b]\) are neither open nor closed.

Definition 9.11. Let \( X \) be a set and \( \mathcal{T}_1, \mathcal{T}_2 \) be two topologies on \( X \).
The topology \( \mathcal{T}_1 \) is smaller (or coarser or weaker) than \( \mathcal{T}_2 \) iff \( \mathcal{T}_1 \subseteq \mathcal{T}_2 \).
The topology \( \mathcal{T}_2 \) is bigger (or finer or stronger) than \( \mathcal{T}_1 \) iff \( \mathcal{T}_1 \subseteq \mathcal{T}_2 \).

Remark 9.12. For any topological space \( (Y, \mathcal{T}_Y) \) and any set \( X \) and function \( f : (Y, \mathcal{T}_Y) \to (X, \{\emptyset, X\}) \) is continuous. Similarly any function \( h : (X, \mathcal{P}(X)) \to (Y, \mathcal{T}_Y) \) is continuous.
**Subspace topology**

**Lemma 9.13.** Let \((X, \mathcal{T}_X)\) is a topological space and \(Y \subseteq X\) is a subset. Then

\[ \mathcal{T}^Y := \{ U \subseteq Y \mid \text{there exists } \tilde{U} \in \mathcal{T}_X \text{ such that } U = \tilde{U} \cap Y \} \]

is a topology on \(Y\). Moreover, \(\mathcal{T}^Y\) is the smallest topology on \(Y\) so that the inclusion map \(i : Y \rightarrow X, y \mapsto y\) is continuous.

**Proof.** Since \(\emptyset \cap Y = \emptyset\) and since \(X \cap Y = Y\), \(\emptyset, Y \in \mathcal{T}^Y\).

If \(U, V \in \mathcal{T}^Y\), then there exists \(\tilde{U}, \tilde{V} \in \mathcal{T}_X\) such that \(U = \tilde{U} \cap Y\) and \(V = \tilde{V} \cap Y\). Then

\[ U \cap V = (\tilde{U} \cap Y) \cap (\tilde{V} \cap Y) = (\tilde{U} \cap \tilde{V}) \cap Y. \]

Since \(\mathcal{T}_X\) is a topology, \((\tilde{U} \cap \tilde{V}) \in \mathcal{T}_X\). Therefore, \(U \cap V \in \mathcal{T}^Y\).

Similarly if \(\{U_\alpha\}_{\alpha \in \Lambda}\) is a family of sets in \(\mathcal{T}^Y\) then there is a family of sets \(\{\tilde{U}_\alpha\}_{\alpha \in \Lambda} \subseteq \mathcal{T}_X\) with \(\tilde{U}_\alpha \cap Y = U_\alpha\) for all \(\alpha\). And then

\[ \bigcup_{\alpha \in \Lambda} U_\alpha = \bigcup_{\alpha \in \Lambda} (\tilde{U}_\alpha \cap Y) = \left( \bigcup_{\alpha \in \Lambda} \tilde{U}_\alpha \right) \cap Y. \]

Since \(\bigcup_{\alpha \in \Lambda} U_\alpha \in \mathcal{T}_X\), the union \(\bigcup_{\alpha \in \Lambda} U_\alpha\) is in \(\mathcal{T}^Y\).

We conclude that \(\mathcal{T}^Y\) is a topology on \(Y\). If \(\mathcal{T}'\) is another topology on \(Y\) so that \(i : Y \rightarrow X\) is continuous, then for any \(\tilde{U} \in \mathcal{T}_X\)

\[ \mathcal{T}' \ni i^{-1}(\tilde{U}) = \tilde{U} \cap Y. \]

Hence \(\mathcal{T}^Y \subseteq \mathcal{T}'\). \(\Box\)

**Definition 9.14.** Let \((X, \mathcal{T}_X)\) is a topological space and \(Y \subseteq X\) is a subset. The topology \(\mathcal{T}^Y\) on \(Y\) defined by (9.8) is called the **subspace topology** on \(Y\).

**A basis for a topology**

**Definition 9.15.** A **basis** for a topology \(\mathcal{T}\) on a set \(X\) is a collection \(\mathcal{B}\) of open sets (i.e., \(\mathcal{B} \subseteq \mathcal{T}\)) so that any \(U \in \mathcal{T}\) is a union of elements of \(\mathcal{B}\).

**Example 9.16.** Consider \(\mathbb{R}^n\) with the standard topology \(\mathcal{T}\). Suppose \(U \subset \mathbb{R}\) is open. Then for any point \(x \in U\) there is \(r(x) > 0\) so that \(B_{r(x)}(x) \subset U\) (where as before \(B_{r(x)}(x) = \{y \in \mathbb{R}^n \mid d(x, y) \equiv \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} < r(x)\}\) is the standard Euclidean open ball of radius \(r(x)\) centered at \(x\)). Consequently

\[ U = \bigcup_{x \in U} B_{r(x)}(x) \]

and therefore

\[ \mathcal{B} = \{B_{r(x)} \mid x \in \mathbb{R}^n, r > 0\} \]

is a basis for the standard topology on \(\mathbb{R}^n\).

This example easily generalizes to any metric space:

**Example 9.17.** Let \((X, d)\) be a metric space. Then the collection

\[ \mathcal{B} = \{B_{r(x)} \mid x \in X, r > 0\} \]

is a basis for the topology \(\mathcal{T}_d\) defined by the metric \(d\) (see Example 8.10).

Last time:

- Notions of continuity: from the $\varepsilon - \delta$ definition of continuity to $f : X \to Y$ is continuous if for any open set $U$ in $Y$ the preimage $f^{-1}(U)$ is open in $X$.
- The subspace topology: If $(X, T_X)$ is a topological space and $Y \subseteq X$ a subset, the subspace topology $T_Y^i$ on $Y$ is the smallest topology making the inclusion $i : Y \to X$ continuous. This amounts to $U \subset Y$ is open $\iff$ there is $\tilde{U} \subset X$ open with $U = \tilde{U} \cap Y$.
- A subset $B$ of a topology $T$ on a set $X$ is a basis for $T$ if every $U \in T$ can be written as a union of some collection of elements of $B$.

**Question 10.1.** Let $X$ be a set and $B$ is a collection of subsets of $X$ (i.e., $B \subseteq \mathcal{P}(X)$ ). Is $B$ a basis for some topology $T$ on $X$? And if it is, what are the elements of $T$ look like?

Lemma 10.2 provides an answer.

**Lemma 10.2.** Let $X$ be a set and $B \subseteq \mathcal{P}(X)$ a set of subsets of $X$. Then $B$ is a basis for a topology $T$ on $X$ if

1. $\bigcup_{B \in B} B = X$ and
2. for any $B_1, B_2 \in B$ the intersection $B_1 \cap B_2$ is a union of a collection of elements of $B$.

**Proof.** If $B$ is a basis for a topology $T$ then

$$(10.9) \quad T = \{ U \subset X \mid U = \bigcup_{B \in A} B \text{ for some } A \subset B \}.$$ 

So define $T$ by (10.9). We need to check that $T$ so defined is a topology.

- $\emptyset \in T$ since $\emptyset \subset B$ and $\bigcup_{B \in \emptyset} B = \emptyset$. On the other hand $X \in T$ since $\bigcup_{B \in B} B = X$.
- Suppose $U, V \in T$. Then there are $A, C \subset B$ so that $U = \bigcup_{B \in A} B$ and $V = \bigcup_{B' \in C} B'$. Then

$$U \cap V = \left( \bigcup_{B \in A} B \right) \cap \left( \bigcup_{B' \in C} B' \right) = \bigcup_{B \in A, B' \in C} B \cap B'.$$

By assumption for each $B, B'$ in $B$ there is $D \subset B$ so that $B \cap B' = \bigcup_{B'' \in D} B''$. It follows that $U \cap V \in T$.

- Similarly if $\{U_\gamma\}_{\gamma \in \Gamma} \subset T$ is a family of open sets then for each $\gamma \in \Gamma$ there is a collection $A_\gamma \subset B$ so that $U_\gamma = \bigcup_{B \in A_\gamma} B$. Therefore

$$\bigcup_{\gamma \in \Gamma} U_\gamma = \bigcup_{B \in \bigcup_{\gamma \in \Gamma} A_\gamma} B,$$

and hence $\bigcup_{\gamma \in \Gamma} U_\gamma \in T$.

\[\square\]

**Remark 10.3.** We will refer to the topology $T$ of Lemma 10.2 as the topology generated by $B$.

The following lemma is very useful.

**Lemma 10.4.** Let $(X, T_X)$ and $(Y, T_Y)$ be topological spaces and $B \subset T_Y$ a basis for $T_Y$. Then a function $f : (X, T_X) \to (Y, T_Y)$ is continuous if and only if for any $B \in B$, $f^{-1}(B)$ is open in $X$.

**Proof.** ($\Longrightarrow$) Suppose that $f$ is continuous. Since any $B \in B$ is open in $Y$, $f^{-1}(B)$ has to be open in $X$. 

29
Since $U$ then for any $p$ projections condition (1) holds. Since (By Lemma 10.2 it is enough to check that (1)

Proof. Let $Z = X \times Y$. Define $p_X$, $p_Y$ to be the projections: $p_X(x, y) = x$ and $p_Y(x, y) = y$. We need to find an appropriate topology $\tau_{\text{prod}}$ (called the product topology) on $X \times Y$ so that $(X, \tau_X) \to (X, \tau_X)$, $p_Y : (Z, \tau_Z) \to (Y, \tau_Y)$ so that for any space $(W, \tau_W)$ and any pair of continuous maps $f_X : (W, \tau_W) \to (X, \tau_X)$, $f_Y : (W, \tau_W) \to (Y, \tau_Y)$ there is a unique continuous map $f : (W, \tau_W) \to (Z, \tau_Z)$ such that $f_X = p_X \circ f$ and $f_Y = p_Y \circ f$.

Claim: The set $B = \{U \times V | U \in \tau_X, V \in \tau_Y\}$ is a basis for a topology on $X \times Y$.

Proof of Claim. By Lemma 10.2 it is enough to check that (1) $\bigcup_{B \in B} = X \times Y$ and (2) for any $U \times V, U' \times V' \in B$ the intersection $(U \times V) \cap (U' \times V')$ is a union of elements of $B$. Since $X \times Y \in B$ condition (1) holds. Since $(U \times V) \cap (U' \times V') = (U \cap U') \times (V \cap V')$ condition (2) holds.

We take $\tau_{\text{prod}}$ to be the topology generated by $B$. Note that by construction of $\tau_{\text{prod}}$ the projections $p_X, p_Y$ are continuous.

It remains to check the universal property of $(X, \tau_X) \to (X, \tau_X)$, $p_Y : (Z, \tau_Z) \to (Y, \tau_Y)$. So let $(W, \tau_W)$ be a topological space with a pair of continuous maps $f_X : (W, \tau_W) \to (X, \tau_X)$, $f_Y : (W, \tau_W) \to (Y, \tau_Y)$. Define $f : W \to X \times Y$ by $f(w) = (f_X(w), f_Y(w))$.

Then for any $U \times V \in B$, $f^{-1}(U \times V) = \{w \in W | (f_X(w), f_Y(w)) \in U \times V\} = f_X^{-1}(U) \cap f_Y^{-1}(V)$.

Since $f_X, f_Y$ are continuous, the sets $f_X^{-1}(U), f_Y^{-1}(V)$ are open in $W$. Hence $f^{-1}(U \times V)$ is open in $W$. By Lemma 10.4 the map $f$ is continuous.

Lemma 10.6. For any two topological spaces $(X, \tau_X)$, $(Y, \tau_Y)$ their coproduct exists in the category Top: there is a topological space $(Z, \tau_Z)$ with two continuous maps $\iota_X : (X, \tau_X) \to (Z, \tau_Z)$, $\iota_Y : (Y, \tau_Y) \to (Z, \tau_Z)$ so that for any space $(W, \tau_W)$ and any pair of continuous maps $f_X : (X, \tau_X) \to (W, \tau_W)$, $f_Y : (Y, \tau_Y) \to (W, \tau_W)$ there is a unique continuous map $f : (Z, \tau_Z) \to (W, \tau_W)$ such that $f_X = f \circ \iota_X$ and $f_Y = f \circ \iota_Y$.

Proof. We take $Z$ to be the disjoint union $X \sqcup Y$ of $X$ and $Y$. We take $\iota_X : X \to X \sqcup Y$, $\iota_Y : Y \to X \sqcup Y$ to be the “inclusion” maps (in other words, we let $Z$ be the coproduct of $X$ and $Y$ in Set). It is not hard to check that the collection $B = \{U \sqcup V \subseteq X \sqcup Y | U \in \tau_X, V \in \tau_Y\}$
satisfies the conditions of Lemma 10.2. Let \( T_{\text{coprod}} \) denote the topology on generated by \( \mathcal{B} \). It is not hard to show that the maps \( \iota_X, \iota_Y \) are continuous. For example, for any \( U \sqcup V \in \mathcal{B} \)
\[
(\iota_X)^{-1}(U \sqcup V) = U
\]
and therefore \( \iota_X \) is continuous by Lemma 10.4.

Finally if \( f_X : (X, T_X) \to (W, T_W), f_Y : (Y, T_Y) \to (W, T_W) \) is a pair of continuous maps then by the universal property of coproducts in \( \textbf{Set} \) there is a unique function \( f : X \sqcup Y \to W \) so that \( f \circ \iota_X = f_X \) and \( f \circ \iota_Y = f_Y \). And then for any open set \( O \subset W \),
\[
f^{-1}(O) = f_X^{-1}(O) \sqcup f_Y^{-1}(O) \in \mathcal{B} \subset T_{\text{coprod}}.
\]
Thus the function \( f : (X \sqcup Y, T_{\text{coprod}}) \to (W, T_W) \) is continuous. \( \square \)

We are now ready to tackle products that are more general than products of two objects.

**Definition 10.7.** Let \( \mathcal{C} \) be a category, \( I \) a set (usually called an indexing set) and \( \{a_i\}_{i \in I} \) a collection of objects of \( \mathcal{C} \) indexed by \( I \). The **product** of the family \( \{a_i\}_{i \in I} \) (if it exists) is an object \( c \in \mathcal{C} \) together with a family of morphisms \( \{c \to a_i\}_{i \in I} \) with the following universal property: for any object \( d \in \mathcal{C} \) and any family of morphisms \( \{d \to a_i\}_{i \in I} \), there exists a unique morphism \( d \to c \) so that \( p_i \circ f = f_i \) for all \( i \in I \).

**Remark 10.8.** It is not hard to show that if \( (c, \{p_i : c \to a_i\}) \) and \( (d, \{q_i : d \to a_i\}) \) are two products of a family \( \{a_i\}_{i \in I} \) of objects in a category \( \mathcal{C} \) then there is a unique isomorphism \( \varphi : c \to d \) so that \( q_i \circ \varphi = p_i \). Hence products are unique up to a unique isomorphism (if/when they exist) and we can talk about “the” product \( \prod_{i \in I} a_i \) of a family \( \{a_i\}_{i \in I} \) of objects in a category \( \mathcal{C} \).

**Remark 10.9.** If \( I = \{1, 2\} \), then the product of \( \{a_1, a_2\} \) is the binary product \( (a_1 \times a_2, p_1, p_2) \).

**Remark 10.10.** If \( \{a_i\}_{i \in I} \) is a family of objects in \( \mathcal{C} \) and \( I = \emptyset \), then the product \( \prod_{i \in I} a_i \) is an object \( c \) so that for any object \( d \in \mathcal{C} \), there exists a unique morphism \( d \to c \). Hence the product of an empty family of objects is a terminal object in \( \mathcal{C} \).

**Remark 10.11.** If \( \{a_i\}_{i \in I} \) is a family of objects in \( \mathcal{C} \) and \( I = \{1\} \), then the product \( \prod_{i \in I} a_i = a_1 \) and \( p_1 = \text{id}_{a_1} \).

**Remark 10.12.** The phrase “a collection of objects \( \{a_i\}_{i \in I} \) of \( \mathcal{C} \) indexed by a set \( I \)” is a somewhat sloppy way of saying that we have a function \( a \) from the set \( I \) to the collection \( \mathcal{C}_0 \) of objects of \( \mathcal{C} \); the function sends \( i \in I \) to \( a_0 \in \mathcal{C}_0 \).

**Example 10.13.** Arbitrary products exist in \( \textbf{Set} \) provided that we accept the axiom of choice. Here is a construction.

Given a family of sets \( \{a_i\}_{i \in I} \), define
\[
\prod_{i \in I} a_i = \{ x : I \to \bigcup_{i \in I} a_i \mid x(i) \in a_i \text{ for every } i \in I \}
\]
and define a family of functions \( p_j : \prod_{i \in I} a_i \to a_j \) by \( p_j(x) = x(j) \) for every \( j \in I \). The universal property is easy to check: given a set \( d \) and a family of functions \( \{f_i : d \to a_i\}_{i \in I} \) define
\[
f : d \to \prod_{i \in I} a_i = \{ x : I \to \bigcup_{i \in I} a_i \}
\]
by
\[
(f(y))(i) := f_i(y)
\]
for all \( y \in d \) and all \( i \in I \). Then \( p_j(f(y)) = (f(y))(j) = f_j(y) \) hence \( p_j \circ f = f_j \) for all \( j \).
Remark 10.14. The construction of the product $\prod_{i \in I} a_i$ in $\mathsf{Set}$ (Example 10.13) implicitly assumes that $I \neq \emptyset$. Convince yourself that when $I = \emptyset$ then $\bigcup_{i \in I} a_i = \emptyset$ and consequently $\prod_{i \in I} a_i$ is the collection of all maps from $\emptyset$ to $\emptyset$. Hence $\prod_{i \in \emptyset} a_i$ is the one elements set $\{\emptyset \to \emptyset\}$. And one element sets are terminal in $\mathsf{Set}$.


Last time:
- Defined a basis $\mathcal{B}$ of a topology $\mathcal{T}$ on a set $X$. Proved a criterion for a subset $\mathcal{B}$ of the power set $\mathcal{P}(X)$ to be a basis for some topology. This allows us to start with a subset $\mathcal{B} \subseteq \mathcal{P}(X)$ and get a topology out of it.
- Proved that the category $\mathsf{Top}$ of topological spaces has products and coproducts of any pair of objects.
- Defined a product $(\prod_{i \in I} a_i, \{p_i : c \to a_i\})$ of a family $\{a_i\}_{i \in I}$ of objects in a category $\mathcal{C}$. It’s characterized (up to a unique isomorphism) by the following universal property: for any object $d \in \mathcal{C}$ and any family of morphisms $\{d \xrightarrow{f_i} a_i\}_{i \in I}$, there exists a unique morphism $d \xrightarrow{f} c$ so that $p_i \circ f = f_i$ for all $i \in I$.

Example 11.1. Let $(P, \leq)$ be a poset and $\{x_i\}_{i \in I}$ a family of elements in $P$. A lower bound for the family is an element $y \in P$ such that for any $i \in I$, $y \leq x_i$. The greatest lower bound of the family, also referred to as the infimum (inf), is an element $z \in P$ such that

1. $z$ is a lower bound for the family;
2. for any other lower bound $y \in P$ of the family $\{x_i\}_{i \in I}$, $y \leq z$.

Now we think of $P$ as a category.

Claim: the product of the family $\{x_i\}_{i \in I}$ is the infimum $\inf_{i \in I}\{x_i\}$.

Proof. The morphisms $\inf_{i \in I}\{x_i\} \xrightarrow{p_j} x_j$ are the “inequalities” $\inf_{i \in I}\{x_i\} \leq x_j$. If $y \in P$ with $y \xrightarrow{q_j} x_j$ for all $i \in I$ or, equivalently, if $y \leq x_j$ for all $j \in I$, then by definition of infimum, $y \leq \inf_{i \in I}\{x_i\}$. The universal property follows from transitivity of $\leq$: there is a unique morphism (there can be at most one morphism between any two objects) $y \xrightarrow{q} \inf_{i \in I}\{x_i\}$ since $y \leq \inf_{i \in I}\{x_i\}$, and we know that $y \leq \inf_{i \in I}\{x_i\} \leq x_j$ for any $j \in I$, therefore, $q \circ p_j = q_j$.

For instance, take $P = \mathbb{R}$ and $\leq$ the usual order. Then $\inf\{x \mid x > 0\} = 0$ but $\inf\{x \mid x < 0\}$ does not exists.

If $X$ is an arbitrary set, take $P = \mathcal{P}(X)$, the powerset, with $\subseteq$ as the partial order:

$$A \leq B \iff A \subseteq B.$$ 

Then given a family $\{A_i\}_{i \in I} \subseteq \mathcal{P}(X)$ of subsets of $X$

$$\inf_{i \in I}\{A_i\} = \bigcap_{i \in I} A_i$$

is the product of the family in the category/poset $(\mathcal{P}(X), \subseteq)$.

Example 11.2. (products in $\mathsf{Vect}_\mathbb{R}$).

I claim that the category $\mathsf{Vect}_\mathbb{R}$ of real vector spaces has (arbitrary) products. Here is a construction. Let $\{V_i\}_{i \in I}$ be a family of vector spaces. If $I$ is empty, we take the product to be the terminal object in $\mathsf{Vect}_\mathbb{R}$, which is the zero dimensional vector space $\{0\}$.

Otherwise we proceed as follows. Let

$$\prod_{i \in I} V_i = \{x : I \to \bigcup_{i \in I} V_i \mid x(i) \in V_i \text{ for every } i \in I\} = \{(x_i)_{i \in I} \mid x_i \in V_i\}$$

32
be the set that will underly the product in \( \text{Vect}_\mathbb{R} \) that we are constructing. Thinks of it as the set of “sequences” of vectors. The scalar multiplication and addition are defined “coordinate-wise”: for any \( \lambda \in \mathbb{R} \), for any \( x, y \in \prod_{i \in I} V_i \), \( (\lambda x)_i = \lambda x_i \) and \( (x + y)_i = x_i + y_i \). This makes \( \prod_{i \in I} V_i \) into a vector space. The projections
\[
 p_j : \prod_{i \in I} V_i \to V_j \quad \text{are defined by } p_j(x) = x_j \text{ for all } j \in I.
\]
By the definition of scalar multiplication and addition, the projections are linear.

It remains to check the universal property. Suppose \( W \) is a vector space with a collection of linear maps \( \{ T_i : W \to V_i \}_{i \in I} \). We define \( T : W \to \prod_{i \in I} V_i \) by
\[
 T(w) := (T_i(w))_{i \in I}
\]
for all \( w \in W \). It is not hard to check that \( T \) is linear.

Arbitrary products exist in the category \( \text{Top} \) of topological spaces. To prove their existence we need some preparation.

**Lemma 11.3.** Let \( X \) be a set, \( S \) a collection of subsets of \( X \) with the property that \( \bigcup_{S \in S} S = X \). Then
\[
 B := \{ B \subset X \mid B \text{ is an intersection of finitely many elements of } S \}
\]
is a basis for a topology on \( X \).

**Proof.** The proof is an exercise. Hint: check that for any \( B_1, B_2 \in B \) the intersection \( B_1 \cap B_2 \) is again in \( B \). Then apply Lemma 10.2

**Definition 11.4.** Let \( (X, T) \) be a topological space. A subset \( S \subseteq T \) is a subbasis of \( T \) if the collection
\[
 B := \{ S_{i_1} \cap \cdots \cap S_{i_k} \mid k > 0, S_{i_1}, \ldots, S_{i_k} \in S \}
\]
is a basis for \( T \). That is, \( S \) is a subbasis for \( T \) iff every open set in \( T \) is a union of finite intersection of elements of \( S \).

**Remark 11.5.** In this language Lemma 11.3 says that if we have a set \( X \) and a collection \( S \) of subsets of \( X \) so that \( \bigcup_{S \in S} S = X \) then \( S \) is a subbasis for a topology \( T \) on \( X \) and the set \( B := \{ B \subset X \mid B \text{ is an intersection of finitely many elements of } S \} \) is a basis for this topology \( T \).

**Example 11.6.** (products in \( \text{Top} \)).

The category \( \text{Top} \) of topological spaces has arbitrary products. This can be seen as follows. Suppose \( \{ (X_i, T_i) \}_{i \in I} \) is a family of topological spaces indexed by \( I \). If \( I = \emptyset \) we take the product \( \prod_{i \in I} (X_i, T_i) \) to be a terminal object in \( \text{Top} \), which is a 1-point set with the only possible topology.

Now suppose \( I \neq \emptyset \). Let \( X = \prod_{i \in I} X_i \), a product in \( \text{Set} \). Let \( p_i : X \to X_i \) be the projections. We need to give \( X \) a topology so that the projections are continuous and so that \( (X, \{ p_i \}_{i \in I}) \) has the required universal property. This topology is commonly called the product topology. We denote it by \( T_{\text{prod}} \).

Since we want the projections \( p_j : \prod_{i \in I} X_i \to X_j \) to be continuous, for any index \( j \in I \) and for any open set \( U \subset X_j \) the preimages \( p_j^{-1}(U) \) must be in \( T_{\text{prod}} \). So consider
\[
 S := \{ p_j^{-1}(U) \mid j \in I, U \subset X_j \text{ open } \}.
\]
The set \( S \) is not a basis for a topology on \( X \) but it is a subbasis since \( \bigcup_{S \in S} S = X \) (why?). Hence \( B := \{ p_{i_1}^{-1}(U_{i_1}) \cap \cdots \cap p_{i_k}^{-1}(U_{i_k}) \mid k > 0, i_1, \ldots, i_k \in I, U_{i_1}, \ldots, U_{i_k} \text{ open in } X_{i_1}, \ldots, X_{i_k} \text{ respectively} \} \) is a basis for a topology \( T_{\text{prod}} \) on the \( \prod_{i \in I} X_i \).
It remains to check that \((\prod_{i \in I} X_i, T_{\text{prod}})\), \(\{p_j : \prod_{i \in I} X_i \to X_j\}_{j \in I}\) is a product of \(\{(X_i, T_{X_i})\}\) in the category \(\text{Top}\). So suppose \((W, T_W)\) is a topological space with a family of continuous functions \(\{f_j : W \to X_j\}_{j \in I}\). Then by the universal property of the product in \(\text{Set}\) we have a function \(f : W \to \prod_{i \in I} X_i\) with \(p_j \circ f = f_j\) for all \(j\). Explicitly \(f(w) = (f_j(w))_{j \in I}\) for all \(w \in W\). By Lemma 10.2 in order check the continuity of \(f\) it’s enough to check that preimages under \(f\) of elements of the basis \(B\) are open in \(W\). This is not hard: for all integer \(k \geq 1\), for all \(j_1, \ldots, j_k \in I\), and for all \(U_{j_1} \in T_{j_1}, \ldots, U_{j_k} \in T_{j_k}\), we have

\[
\left( \bigcap_{i=1}^k p_{j_i}^{-1}(U_{j_i}) \right) = \bigcap_{i=1}^k (p_{j_i} \circ f)^{-1}(U_{j_i}) = \bigcap_{i=1}^k f^{-1}(p_{j_i})^{-1}(U_{j_i}) = \bigcap_{i=1}^k f^{-1}_{j_i}(U_{j_i}),
\]

which is open in \(W\) since each function \(f_{j_i}\) is continuous.

There are categories that have finite products but no infinite products. First a definition.

**Definition 11.7.** A subcategory \(B\) of a category \(C\) is a subcollection \(B_0 \subseteq C_0\) of objects and a subcollection \(B_1 \subseteq C_1\) of morphisms which are closed under the structure maps source, target \(s, t : C_1 \to C_0\), unit \(u : C_0 \to C_1\) and the composition \(\circ : C_2 = \{((\alpha, \beta) \in C_1 \times C_2 \mid s(\alpha) = t(\beta))\} \to C_1\). Thus

- for any \(x \xrightarrow{\alpha} y \in B_1\) the objects \(x\) and \(y\) are in \(B_1\),
- for any \(x \in B_0\), \(\text{id}_x \in B_1\) and
- for any \(x \xrightarrow{\alpha} y\), \(y \xrightarrow{\beta} z\) in \(B_1\) the composite \(x \to \beta \circ \alpha z\) is also in \(B_1\).

**Example 11.8.** The category \(\text{Ab}\) of abelian groups is a subcategory of \(\text{Group}\).

The category \(\text{FinSet}\) of finite sets is a subcategory of \(\text{Set}\).

The category \(\text{FinAb}\) of finite abelian groups is a subcategory of \(\text{Ab}\).

**Remark 11.9.** If \(B\) is a subcategory of \(C\), then there is an inclusion functor \(i : B \to C\), \(i(a \xrightarrow{\ell} b) = (a \xrightarrow{\ell} b)\)

The functor \(i\) is injective on both objects and morphisms. In particular, \(i\) is faithful. Note that the definition of a subcategory does not require the functor \(i\) to be full.

**Exercise 11.10.** The category \(\text{FinSet}\) of finite sets does not have finite products: there is an infinite

- set \(I\) and
- and a collection \(\{A_i\}_{i \in I}\) of finite sets indexed by \(I\) so that \((\prod_{i \in I} A_i, \{p_i\}_{i \in I})\) does not exist in \(\text{FinSet}\). Note that this is not completely obvious.

**Remark 11.11.** The image of a faithful functor \(F : A \to B\) need not be a subcategory. Here is an example:

\[
\begin{array}{ccccc}
& b & \xrightarrow{f} & c & \xrightarrow{g} & \text{d} \\
\downarrow & & & \downarrow & & \downarrow \\
\text{a} & \xrightarrow{F} & \text{F(b)} = \text{F(c)} & \xrightarrow{F(f)} & \text{F(d)} \\
& & \downarrow & & \downarrow \\
& & \text{F(A)} & \xrightarrow{\text{F(g)}} & \text{F(d)}
\end{array}
\]

The image of \(F\) is not a subcategory because the composite \(F(g) \circ F(f)\) is not in the image of \(F\) while \(F(f)\) and \(F(g)\) are and are composable.

Last time:
- In a poset the product $\prod_{i \in I} x_i$ is $\inf_{i \in I} x_i$, the greatest lower bound of the set $\{x_i\}_{i \in I}$.
- In the category $\textbf{Vect}$ of real vector spaces arbitrary products exist.
- Arbitrary products also exist in the category $\textbf{Top}$ of topological spaces.
- The notion of a subcategory.

Remark 12.1. Recall that $\left( \prod_{i \in I} a_i, \{p_j : \prod_{i \in I} a_i \to a_j\}_{j \in I} \right)$ is a product in $\mathcal{C}$ if for any $d \in \mathcal{C}$ and $\{f_i : d \to a_i\}_{i \in I}$, there exists a unique morphism $f : d \to \prod_{i \in I} a_i$ such that $p_j \circ f = f_j$ for any $j \in I$. If the category $\mathcal{C}$ is locally small then this is equivalent to: for any $d \in \mathcal{C}$ the map

$$p_* : \text{Hom}_\mathcal{C}(d, \prod_{i \in I} a_i) \to \prod_{i \in I} \text{Hom}_\mathcal{C}(d, a_i), \quad p_*(f) = (p_i \circ f)_{i \in I}$$

is a bijection. Note that the product on the right hand side is the Cartesian product of sets.

The remark can be used to solve Exercise 11.10. See supplementary text at the end of the lecture.

Coproducts

Definition 12.2. Let $\mathcal{C}$ be a category, $\{a_i\}_{i \in I}$ a family of objects in $\mathcal{C}$ indexed by $I$. A coproduct of $\{a_i\}_{i \in I}$, if it exists, is their product in the opposite category $\mathcal{C}^{\text{op}}$. Explicitly, the coproduct is and object $c$ of $\mathcal{C}$ together with a family of morphisms $\{\iota_j : a_j \to c\}_{j \in I}$ so that given any object $d$ of $\mathcal{C}$ and any family of morphisms $\{f_j : a_j \to d\}_{j \in I}$, there exists a unique morphism $f : c \to d$ such that $f \circ \iota_j = f_j$ for every $j$, ie. the diagram

$$\begin{array}{ccc}
\vspace{5pt} & a_j & \\
\iota_j & \downarrow & f_j \\
\downarrow & & \downarrow \\
c & \exists f & d
\end{array}$$

commutes (for every $j$).

Since coproducts are products in the opposite category, coproducts (when they exist) are unique up to a unique isomorphism. We write $\left( \bigsqcup_{i \in I} a_i, \{\iota_j : a_j \to \bigsqcup_{i \in I} a_i\}_{j \in I} \right)$ or just $\bigsqcup_{i \in I} a_i$ for “the” coproduct of $\{a_i\}_{i \in I}$.

Example 12.3. Coproducts exist in the category $\textbf{Set}$ of sets; they are disjoint unions. For example given a collection $\{a_i\}_{i \in I}$ of sets, we can defined their disjoint union to be

$$\bigsqcup_{i \in I} a_i = \bigcup_{i \in I} a_i \times \{i\}$$

together with a family of functions $\iota_j : a_j \to \bigsqcup_{i \in I} a_i$ which are given by $\iota_j(x) := (x, j)$. Given a collection of functions $\{f_j : a_j \to d\}_{j \in I}$, we define $f : \bigsqcup_{i \in I} a_i \to d$ by $f(x, j) = f_j(x)$ for every $j \in I$ and $x \in a_j$.

Example 12.4. Coproducts exist in the category $\textbf{Vect}$ of vector spaces; they are direct sums and are usually constructed as follows. Given a collection $\{V_i\}_{i \in I}$ of vector spaces, let

$$\bigoplus_{i \in I} V_i = \left\{ (x_i)_{i \in I} \in \prod_{i \in I} V_i \mid x_i = 0 \text{ for all but finitely many } i \in I \right\}$$

The linear maps $\iota_j : V_j \to \bigoplus_{i \in I} V_i$ are given by $\iota_j(v) = (x_i)_{i \in I}$ where

$$x_i = \begin{cases} 
0 & \text{if } i \neq j \\
v & \text{if } i = j
\end{cases}$$
Given a family of linear maps $T_j : V_j \rightarrow W$ the corresponding map $T : \bigoplus_{i \in I} V_i \rightarrow W$ should satisfy $T \circ J = T_j$. So we define $T$ by

$$T((x_i)_{i \in I}) = \sum_{i \in I} T_i(x_i).$$

Note that because only finitely many $x_j$ are not zero in any “tuple” $(x_i)_{i \in I} \in \bigoplus_{i \in I} V_i$, the sum on the right makes sense.

**Remark 12.5.** The free vector space $F(X)$ on a set $X$ is the direct sum $\bigoplus_{x \in X} \mathbb{R}$ (one copy of $\mathbb{R}$ for each $x \in X$). That is, $F(X)$ is the coproduct in $\text{Vect}$ of the family $\{V_i\}_{i \in X}$ of vector spaces with $V_i = \mathbb{R}$ for all $i \in X$.

**Remark 12.6.** For any category $\mathcal{C}$ if $I = \emptyset$ then the coproduct $\bigcup_{i \in \emptyset} a_i$ is a terminal object in $\mathcal{C}^{\text{op}}$, hence an initial object in $\mathcal{C}$.

**Example 12.7.** If $(P, \leq)$ is a poset and $\{a_i\}_{i \in I}$ a family of elements in $P$. Then $\bigcup_{i \in I} a_i$, if it exists, is the least upper bound of the collection $\{a_i\}_{i \in I}$.

**Example 12.8.** Coproducts exist in the category $\text{Top}$: given a collection $\{(X_i, T_i)\}_{i \in I}$ of topological spaces, their coproduct is $(\bigcup_{i \in I} X_i, T)$ where the topology $T$ is defined by

$$U \in T \iff U \cap X_j \in T_j \text{ for each } j.$$

Here we identified each $X_j$ with the corresponding subset of the coproduct $\bigcup_{i \in I} X_i$. Note that the topology $T$ is generated by the basis $\mathcal{B} = \bigcup_{i \in I} T_i$.

---

**Size (sometimes) matters.**

So far we have been ignoring the theories of sets/collections. But sometimes the size of collections matter and we would need to be more careful. First of all, what does one mean by “size”?

**Definition 12.9.** Two collections $X$ and $Y$ have the same size (we write $|X| = |Y|$) if there exists an invertible map $f : X \rightarrow Y$. The size of $X$ is less than or equal to the size of $Y$ (we write $|X| \leq |Y|$) if there is an injective map $f : X \rightarrow Y$.

Schröder-Bernstein theorem guarantees that if $|X| \leq |Y|$ and $|Y| \leq |X|$ then $|X| = |Y|$. This is not obvious.

**Definition 12.10.** Given two collections $X$ and $Y$, we say $Y$ is strictly bigger than $X$ and write $|X| \not\leq |Y|$ if there exists an injection $X \rightarrow Y$ but no bijection between them.

**Theorem 12.11** (Cantor). For any collection $X$, $|X| \not\leq |\mathcal{P}(X)|$. That is, any collection $X$ is strictly smaller than the collection $\mathcal{P}(X)$ of all of its subcollections/subsets.

**Proof.** The function $h : X \rightarrow \mathcal{P}(X)$ given by $h(x) = \{x\}$ is an injective map. Hence $|X| \leq |\mathcal{P}(X)|$.

Suppose there exists an invertible function $f : X \rightarrow \mathcal{P}(X)$. Then $f$ is surjective. Consider

$$Y = \{x \in X \mid x \notin f(x)\} \subseteq X.$$ 

Since $f$ is surjective, $Y = f(x_0)$ for some $x_0 \in X$. Now if $x_0 \in Y$, then by construction of $Y$, $x_0 \notin f(x_0) = Y$. Contradiction.

If $x_0 \notin Y$, then by construction, $x_0 \in f(x_0) = Y$. Contradiction again. \qed

36
Supplementary material for Lecture 12.

Here are some additional remarks and examples that don’t quite fit into the lecture. You don’t need to know it for homework assignments or exams.

Remark 12.12. The coproduct of a family \( \{c_i\}_{i \in I} \) in a subcategory \( D \) of \( C \) may not agree with its coproduct in \( C \) even when both coproducts exist. Here is an example that assumes that you know what a free product of groups is.

Consider \( \text{Ab} \subseteq \text{Group} \). The coproduct in \( \text{Ab} \) is the direct sum while the coproduct in \( \text{Group} \) is the free product of groups. They generally do not agree. For instance \( \mathbb{Z} \oplus \mathbb{Z} \neq \mathbb{Z} * \mathbb{Z} \) since \( \mathbb{Z} \oplus \mathbb{Z} \) is abelian and the free product \( \mathbb{Z} * \mathbb{Z} \) is not abelian.

Proposition 12.13. Let \( \{X_n\}_{n \in \mathbb{N}} \) be a countable collection of sets with \( X_n = \{0,1\} \) for all \( n \). The product \( \prod_{n \in \mathbb{N}} X_n \) does not exist in the category \( \text{FinSet} \) of finite sets.

Proof. Suppose the product \( \prod_{n \in \mathbb{N}} X_n \) does exist in \( \text{FinSet} \). Then by Remark 12.1 for any finite set \( F \) there exists a bijection \( p_* : \text{Hom}_{\text{FinSet}}(F, \prod_{n \in \mathbb{N}} X_n) \to \prod_{n \in \mathbb{N}} \text{Hom}_{\text{FinSet}}(F, X_n) \).

For any two objects \( F, G \in \text{FinSet} \) the set of functions \( \text{Hom}_{\text{FinSet}}(F, G) \) is finite. Hence the set \( \text{Hom}_{\text{FinSet}}(F, \prod_{n \in \mathbb{N}} X_n) \) is finite. Since \( |\text{Hom}_{\text{FinSet}}(F, X_n)| = 2^{|F|} \) (where \( |F| \) denotes the number of elements of \( F \)) the Cartesian product \( \prod_{n \in \mathbb{N}} \text{Hom}_{\text{FinSet}}(F, X_n) \) is the set of sequences \( (2^{|F|})^\mathbb{N} \) in \( 2^{|F|} \). It is not be a finite set. Contradiction. \( \square \)

Remark 12.14. Let \( \mathcal{A} \) be a subcategory of \( \mathcal{B} \) and \( \{a_i\}_{i \in I} \) a collection of objects in \( \mathcal{A} \). The existence of the product of the family in \( \mathcal{A} \) does not guarantee the existence of their product in \( \mathcal{B} \). Similarly, the product of the family may exists in \( \mathcal{B} \) but not in \( \mathcal{A} \). Here are some examples:

Consider the following three categories and their embeddings (we omit identity morphisms):

\[
\mathcal{A} = \begin{array}{ccc}
f & a \\
g & b \\
\end{array} \quad \leftrightarrow \quad \mathcal{B} = \begin{array}{ccc}
f & a & h \\
g & b & k \\
l & d \\
\end{array} \quad \leftrightarrow \quad \mathcal{C} = \begin{array}{ccc}
f & a & h = f o l \\
g & b & k = g o l \\
\end{array}
\]

\( \mathcal{A} \) is a subcategory of \( \mathcal{B} \) and \( \mathcal{B} \) is a subcategory of \( \mathcal{C} \). The product of \( a \) and \( b \) is \( c \) in \( \mathcal{A} \) and \( \mathcal{C} \) but the product of \( a \) and \( b \) does not exist in \( \mathcal{B} \).

Lecture 13. Russel’s paradox, Freyd’s theorem, universes.

Last time:

- Universal property of products in terms of Hom:
  \[
p_* : \text{Hom}_\mathcal{C}(d, \prod_{i \in I} a_i) \to \prod_{i \in I} \text{Hom}_\mathcal{C}(d, a_i), \quad p_*(f) = (p_i \circ f)_{i \in I}
\]
  is a bijection.
- Coproducts. They exist in \( \text{Set}, \text{Vect}, \text{Top} \) (and many other categories).
- We informally treated the size of a collection and proved Cantor’s theorem: \( |X| \leq |\mathcal{P}(X)| \) for any collection \( X \).
**Notation 13.1.** For any two collections $X$ and $Y$, 

$$Y^X := \{ f : X \to Y \},$$

the collection of all functions from $X$ to $Y$.

**Lemma 13.2.** For any collection $X$, there is a bijection between the powerset $\mathcal{P}(X)$ and $2^X$, where $2 = \{0, 1\}$, the set with exactly two elements.

**Proof.** Given $Y \in \mathcal{P}(X)$, the characteristic function $\chi_Y$ of $Y$ is defined by

$$\chi_Y(x) = \begin{cases} 0 & \text{if } x \notin Y \\ 1 & \text{if } x \in Y \end{cases}.$$ 

This gives us a map from $\mathcal{P}(X)$ to $2^X$. For any function $f \in 2^X$, $f^{-1}(1) \in \mathcal{P}(X)$. This gives us a map the other way.

It’s not hard to check that the maps $Y \mapsto \chi_Y$ and $f \mapsto f^{-1}(1)$ are inverses of each other. □

**Theorem 13.3** (Russel’s Paradox). The assumption that the collection of all sets is a set leads to a contradiction. Equivalently the collection of all sets cannot be a set.

**Proof.** Suppose the collection $S$ of all sets (i.e., the collection of objects in $\text{Set}$) is a set. Let $A = \{ X \in S \mid X \notin X \}$, the collection of sets that are not members of themselves. This collection is not empty since, for example, $\emptyset \in A$.

Since $S$ is a set and $A$ is a subcollection of $S$, $A$ is a set (any subcollection of a set is a set). We now ask if $A$ is an element of $A$.

If $A \in A$, then $A \notin A$ by definition of $A$. Contradiction.

If $A \notin A$, then $A \in A$ by definition of $A$. Contradiction again.

We conclude that the collection $S$ of all sets cannot be a set. □

**Theorem 13.4** (Freid). Let $C$ be a category. Suppose that for any function $c : C_1 \to C_0$, $c(i) = c_i$, the product $\prod_{i \in C_1} c_i$ exists. Then $C$ is a preorder: for any two objects $a, b \in C$, the collection $\text{Hom}_C(a, b)$ has at most one element.

**Proof.** Suppose $C$ is not a preorder. Then there exist two objects $a, b \in C_0$ and two morphisms $f, g : a \to b$ so that $f \neq g$. Consider the function $j : C_1 \to C_0$ given by $j(i) = b$ for any $i \in C_1$. Let $(\prod_{i \in C_1} j(i), \{p_k : \prod_{i \in C_1} b \to b\}_{k \in C_1})$ be the corresponding product. That is, $\prod_{i \in C_1} j(i) = \prod_{i \in C_1} b$.

By the universal property of products, we have a bijection

$$\text{Hom}_C(a, \prod_{i \in C_1} b) \to \prod_{i \in C_1} \text{Hom}_C(a, b).$$

Hence $|\text{Hom}_C(a, \prod_{i \in C_1} b)| = |\prod_{i \in C_1} \text{Hom}_C(a, b)|$. Since $\text{Hom}(a, b)$ has at least two elements,

$$\left| \prod_{i \in C_1} \text{Hom}_C(a, b) \right| \geq 2^{\left| C_1 \right|}.$$

On the other hand, $\text{Hom}_C(a, \prod_{i \in C_1} b) \subseteq C_1$, so

$$\left| \text{Hom}_C(a, \prod_{i \in C_1} b) \right| \leq \left| C_1 \right|.$$

Hence $\left| C_1 \right| \geq 2^{\left| C_1 \right|}$, which contradicts Cantor’s theorem: for any collection $X$

$$|X| \leq |\mathcal{P}(X)| = 2^{|X|}. $$
Therefore, \( \mathcal{C} \) is a preorder.

\[\Box\]

**Example 13.5.** The category \( \text{Vect} \) of vector spaces is not a preorder, so it has no products indexed by \( \text{Vect}_1 \). Note that \( \text{Vect}_1 \) is at least as big as the collection \( S \) of all sets, which is too big to be a set. So non-existence of products indexed by \( \text{Vect}_1 \) does not contradict existence of products in \( \text{Vect} \) indexed by sets.

The most widely taught theory of collections is the Zermelo-Fraenkel (ZF) set theory. ZF is not designed to deal with collections that are too big to be sets. There are other theories of collections. For example, the von Neumann-Bernays-Gödel (NBG) set theory allows, in addition to sets, larger collections called (proper) classes. In NBG the collection of objects and the collection of morphisms of the category \( \text{Set} \) are classes in NBG. But in NBG a large enough collection of classes is not a class, which will cause us problems. We will soon see that given two categories \( \mathcal{C}, \mathcal{D} \), the collection of all functors \([\mathcal{C}, \mathcal{D}]\) from \( \mathcal{C} \) to \( \mathcal{D} \) can be turned into a category. If \( \mathcal{C} = \text{Vect} \) and \( \mathcal{D} = \text{Set} \), the objects and morphisms of \([\text{Vect}, \text{Set}]\) will be too big to be classes of NBG.

But do we really need the collections of all sets, groups, vector spaces etc? In practice, all we need are large enough sets of these objects, which is what Grothendieck’s universes give us. We want a set \( V \) big enough so that \( \mathbb{N} \in V \), all subsets of \( \mathbb{N} \) are in \( V \) and all sets on can build from \( \mathbb{N} \) are in \( V \): the powerset \( \mathcal{P}(\mathbb{N}) \), its powerset \( \mathcal{P}(\mathcal{P}(\mathbb{N})) \), the union \( \mathbb{N} \cup \mathcal{P}(\mathbb{N}) \cup \mathcal{P}(\mathcal{P}(\mathbb{N})) \) and so on.

**Definition 13.6** (Following McLarty, *Elementary Categories, Elementary Toposes*). A **Grothendieck universe** is a set \( V \) which satisfies the Zermelo-Fraenkel axioms. More precisely

1. The set \( \mathbb{N} \) of natural numbers is in \( V \), ie \( \mathbb{N} \in V \);
2. if \( y \in x \) and \( x \in V \), then \( y \in V \);
3. if \( x, y \in V \), then \( \{x, y\} \in V \);
4. if \( x \in V \) then the power set \( \mathcal{P}(x) \in V \) and the union \( \bigcup_{y \in x} y \in V \);
5. if \( x \in V \) and \( f : x \to y \) is a surjection then \( y \in V \).

It is known that one cannot deduce from ZF axioms the existence of a Grothendieck universe. So we adopt the following axiom:

Every set \( x \) is a member of some universe \( V \).

Note that in particular every universe \( V \) is a member of some larger universe \( V' \).

**Definition 13.7.** Fix a universe \( V \). A **\( V \)-set** is a member (element) of \( V \). A category \( \mathcal{C} \) is **\( V \)-small** if \( \mathcal{C}_0 \) and \( \mathcal{C}_1 \) are \( V \)-sets, i.e., members of \( V \). A category \( \mathcal{C} \) is call **\( V \)-locally small** if for any \( a, b \in \mathcal{C}_0 \), \( \text{Hom}_\mathcal{C}(a, b) \) is \( V \)-small.

**Example 13.8.** Fix a universe \( V \). We have a (\( V \)-)locally small category \( \text{Set} \) of \( V \)-sets: the objects of \( \text{Set} \) are elements of \( V \) and the collection of objects \( \text{Set}_0 \) is a subset of \( V \). Similarly, we have (\( V \)-)locally small category \( \text{Group} \) of groups, \( \text{Mon} \) of monoids, \( \text{Vect}_\mathbb{R} \) of real vector spaces, etc., whose underlying sets are \( V \)-sets.

From now on our convention is that any collection is a (\( V \)-)set for some universe \( V \). Most of the time we won’t need to keep track of universes.

**Lecture 14. Limits, diagrams, equalizers.**

**Last time:**
• Russell’s paradox: the collection of all sets is not a set

• Freyd’s theorem: if a category $\mathcal{C}$ has products indexed by the collection $\mathcal{C}_1$ of all morphisms in $\mathcal{C}$ then $\mathcal{C}$ is a preorder.

• Grothendieck’s universe and Grothendieck’s axiom: every set is a member of some universe.

**Definition 14.1.** A category $\mathcal{C}$ is discrete if the only morphisms in $\mathcal{C}$ are the identity morphisms:

\[
\text{Hom}_\mathcal{C}(a, b) = \begin{cases} 
\emptyset & \text{if } a \neq b \\
\{\text{id}_a\} & \text{if } a = b
\end{cases}
\]

for any two objects $a, b$ of $\mathcal{C}$.

**Remark 14.2.** Any set $C$ gives rise to a discrete category $\mathcal{C}$: the set of objects of $\mathcal{C}$ is $C$, and the set of morphisms in $\mathcal{C}$ is $\mathcal{C}_1 = \{\text{id}_a \mid a \in C\}$. And any function between two sets $f : C \to D$ gives rise to a functor $f : \mathcal{C} \to \mathcal{D}$, where $f_0 = f$ and

\[
f_1 \left(a \xrightarrow{\text{id}_a} a\right) = f(a) \xrightarrow{\text{id}_{f(a)}} f(a).
\]

This gives us a functor $i : \text{Set} \to \text{Cat}$, $i(C \xrightarrow{f} D) = C \xrightarrow{f} D$.

**Exercise 14.3.** Prove that the functor $i : \text{Set} \to \text{Cat}$ of Remark 14.2 is fully faithful. Explain why this allows us to identify sets with small discrete categories.

**Remark 14.4.** The notion of a discrete category now allows us define “a collection of objects $\{a_i\}_{i \in I}$ of a category $\mathcal{A}$ indexed by a set $I$” as a functor

\[
a : I \to \mathcal{A},
\]

where $I$ is the discrete category defined by the set $I$.

We are now in position to introduce limits. Limits generalize products. First we define the category of cones on a functor.

**Definition 14.5** (Cone on a functor). Let $F : I \to \mathcal{C}$ be a functor from a (small) category $I$ to a category $\mathcal{C}$. A cone on $F$ is an object $c \in \mathcal{C}$ (the vertex of the cone) together with a collection of morphisms $\{p_i : c \to F(i)\}_{i \in I}$, one for each object $i$ of $I$, so that for every morphism $i \xrightarrow{\gamma} j$ in $I$ the diagram

\[
\begin{array}{ccc}
F(i) & \xrightarrow{F(\gamma)} & F(j) \\
p_i & & p_j \\
c & \xrightarrow{\varphi} & d
\end{array}
\]

commutes.

**Definition 14.6** (The category of cones on a functor). Cones on a functor $F : I \to \mathcal{C}$ forms a category $\text{Cone}(F)$: the objects are cones on $F$ and a morphism in $\text{Cone}(F)$ from $(c, \{p_i : c \to F(i)\}_{i \in I})$ to $(d, \{q_i : d \to F(i)\}_{i \in I})$ is a morphism $c \xrightarrow{\varphi} d$ in $\mathcal{C}$ such that the diagrams

\[
\begin{array}{ccc}
F(i) & \xrightarrow{p_i} & F(i) \\
& \downarrow & \downarrow \\
& c & \xrightarrow{\varphi} & d \\
& \downarrow & \downarrow \\
& F(i) & \xrightarrow{q_i} & F(i)
\end{array}
\]

commute for every $i \in I$.

**Definition 14.7.** A limit of a functor $F : I \to \mathcal{C}$ is a terminal object in the category $\text{Cone}(F)$ of cones on $F$. That is, a limit is a cone $(L, \{\pi_i : L \to F(i)\}_{i \in I})$ such that for any cone $(d, \{q_i : d \to F(i)\}_{i \in I})$
on $F$, there exists a unique morphism $f : d \to L$ making the diagram 
\[
d \xrightarrow{f} L \quad \xrightarrow{\pi_i} F(i)
\]
commute for every $i \in I$.

**Remark 14.8.** Since a limit of a functor $F : I \to C$ (if it exists) is a terminal object in $\text{Cone}(F)$, limits are unique up to a unique isomorphism in $\text{Cone}(F)$. We write $(\lim F, \{\pi_i : \lim F \to F(i)\}_{i \in I})$ or just $\lim F$ to denote "the" limit of a functor $F$. Several other notations are used for limits.

**Example 14.9 (Products are limits).** If $F : I \to C$ is a functor and $I$ is a discrete category, then the limit of $F$ is the product $\prod_{i \in I} F(i)$. Why?

**Definition 14.10 (Equalizers).** An **equalizer** in a category $C$ is a limit of a functor $F : I \to C$ where $I$ is the category with two objects $x, y$ and two non-identity morphisms $\alpha, \beta : x \to y$:

$$I = x \xleftarrow{\alpha} y \xrightarrow{\beta} y.$$

Let’s unpack the definition. A functor $F : I \to C$ is the same data as $F(x) \xrightarrow{F(\alpha)} F(y)$, i.e., a pair of objects $F(x), F(y)$ and a pair of parallel morphisms $F(\alpha), F(\beta) : F(x) \to F(y)$ in the category $C$. A cone on $F$ is an object $c \in C$ and a pair of morphisms $q_x : c \rightarrow F(x)$ and $q_y : c \rightarrow F(y)$ so that the triangles

$$\xymatrix{ c \ar[dr]^{q_x} \ar[rr]^{F(\alpha)} & & F(y) \ar[dl]_{q_y} \ar[ll]_{F(\beta)} }$$

and

$$\xymatrix{ c \ar[dr]^{q_x} \ar[rr]^{F(\alpha)} & & F(y) \ar[dl]_{q_y} \ar[ll]_{F(\beta)} }$$

commute. Note that $q_y$ is completely determined by $q_x$ since

$$q_y = F(\alpha) \circ q_x = F(\beta) \circ q_x.$$

So a cone on $F : x \xleftarrow{\alpha} y \to C$ is a morphism $c \xrightarrow{q} F(x)$ so that $F(\alpha) \circ q = F(\beta) \circ q$.

The limit of $F$ is then an object $L$ of $C$ together with a morphism $\pi : L \to F(x)$ so that

- $F(\alpha) \circ \pi = F(\beta) \circ \pi$;
- if $q : c \to F(x)$ is a cone on $F$, then there exists a unique morphism $m : c \to L$ in $C$ so that

$$\xymatrix{ c \ar[r]^{m} \ar@{-->}[d]_{q} & L \ar[r]_{\pi} & F(x) }$$

commutes.

Before proceeding to examples it will be useful to record a definition.
Definition 14.11. A diagram of shape $I$ in a category $C$ is a functor $F : I \to C$.

A diagram in a category $C$ is a diagram of shape $I$ for some (possibly unspecified) category $I$. Thus a diagram in $C$ is a functor with values in $C$.

We think of the category $I$ in Definition 14.11 as an indexing category. It is usually small and often finite.


Remark 14.13. A functor $F : I \to C$ where $I = \xymatrix{x \ar@<0.5ex>[r]^{\alpha} & y \ar@<0.5ex>[l]^{\beta}}$, is the same set of data as a pair of objects $a, b \in C$ and a pair of morphisms $f, g : a \to b$.

[Given a functor $F$ let $a = F(x), b = F(y), f = F(\alpha)$ and $g = F(\beta)$. Conversely given $f, g : a \to b$ in $C$ define $F(x \xrightarrow{\alpha} y) = a \xrightarrow{f} b, F(x \xrightarrow{\beta} y) = a \xrightarrow{g} b, F(id_x) = id_a$ and $F(id_y) = id_b$. It is easy to check that $F$ so defined is a functor.]

In other words there is a bijection between functors $F : I \to C$ and, well, diagrams like this:

\[
\begin{array}{ccc}
a & \xrightarrow{f} & b \\
\downarrow{g} & & \\
\end{array}
\]

It is a common abuse of terminology to talk about an equalizer of a diagram $\xymatrix{a \ar@<0.5ex>[r]^{f} & b \ar@<0.5ex>[l]^{g}}$ in a category $C$. What is meant by this a limit of the corresponding functor. Again, we don’t assume that $f = g$.

An equalizer of $\xymatrix{a \ar@<0.5ex>[r]^{f} & b \ar@<0.5ex>[l]^{g}}$ is often displayed as $\xymatrix{L \ar[r]^\pi & a \ar@<0.5ex>[r]^{f} & b \ar@<0.5ex>[l]^{g}}$ and its universal property

as $\exists m \xymatrix{d \ar[r]^q \ar[dr]_m & a \ar@<0.5ex>[r]^{f} & b \ar@<0.5ex>[l]^{g}}$.

Example 14.14. Kernels of homomorphisms of groups are equalizers. More precisely let $f : G \to H$ be a homomorphism of groups and $i : \ker f \to G$ the inclusion homomorphism. Then $i : \ker f \to G$ is an equalizer of the diagram

\[
\begin{array}{ccc}
G & \xrightarrow{f} & H \\
\downarrow{e} & & \\
\end{array}
\]

in the category Group of groups. Here $e : G \to H$ is the map that sends everything to the identity $e_H \in H$.

Proof. Let $K$ be a group and $\psi : K \to G$ a homomorphism so that $e \circ \psi = f \circ \psi$. Then for any $k \in K$,

\[
f(\psi(k)) = e(\psi(k)) = e_H.
\]

Hence $\psi(k) \in \ker f$ for all $k \in K$. In other words $\psi$ factors (uniquely!) as $\psi = i \circ \varphi$ where $i$ is the inclusion and $\varphi : K \to \ker f$ “is” $\psi : \varphi(k) = \psi(k)$ for all $k \in K$. \qed
Proposition 14.15. Equalizers exist in $\mathsf{Set}$. That is, given any two sets $X, Y$ and any two functions $f, g : X \to Y$, an equalizer of the diagram $\xymatrix{X \ar@{>->}[r]_-f & Y \ar@{>->}[l]^-g}$ exists in $\mathsf{Set}$. 

Proof. Let $L = \{ x \in X \mid f(x) = g(x) \} \subseteq X$ and let $i : L \to X$ be the inclusion map: $i(x) = x$. If $D$ is any set and $q : D \to X$ is a function with $f \circ q = g \circ q$ then $f(q(d)) = g(q(d))$ for every $d \in D$. Hence $q(D) \subseteq L$. Therefore $q : D \to X$ factors uniquely as $q = i \circ m$ where $m$ “is” $q$: $m(d) = q(d)$ for all $d \in D$. 

Proposition 14.16. Equalizers exist in $\mathsf{Vect}$: Given any two vector spaces $V, U$ and any two linear maps $T, S : V \to U$, an equalizer of the diagram $\xymatrix{V \ar@{>->}[r]_-T & U \ar@{>->}[l]^-S}$ exists in $\mathsf{Vect}$. 

Proof. Let $L = \{ v \in V \mid T(v) = S(v) \}$. It is not hard to check that $L$ is a subspace of $V$. Then the inclusion map $i : L \to V$, $i(x) = x$, is linear. 

The rest of the proof is the same as our proof of Proposition 14.15. If $W$ is any vector space and $R : W \to V$ is a linear map with $T \circ R = S \circ R$ then $T(R(w)) = S(R(w))$ for every $w \in W$. Hence $R(W) \subseteq L$. Therefore $R : W \to V$ factors uniquely as $R = i \circ m$ where $m$ “is” $R$: $m(w) = R(w)$ for all $w \in W$. 

Proposition 14.17. Equalizers exist in $\mathsf{Top}$: Given any two topological spaces $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ and any two continuous maps $f, g : X \to Y$, the equalizer of the diagram $\xymatrix{X \ar@{>->}[r]_-f & Y \ar@{>->}[l]^-g}$ exists in $\mathsf{Top}$. 

Proof. Let $L = \{ x \in X \mid f(x) = g(x) \}; L$ is a subset of $X$. Give $L$ the subspace topology: $U \subseteq L$ is open iff there is an open set $\tilde{U} \subseteq X$ with $U = \tilde{U} \cap L$. Then the inclusion map $i : L \hookrightarrow X$ is continuous since $i^{-1}(\tilde{U}) = \tilde{U} \cap L$ for any open set $\tilde{U} \subseteq X$. 

Suppose that $Z$ is a topological space and $h : Z \to X$ a continuous function with $f \circ h = g \circ h$. Then $h(Z)$ is contained in $L$ hence the function $h$ factors (uniquely) as $h = m \circ i$ where $m$ “is” $h$. It remains to check that $m : Z \to L$ is continuous. Let $U \subseteq L$ be open. Then there is an open set $\tilde{U} \subseteq X$ with $U = L \cap \tilde{U} = i^{-1}(\tilde{U})$. Consequently

$$m^{-1}(U) = m^{-1}(i^{-1}(\tilde{U})) = (i \circ m)^{-1}(\tilde{U}) = h^{-1}(\tilde{U})$$

and $h^{-1}(\tilde{U})$ is open since $h$ is continuous. 


Last time:

- Defined the category $\mathsf{Cone}(F)$ of cones on a functor $F$.
- Defined a limit of a functor $F$ as a terminal object in $\mathsf{Cone}(F)$. This amounts to: a limit of a functor $F : I \to \mathcal{C}$ is a pair $(L, \{ \pi_i : L \to F(i) \}_{i \in I_0})$ where $L$ is an object of $\mathcal{C}$ and $\{ \pi_i : L \to F(i) \}_{i \in I_0}$ is a collection of morphisms of $\mathcal{C}$ so that

  - for any morphism $i \to j$ in $I$ the diagram

    \[
    \xymatrix{ L \ar@{.>}[dr]_{\pi_i} \ar@{>->}[r]^-{F(i)} & F(j) \ar@{>->}[l]^-{F(j)} \ar@{.>}[ur]_{\pi_j} \\
    \} \]

    (i.e., $(L, \{ \pi_i : L \to F(i) \}_{i \in I_0})$ is a cone on $F$) and
– for any cone \((c, \{q_i : c \to F(i)\}_{i \in I})\) on \(F\) there is a unique morphism \(f : c \to L\) so that
\[
\begin{array}{c}
c \\[0.5cm]
\downarrow \quad f \\[0.5cm]
L
\end{array}
\begin{array}{c}
q_i \\[0.5cm]
\downarrow \quad \pi_i \\[0.5cm]
F(i)
\end{array}
\]
commute for every \(i \in I\).

- Defined diagrams of shape \(I\) in a category \(C\): they are functors \(F : I \to C\) for a small category \(I\).
- If \(I\) is a discrete category (i.e., a set) then a diagram \(F : I \to C\) is a collection of objects of \(C\) indexed by \(I\).
- Defined equalizers in a category \(C\): they are limits of functors of the form
\[
F : x \begin{array}{c} \alpha \end{array} y \rightarrow C
\]
or, equivalently, they are limits of diagrams of the form \(a \begin{array}{c} \alpha \end{array} b\).

- We saw that kernels of group homomorphisms are equalizers. We also proved that equalizers exist in \(\text{Set}, \text{Vect}\) and \(\text{Top}\).

**Exercise 15.1.** An equalizer \(e \quad \pi \rightarrow a\) of \(a \begin{array}{c} f \end{array} b\) is monic.

**Definition 15.2** (Fiber product). A fiber product in a category \(C\) is a limit of a functor 
\[F : I \rightarrow C\]
where \(I = \begin{array}{c} x \\ y \\ z \end{array}\) is a category with three objects \(x, y, z\) and two non-identity morphisms
\[
\begin{aligned}
\alpha &: x \to z \\
\beta &: y \to z.
\end{aligned}
\]
Equivalently a fiber product \(a \times_b c \equiv a \times_{f,b,g} c\) in a category \(C\) is a limit of the diagram
\[
\begin{array}{c} \alpha \end{array} a \rightarrow b
\]

Let’s unpack the definition. A cone on the diagram \(a \begin{array}{c} f \end{array} b\) is an object \(d\) of \(C\) together with a triple of morphisms \(\varphi_a : d \to a, \varphi_b : d \to b, \varphi_c : d \to c\) so that the diagram
\[
\begin{array}{c}
d \\[0.5cm]
\downarrow \quad \varphi_c \\
\varphi_a \\
\downarrow \quad \varphi_b \\
a \\[0.5cm]
\downarrow \quad f \\
b
\end{array}
\]
commutes. Note that since \(\varphi_b = f \circ \varphi_a = g \circ \varphi_c\) we may omit \(\varphi_b\).

Therefore a limit of the diagram \(a \begin{array}{c} f \end{array} b\) is an object \(a \times_b c\) of \(C\) together with a pair of morphisms \(p_a : a \times_b c \to a, p_c : a \times_b c \to c\) so that
• the diagram \( \begin{array}{ccc} a \times_b c & \xrightarrow{p_c} & c \\ \downarrow{p_a} & & \downarrow{g} \\ a & \xrightarrow{f} & b \end{array} \) commutes;

• given a commuting diagram \( \begin{array}{ccc} d & \xrightarrow{c} & \varphi_c \\ \downarrow{\varphi_a} & & \downarrow{g} \\ a & \xrightarrow{f} & b \end{array} \), there exists a unique morphism \( d \xrightarrow{\varphi} a \times_b c \) so that the diagram \( \begin{array}{ccc} d & \xrightarrow{\varphi_c} & c \\ \downarrow{\varphi_a} & & \downarrow{g} \\ a \times_b c & \xrightarrow{p_c} & c \\ \downarrow{p_a} & & \downarrow{f} \\ a & \xrightarrow{f} & b \end{array} \) commutes.

Proposition 15.3. Fiber products exist in the category \( \text{Set} \) of sets and functions.

Proof. Let \( f : X \to Z \) and \( g : Y \to Z \) be two functions. Consider the set

\[ X \times_Z Y = \{(x, y) \mid f(x) = g(y)\}. \]

We define \( \pi_X : X \times_Z Y \to X \) by \( \pi_X(x, y) = x \) and \( \pi_Y : X \times_Z Y \to Y \) by \( \pi_Y(x, y) = y \). These functions are restriction of the projections from \( X \times Y \) to \( X \) and \( Y \), respectively.

We check the universal properties. So suppose we are given two functions \( \varphi_X : W \to X \) and \( \varphi_Y : W \to Y \) with \( f \circ \varphi_X = g \circ \varphi_Y \). By the universal property of the product \( (X \times Y, p_X : X \times Y \to X, p_Y : X \times Y \to Y) \) there exists a unique function \( \varphi : W \to X \times Y \) such that \( p_X \circ \varphi = \varphi_X \) and \( p_Y \circ \varphi = \varphi_Y \). Since \( f \circ \varphi_X = g \circ \varphi_Y \), for every \( w \in W \), \( \varphi(w) = (\varphi_X(w), \varphi_Y(w)) \) satisfies \( f(\varphi_X(w)) = g(\varphi_Y(w)) \), that is, \( \varphi(w) \in X \times_Z Y \). Hence \( \varphi : W \to X \times_Z Y \) is the desired unique function. \( \square \)

Remark 15.4. The fiber product \( a \times_b c \) of the diagram \( \begin{array}{ccc} b & \xrightarrow{g} & c \\ \downarrow{a} & & \downarrow{f} \\ a & \xrightarrow{f} & c \end{array} \) depends on the functions \( f \) and \( g \) even if we omit them from our notation.

Remark 15.5. Given a diagram \( \begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\iota} & & \downarrow{id_Y} \\ X & \xrightarrow{f} & Y \end{array} \) in \( \text{Set} \), the fiber product \( X \times_{f, Y, \text{id}_Y} Y \) “is” the preimage \( f^{-1}(Y) \). This is because the diagram

\[ \begin{array}{ccc} f^{-1}(Y) & \xrightarrow{f} & Y \\ \downarrow{\iota} & & \downarrow{id_Y} \\ X & \xrightarrow{f} & Y \end{array} \]

commutes (where \( \iota \) is the inclusion map) and has the desired universal property which we will now check.

Suppose we are given any pair of functions \( \varphi_X : W \to X \) and \( \varphi_Y : W \to Y \) such that \( f \circ \varphi_X = \text{id}_Y \circ \varphi_Y \). Then \( f(\varphi_X(w)) = \varphi_Y(w) \in Y \) for all \( w \in W \). Hence

\[ \varphi_X(w) \in f^{-1}(\varphi_Y(w)) \subseteq f^{-1}(Y). \]
Therefore $\varphi_X : W \to X$ factors as
\[ \varphi_X = \iota \circ \varphi \]
where $\varphi : W \to f^{-1}(Y)$ “is” $\varphi_X$ (that is, $\varphi(w) = \varphi_X(w)$ for all $w \in W$). Since
\[ \varphi_Y(w) = f(\varphi_X(w)) = f(\varphi(w)) \]
$\varphi_Y = f \circ \varphi$ and the diagram
\[
\begin{array}{ccc}
W & \xrightarrow{\varphi} & Y \\
\downarrow{\varphi_X} & & \downarrow{f} \\
X & \xrightarrow{id_Y} & Y
\end{array}
\]
commutes. Hence $f^{-1}(Y) = X \times_{f, Y, id_Y} Y$.

Remark 15.6. If $f : X \to Z$ is the inclusion of a subset and $g : Y \to Z$ is a function, then the fiber product is $X \times_{f, Z, g} Y = \{ y \in Y \mid g(y) \in X \} = g^{-1}(X)$, with $\pi_X : g^{-1}(X) \to X$ and $g^{-1}(X) \xrightarrow{i} Y$ $\pi_Y : g^{-1}(X) \to Y$ being $g$ and the inclusion $i$, respectively: the diagram
\[
\begin{array}{ccc}
X & \xrightarrow{f} & Z \\
\downarrow{g} & & \downarrow{g} \\
Y & \xrightarrow{id_Y} & Y
\end{array}
\]
and has the desired universal properties.

Note that if $g(Y) \cap f(X) = \emptyset$ then the fiber product $X \times_{f, Z, g} Y$ is the empty set.

**Proposition 15.7.** Fiber products exist in Top.

**Proof.** Let $f : X \to Z$ and $g : Y \to Z$ be a pair of continuous functions (morphisms in Top.) Consider the set
\[ X \times_{f, Z, g} Y = \{ (x, y) \in X \times Y \mid f(x) = g(y) \} \]
and give it the subspace topology (the topology on the product $X \times Y$ is, of course, the product topology). It is not hard to check that the functions $\pi_X : X \times_{f, Z, g} Y \to X$, $\pi_X(x, y) = x$ and $\pi_Y : X \times_{f, Z, g} Y \to Y$, $\pi_Y(x, y) = y$ are continuous and that $(X \times_{f, Z, g} Y, \{ \pi_X, \pi_Y \})$ has the desired universal property. \qed

**Definition 15.8.** A category $C$ is **complete** if for any small category $I$, and any functor $F : I \to C$, the limit of $F$ exists.

It is often useful to know that a given category is complete. In the next lecture we will prove the following theorem:

**Theorem 15.9.** If a category $C$ has all equalizers and all small products (i.e., products of families of objects indexed by sets), then $C$ is complete.

**Example 15.10.** The category Set of sets and functions has all products and equalizers hence is complete.

**Example 15.11.** The category Top of topological spaces has all products and equalizers hence is complete.
Example 15.12. The category $\text{Vect}$ of real vector spaces and linear maps has equalizers and all products. Hence $\text{Vect}$ is complete.

Supplementary material for Lecture 15.

Example 15.13. The category $\text{Group}$ of groups and homomorphisms is complete.

Sketch of proof. We need to check that $\text{Group}$ has arbitrary small products and that it has equalizers.

Let $\{G_i\}_{i \in I}$ be a family of groups indexed by a set $I$. Consider the set

$$\prod_{i \in I} G_i = \left\{ x : I \to \bigcup_{i \in I} G_i \mid x(i) \in G_i \text{ for all } i \in I \right\} = \{(x_i)_{i \in I} \mid x_i \in G_i \text{ for all } i \in I\}.$$

Define multiplication on $\prod_{i \in I} G_i$ “coordinate-wise”:

$$(x_i) \cdot (y_i) := (x_i y_i).$$

It is not hard to check that the product $\prod_{i \in I} G_i$ with this multiplication is a group (what is the multiplicative identity?) and that the projections $p_j : \prod_{i \in I} G_i \to G_j$, $p_j((x_i)) := x_j$ are homomorphisms. The universal properties of the product follow easily.

Next we take up the existence of equalizers. So let $f, k : G \to H$ be a pair of homomorphisms. Consider

$$L = \{ g \in G \mid f(g) = k(g) \}.$$

Then $L$ is a subgroup of $G$ and the inclusion map $\iota : L \hookrightarrow G$ is an equalizer of $G \xrightarrow{f} H \xleftarrow{k} H$. □

Lecture 16. Completeness of categories with products and equalizers.

Last time:

- Defined fiber products in a category. Showed that fiber products exist in $\text{Set}$, $\text{Top}$, ...
- Defined a category $C$ to be complete if it has all small limits.
- Stated but didn’t prove Theorem 15.9 if a category $C$ has all small products and equalizers then $C$ is complete.

Remark 16.1. By Freyd’s theorem any small complete category is a preorder. Since $\text{Top}$ is complete and is not a preorder, it cannot be small. Similarly $\text{Vect}$ cannot be small.

Proof of Theorem 15.9. Let $C$ be a category with all products and equalizers, $I$ a small category, and $F : I \to C$ a functor. We’d like to show that the limit of $F$ exists in $C$.

Recall that for any category $D$ we have the source and target functions $s, t : D_1 \to D_0$. They are defined by

$$s(x \xrightarrow{\gamma} y) = x, \quad t(x \xrightarrow{\gamma} y) = y$$

for any morphism $x \xrightarrow{\gamma} y$ in $D$. By our assumptions on $C$ the products

$$\left( \prod_{i \in I_0} F(i), \left\{ p_j : \prod_{i \in I_0} F(i) \to F(j) \right\}_{j \in I_0} \right) \quad \text{and} \quad \left( \prod_{\gamma \in I_1} F(t(\gamma)), \left\{ \pi_\alpha : \prod_{\gamma \in I_1} F(t(\gamma)) \to F(t(\alpha)) \right\}_{\alpha \in I_1} \right)$$

exist in $C$. We next construct a pair of morphisms

$$\prod_{i \in I_0} F(i) \xrightarrow{S} \prod_{\gamma \in I_1} F(t(\gamma))$$

47
and take their equalizer.

Since \( \left( \prod_{\gamma \in I_1} F(t(\gamma)), \{\pi_\alpha\}_{\alpha \in I_1} \right) \) is a product, a morphism into \( \prod_{\gamma \in I_1} F(t(\gamma)) \) is uniquely determined by the set of morphisms into its factors. Given a morphism \( s(\alpha) \xrightarrow{\alpha} t(\alpha) \) in \( I_1 \), we have a morphism \( p_{t(\alpha)} : \prod_{i \in I_0} F(i) \to F(t(\alpha)) \) in \( C \). Therefore, there exists a unique morphism \( T : \prod_{i \in I_0} F(i) \to \prod_{\gamma \in I_1} F(t(\gamma)) \) with

\[
\pi_\alpha \circ T = p_{t(\alpha)}
\]

for all \( \alpha \in I_1 \). For every \( s(\alpha) \xrightarrow{\alpha} t(\alpha) \) in \( I_1 \) is also a morphism \( F(\alpha) \circ p_{s(\alpha)} : \prod_{i \in I_0} F(i) \to F(t(\alpha)) \) in \( C \). Therefore there exists a unique morphism \( S : \prod_{i \in I_0} F(i) \to \prod_{\gamma \in I_1} F(t(\gamma)) \) with

\[
\pi_\alpha \circ S = F(\alpha) \circ p_{s(\alpha)}
\]

for all \( \alpha \in I_1 \).

By assumption there is an equalizer \( L \xrightarrow{q} \prod_{i \in I_0} F(i) \) of the diagram \( \xymatrix{ \prod_{i \in I_0} F(i) \ar[r]^-S \ar[dr]^-{T} & \prod_{\gamma \in I_1} F(t(\gamma)) \ar[d]^-{} \cr & } \). So in particular,

\[
T \circ q = S \circ q.
\]

For each \( i \in I_0 \) define \( q_i : L \to F(i) \) by

\[
q_i = p_i \circ q
\]

We now argue that \( (L, \{q_i\}_{i \in I}) \) is a limit cone of the functor \( F \). Let \( i \xrightarrow{\alpha} j \) be a morphism in \( I \). We need to check that the diagram \( \xymatrix{ L \ar[dr]^-{q_j} \ar[r]^-{q_i} & F(i) \ar[r]^-{F(\alpha)} \ar[d]^-{q_j} & F(j) \cr & } \) commutes. Since \( \ref{eq:16.12} \) holds

\[
\pi_\alpha \circ S \circ q = \pi_\alpha \circ T \circ q
\]

for all \( \alpha \in I_1 \). Since \( \pi_\alpha \circ S = F(\alpha) \circ p_{s(\alpha)} \) and \( \pi_\alpha \circ T = p_{t(\alpha)} \)

\[
F(\alpha) \circ p_{s(\alpha)} \circ q = p_{t(\alpha)} \circ q.
\]

Since \( s(\alpha) = i \) and \( t(\alpha) = j \) the above equation amounts to

\[
F(\alpha) \circ p_i \circ q = p_j \circ q.
\]

Since \( q_i = p_i \circ q \) and \( q_j = p_j \circ q \) this just says that \( \xymatrix{ L \ar[dr]^-{q_j} \ar[r]^-{q_i} & F(i) \ar[r]^-{F(\alpha)} \ar[d]^-{q_j} & F(j) \cr & } \) commutes. We conclude that \( (L, \{q_i\}_{i \in I}) \) is a cone on \( F \). It remains to check that the cone is terminal.

Consider a cone \( (c, \{r_i : c \to F(i)\}_{i \in I}) \) on \( F \). By the universal property of the product \( (\prod_{i \in I} F(i), \{p_i\}) \), there exists a unique morphism \( r : c \to \prod_{i \in I} F(i) \) with

\[
p_i \circ r = r_i
\]

for every object \( i \) of \( I \).

**Claim** \( S \circ r = T \circ r \).
Proof of claim. Since \((c, \{r_i : c \to F(i)\})_{i \in I}\) is a cone on \(F\), the diagram \[
\begin{array}{c}
F(i) \\
\downarrow r_i \quad \downarrow r_j \\
F(\alpha) \\
\end{array}
\]
commutes for every morphism \(\alpha\) in \(I\). That is,
\[
(16.16) \quad r_{t(\alpha)} = F(\alpha) \circ r_{s(\alpha)}
\]
for all morphisms \(\alpha\) in \(I\). Now
\[
\begin{align*}
\pi_\alpha & \circ S \circ r = F(\alpha) \circ p_{s(\alpha)} \circ r \\
& = F(\alpha) \circ r_{s(\alpha)} \quad \text{by } 16.15 \\
& = r_{t(\alpha)} \quad \text{by } 16.16 \\
& = p_{t(\alpha)} \circ r \quad \text{by } 16.15 \\
& = \pi_\alpha \circ T \circ r \quad \text{by } 16.10
\end{align*}
\]
for every \(\alpha\). Since \((\prod_{\gamma \in I_1} F(t(\gamma)), \{\pi_\alpha\})\) is a product, \(S \circ r = T \circ r\) as desired. \(\square\)

Since \((L, q)\) is an equalizer of \(\prod_{i \in I_0} F(i) \xrightarrow{S} \prod_{\gamma \in I_1} F(t(\gamma))\), there is a unique morphism \(m : c \to L\) so that
\[
(16.17) \quad q_i \circ m = p_i \circ q \circ m = p_i \circ r = r_i
\]
for all \(i \in I_0\). Hence the diagrams \[
\begin{array}{c}
F(i) \quad \downarrow q_i \\
\downarrow r_i \\
\Pi_{i \in I_0} F(i)
\end{array}
\]
commute for all \(i \in I_0\). Hence \(m\) gives rise to a morphism of cones \(m : (c, \{r_i\}) \to (L, \{q_i\})\). Conversely any morphism of cones \(m : (c, \{r_i\}) \to (L, \{q_i\})\) will make \((16.17)\) commute and therefore has to be unique.

Therefore, \((L, \{q_i\})\) is terminal in \(\text{Cone}(F)\), that is, it is the limit of the functor \(F : I \to \mathcal{C}\). \(\square\)

Example 16.2. Let’s work out an example of what the proof of Theorem 15.9 would produce when \(I = x \xrightarrow{\alpha} y \xrightarrow{\beta} z\). That is, when \(I\) is a category with three objects \(x, y, z\) and two non-identity morphisms \(\alpha : x \to z\) and \(\beta : y \to z\). For concreteness (no pun intended) let’s take \(\mathcal{C} = \text{Set}\). In other words let’s see what the proof will construct when we apply the proof to the diagram \(X \xrightarrow{f} Z \xrightarrow{g} Y\) is \(\text{Set}\). Note that once we identify the indexing category \(I\) with \(F(I) = \{X \xrightarrow{f} Z \xrightarrow{g} Y\} \subset \text{Set}\), the functor \(F\) becomes the inclusion. The construction should, of course, produce the/a fiber product \(X \times_{f, Z, g} Y\).

Note first that \(I_0 = \{X, Y, Z\}\) and that \(I_1 = \{f, g, \text{id}_X, \text{id}_Y, \text{id}_Z\}\). Hence the two products are
\[
\prod_{i \in I_0} F(i) = X \times Y \times Z
\]
and \[ \prod_{\gamma \in I_1} F(t(\gamma)) = t(\text{id}_X) \times t(\text{id}_Y) \times t(\text{id}_Z) \times t(f) \times t(g) = X \times Y \times Z \times Z \times Z. \]

The function \( S : \prod_{i \in I_0} F(i) \to \prod_{\gamma \in I_1} F(t(\gamma)) \) is defined by \( \pi_\alpha \circ S = F(\alpha) \circ p_s(\alpha) \) for every \( \alpha \in I_1 \).

Hence
\[
\begin{align*}
(\pi_{\text{id}_X} \circ S)(x, y, z) &= (F(\text{id}_X) \circ p_X)(x, y, z) = x \\
(\pi_{\text{id}_Y} \circ S)(x, y, z) &= (F(\text{id}_Y) \circ p_Y)(x, y, z) = y \\
(\pi_{\text{id}_Z} \circ S)(x, y, z) &= (F(\text{id}_Z) \circ p_Z)(x, y, z) = z \\
(\pi_f \circ S)(x, y, z) &= (F(f) \circ p_X)(x, y, z) = f(x) \\
(\pi_f \circ S)(x, y, z) &= (F(g) \circ p_Y)(x, y, z) = g(y)
\end{align*}
\]

for every \((x, y, z) \in X \times Y \times Z\). Therefore
\[
S(x, y, z) = (x, y, z, f(x), g(y)).
\]

Similarly, \( T : \prod_{i \in I_0} F(i) \to \prod_{\gamma \in I_1} F(t(\gamma)) \) is defined by
\[
\pi_\alpha \circ T = p_s(\alpha)
\]

for every \( \alpha \in I_1 \). Hence
\[
\begin{align*}
(\pi_{\text{id}_X} \circ T)(x, y, z) &= p_X(x, y, z) = x \\
(\pi_{\text{id}_Y} \circ T)(x, y, z) &= p_Y(x, y, z) = y \\
(\pi_{\text{id}_Z} \circ T)(x, y, z) &= p_Z(x, y, z) = z \\
(\pi_f \circ T)(x, y, z) &= p_Z(x, y, z) = z \\
(\pi_f \circ T)(x, y, z) &= p_Z(x, y, z) = z.
\end{align*}
\]

Therefore
\[
T(x, y, z) = (x, y, z, z, z).
\]

The equalizer \( L \) of \( \prod_{i \in I_0} F(i) \xrightarrow{T} \prod_{\gamma \in I_1} F(t(\gamma)) \) is then the set
\[
L = \{(x, y, z) \mid S(x, y, z) = T(x, y, z)\}
\]
\[
= \{(x, y, z) \mid (x, y, z, f(x), g(y)) = (x, y, z, z, z)\}
\]
\[
= \{(x, y, z) \mid f(x) = z = g(y)\}
\]

together with the inclusion map \( L \mapsto X \times Y \times Z \). The limit cone \( \{q_i : L \to F(i)\}_{i \in \{X,Y,Z\}} \) is
\[
\begin{array}{c}
L \\
\downarrow q_x \\
Y \\
\downarrow q_y \\
X \\
\downarrow q_z \\
Z
\end{array}
\]

Note that there is a bijection \( h \) from \( L' = \{(x, y) \mid f(x) = g(y)\} \) to \( L = \{(x, y, z) \mid f(x) = z = g(y)\} \):
\[
h(x, y) = (x, y, f(x)) = (x, y, g(y)).
\]
Lecture 17. Colimits and coequalizers.

Last time:
- Proved that if a category $\mathcal{C}$ has all equalizers and all small products then it has all small limits: for any functor $F : I \to \mathcal{C}$, where $I$ is a small category, the limit $\lim I F$ exists.

We now tackle colimits. They are defined by dualizing limits.

**Definition 17.1.** Let $F : I \to \mathcal{C}$ be a functor and $F^{\text{op}} : I^{\text{op}} \to \mathcal{C}^{\text{op}}$ the corresponding functor between the opposite categories. A cocone on $F$ is a cone on/over $F^{\text{op}}$ in $\mathcal{C}^{\text{op}}$. A colimit of $F$ is the limit of $F^{\text{op}}$.

Equivalently and more explicitly:

**Definition 17.2.** A cocone on a functor $F : I \to \mathcal{C}$ is the pair $(c, \{q_i : F(i) \to c\}_{i \in I})$ so that for every morphism $i \overset{f}{\to} j$ in $I$ the diagram

\[
\begin{array}{ccc}
F(i) & \xrightarrow{p_i} & F(j) \\
\downarrow{F(f)} & & \downarrow{p_j} \\
c & \xrightarrow{q_i} & c
\end{array}
\]

commutes.

A morphism of cocones from $(c, \{q_i : F(i) \to c\}_{i \in I})$ to $(c', \{q'_i : F(i) \to c'\}_{i \in I})$ is a morphism $c \xrightarrow{f} c'$ in $\mathcal{C}$ such that for every $i \in I$, the diagram

\[
\begin{array}{ccc}
c & \xrightarrow{f} & c' \\
\downarrow{q_i} & & \downarrow{q'_i} \\
F(i) & \xrightarrow{p_i} & F(j)
\end{array}
\]

commutes.

Cocones and their morphism form a category $\text{Cocone}(F)$.

A colimit of a functor $F : I \to \mathcal{C}$ is an initial object in $\text{Cocone}(F)$. That is, it is a cocone $(d, \{i_i : F(i) \to d\}_{i \in I})$ such that for any cocone $(c, \{q_i : F(i) \to c\}_{i \in I})$ on $F$, there exists a unique morphism $d \xrightarrow{f} c$ so that for every $i \in I$, the diagram

\[
\begin{array}{ccc}
d & \xrightarrow{f} & c \\
\downarrow{i_i} & & \downarrow{q_i} \\
F(i) & \xrightarrow{p_i} & F(j)
\end{array}
\]

commutes for every $i \in I$.

**Remark 17.3.** Since colimits of a functor $F : I \to \mathcal{C}$ are initial in $\text{Cocone}(F)$, colimits are unique up to a unique isomorphism in $\text{Cocone}(F)$.

**Notation 17.4.** We write $(\text{colim} F, \{p_i : F(i) \to \text{colim} F\}_{i \in I})$ or just $\text{colim} F$ to denote “the” colimit of a functor $F$.

**Example 17.5.** If $I$ is a discrete category the colimit of a functor $F : I \to \mathcal{C}$ is the coproduct of $\{F(i)\}_{i \in I}$: $\text{colim} F = \bigsqcup_{i \in I} F(i)$.

**Example 17.6.** Let $I = \emptyset$, the empty category. The colimit of $F : \emptyset \to \mathcal{C}$ is an object $e \in \mathcal{C}$ such that for every $c \in \mathcal{C}$, there exists a unique morphism $q : e \to c$. Therefore, $\text{colim}(F : \emptyset \to \mathcal{C})$ in an initial object in $\mathcal{C}$.

**Example 17.7.** Let $X$ be a set, $R \subseteq X \times X$ an equivalence relation, $X/R$ the set of equivalence classes of $R$, and $q : X \to X/R$, $q(x) = [x]$ the quotient map. We have two functions $p_1, p_2 : R \to X$ given by $p_1(x_1, x_2) = x_1$ and $p_2(x_1, x_2) = x_2$ which give us a diagram

\[
(17.18) \quad R \xrightarrow{p_1, p_2} X.
\]

**Claim:** The quotient map $q : X \to X/R$ is a colimit of the the diagram $(17.18)$. Hence quotients are colimits.

51
Lemma 17.12. Set \(\mathbf{Set}\) has coequalizers. \(\mathbf{Set}\) is an example of a category with all small colimits and all small coproducts by Theorem 15.9.

Proof. Suppose \(\xymatrix{ C \ar[r]^-{f} & R \ar[r]_-{p_1}^\sim & X \ar[r]_-{p_2} & X }\) is a cocone in \(\mathbf{Set}\) on the inclusion functor \(F : \left\{ \begin{array}{c} R \ar[r]^-{p_1} & X \end{array} \right\} \rightarrow \mathbf{Set}\). Then \(g \circ p_1 = f = g \circ p_2\). Hence if \((x_1, x_2) \in R\) then \(g(x_1) = g(x_2)\). Therefore, there exists a well-defined map \(\bar{g} : X/R \rightarrow C\) given by \(\bar{g}(q(x)) = g(x)\) for any equivalence class \([x] = q(x)\). Moreover, such map \(\bar{g} : X/R \rightarrow C\) is unique: if \(h : X/R \rightarrow C\) is another function so that \(h \circ q = g\), then \(h([x]) = g(x) = \bar{g}(q(x))\), so \(h = \bar{g}\).

Definition 17.8. Let \(C\) be a category, \(a, b \in C\) two objects, \(f, g : a \rightarrow b\) two morphisms. The coequalizer of \(a \xleftarrow{f} b\) in \(C\) is the colimit of the corresponding functor.

That is, the coequalizer of a diagram \(a \xleftarrow{f} b\) in a category \(C\) is an object \(d \in C\) and a morphism \(b \xrightarrow{g} d\) with the following universal property: given an objects \(c \in C\) and a morphism \(b \xrightarrow{h} c\) so that \(h \circ f = h \circ g\), there exists a unique morphism \(d \xrightarrow{m} c\) so that the diagram \(\begin{array}{ccc} a & \xrightarrow{f} & b & \xrightarrow{g} & c \\ \downarrow{h} \downarrow{g} \downarrow{h} \downarrow{m} \end{array}\)

commutes. We picture this as

\[
\begin{array}{ccc}
  a & \xrightarrow{f} & b \\
  \downarrow{g} & \downarrow{h} & \downarrow{\circ m} \\
  b & \xrightarrow{\circ f} & c \\
\end{array}
\]

Example 17.9. Let \(X\) be a set, \(R \subseteq X \times X\) an equivalence relation, and \(q : X \rightarrow X/R\) the quotient map. Then \(q\) is the coequalizer of the diagram \((17.18)\).

Definition 17.10. A category \(C\) is cocomplete if it has all small colimits: for any small category \(I\) and any functor \(F : I \rightarrow C\), the colimit \((\text{colim}F, \{1_j : F(j) \rightarrow \text{colim}F\}_{j \in I})\) exists.

Theorem 17.11. A category \(C\) is cocomplete if and only if it has all small coproducts and all coequalizers.

Proof. A category \(C\) has all coequalizers and all small coproducts

\[
\iff C^{\text{op}} \text{ has all small products and equalizers} \\
\iff C^{\text{op}} \text{ is complete (by Theorem 15.9)} \\
\iff C \text{ is cocomplete.} \]

We would like to show that the category \(\mathbf{Set}\) of sets is cocomplete. We know that \(\mathbf{Set}\) has arbitrary small coproducts: they are disjoint unions. We need to show that \(\mathbf{Set}\) has coequalizers. We’ll construct the coequalizers in \(\mathbf{Set}\) as quotients. To carry this out we’ll need two results.

Lemma 17.12. Let \(X\) be a set and \(\{R_\alpha\}_{\alpha \in A}\) a family of equivalence relations on \(X\). Then the intersection \(S = \bigcap_{\alpha \in A} R_\alpha \subseteq X \times X\) is also an equivalence relation.

Proof. Suppose \((a, b), (b, c) \in S\), then \((a, b), (b, c) \in R_\alpha\) for every \(\alpha \in A\). Since \(R_\alpha\) is an equivalence relation, \((a, c) \in R_\alpha\) for every \(\alpha\), then \((a, c) \in \bigcap_{\alpha \in A} R_\alpha = S\). So \(S\) is transitive.

Similarly, \(S\) is reflexive and symmetric. So \(S\) is an equivalence relation on \(X\).

Corollary 17.13. Suppose \(X\) is a set and \(S \subseteq X \times X\) a relation. Let \(A\) be the set of all equivalence relation \(R\) containing \(S\). Then \(\langle S \rangle = \bigcap_{R \in A} R\) is the smallest equivalence relation containing \(S\).
Proof. The set \( A \) is not empty since \( X \times X \in A \). By Lemma 17.12 \( \langle S \rangle \) is an equivalence relation. Since for every \( R \in A \), \( S \subseteq R \), we have that \( S \subseteq \bigcap_{R \in A} = \langle S \rangle \).

If \( U \subseteq X \times X \) is an equivalence relation with \( S \subseteq U \), then \( U \in A \) hence \( \langle S \rangle = \bigcap_{R \in A} R \subseteq U \). We conclude that \( \langle S \rangle \) is the smallest equivalence relation containing \( S \).

\[ \tag{18.19} \]

Lecture 18. Cocompleteness of Set and of other categories.

Last time:

- colimits and, in particular, coequalizers;
- definition of a cocomplete category — a category is cocomplete iff it has all small limits;
- existence of coproducts and coequalizers is equivalent to cocompleteness;
- quotients of sets by equivalence relations and quotient maps are colimits, in fact, coequalizers;
- equivalence relations generated by relations.

Lemma 18.1. The category \( \text{Set} \) of sets and functions has all coequalizers, hence is cocomplete.

Proof. Consider a diagram

\[
\begin{align*}
X & \xrightarrow{f} Y \\
\downarrow{g} & \downarrow{g} \\
\end{align*}
\]

in the category \( \text{Set} \) (so \( X, Y \) are sets and \( f, g \) are functions). Let

\[
S = \{(f(x), g(x)) \in Y \times Y \mid x \in X\};
\]

it is a relation on the set \( Y \). Let \( \langle S \rangle \) be the equivalence relation generated by \( S \). As usual, we write \( y_1 \sim y_2 \) if \( (y_1, y_2) \in \langle S \rangle \). Let \( Z \) denote the set of equivalence classes defined by the equivalence relation \( \langle S \rangle \):

\[
Z = \{[y] \mid y \in Y\}.
\]

Let \( q : Y \to Z \), \( q(y) = [y] \) be the quotient map. Since \( f(x) \sim g(x) \) for all \( x \in X \),

\[
(q \circ f)(x) = q(f(x)) = q(g(x)) = (q \circ g)(x).
\]

Therefore \( q \circ f = q \circ g \).

It remains to show that \( q : Y \to Z \) is the coequalizer of the diagram (18.19). Suppose \( W \) is a set and \( h : Y \to W \) is a function such that \( h(f(x)) = g(g(x)) \) for every \( x \in X \). Let \( R = \{(y_1, y_2) \in Y \times Y \mid h(y_1) = h(y_2)\} \). Then \( R \) is an equivalence relation on \( Y \) and \( S \subseteq R \). Since \( \langle S \rangle \) is the smallest equivalence relation containing \( S \), \( \langle S \rangle \subseteq R \). Hence the function \( h \) is constant on each equivalence class of \( \langle S \rangle \). Therefore there exists a well-defined map \( \tilde{h} : Z \to W \) such that

\[
\tilde{h}(q(y)) = h(y) \text{ for all } y \in Y, \text{i.e., the diagram } Y \xrightarrow{\tilde{h}} W \text{ commutes. If } \tilde{h} : Z \to W \text{ is another function so that } \tilde{h} \circ q = h, \text{ then }
\]

\[
\tilde{h} \circ q = \tilde{h} \circ q.
\]

Since \( q \) is onto, it is an epimorphism in \( \text{Set} \). Hence \( \tilde{h} = \tilde{h} \) and we get uniqueness.

\[ \square \]

Colimits in \( \text{Vect} \), the category of vector spaces.

We have seen that coproducts in \( \text{Vect}_\mathbb{R} \) exists — they are direct sums. Therefore, to show that \( \text{Vect}_\mathbb{R} \) is cocomplete, we only need to prove that \( \text{Vect}_\mathbb{R} \) has coequalizers. To construct coequalizers, we need quotients. “Recall” the construction of a quotient vector space: Given a subspace \( U \) of a
vector space $W$, the relation $\sim$ on $W$ defined by $w_q \sim w_2 \iff w_1 - w_2 \in U$ is an equivalence relation. The equivalence classes are the cosets, that is, they are all of the form $[w] = w + U := \{w + u \mid u \in U\}$.

Denote the set of equivalence classes by $W/U$. The set $W/U$ has well-defined addition $+$ and scalar multiplication: for every $w_1, w_2, w \in W$ and $\lambda \in \mathbb{R}$,

$$(w_1 + U) + (w_2 + U) = (w_1 + w_2) + U \quad \text{and} \quad \lambda(w + U) = \lambda w + U$$

The zero vector in $W/U$ is the coset $0 + U$. Moreover, the quotient map

$$q : W \rightarrow W/U, \quad q(w) = w + U$$

is a surjective linear map.

**Proposition 18.2.** The category $\text{Vect}_\mathbb{R}$ of real vector spaces and linear maps has all coequalizers. Consequently $\text{Vect}_\mathbb{R}$ is cocomplete.

**Proof.** Consider a diagram $V \xrightarrow{T} W \xleftarrow{S} W$ in $\text{Vect}$. Let $U$ be the image of the linear map $T - S$:

$$U = (T - S)(V) = \{T(v) - S(v) \mid v \in V\}.$$ 

Let $q : W \rightarrow W/U$ be the quotient map. Then

$$(q \circ T)(v) - (q \circ S)(v) = (T(v) + U) - (S(v) + U) = (T(v) - S(v)) + U = 0 + U$$

for any vector $v \in V$. Hence $q \circ T = q \circ S$.

Suppose $X$ is a vector space and $F : W \rightarrow X$ a linear map with $F \circ T = F \circ S$. Then $F \circ (T - S) = 0$ hence $F(u) = 0$ for every $u \in U$. It follows that

$$\bar{F} : W/U \rightarrow X, \quad \bar{F}(w + U) = F(w)$$

is a well-defined linear map making the diagram $W \xleftarrow{F} X \xrightarrow{\bar{F}} W/U$ commute. Since $q : W \rightarrow Z$ is onto, it is an epimorphism in $\text{Vect}$. Consequently the map $\bar{F}$ is unique. We conclude that

$q : W \rightarrow W/((T - S)(V))$ is the coequalizer of our diagram $V \xrightarrow{T} W \xleftarrow{S} W$ in $\text{Vect}$. \qed

**Colimits in $\text{Top}$, the category of topological spaces.**

We know that the category $\text{Top}$ has all small coproducts — they are disjoint unions topologized appropriately. So to show that $\text{Top}$ is cocomplete, we only need to show that $\text{Top}$ has all coequalizers. As we have seen in the case of $\text{Set}$ and $\text{Vect}_\mathbb{R}$, the key to the existence of coequalizers is the existence of quotients.

**Lemma 18.3.** The category $\text{Top}$ of topological spaces has quotients: given an equivalence relation $R$ on a topological space $X$ there exists a topology $\mathcal{T}$ on the quotient $X/R$ so that

1. The quotient map $q : X \rightarrow X/R$ is continuous;
2. If $f : X \rightarrow Y$ is a continuous function that is constant on the equivalence classes of $R$ then there is a unique continuous function $\bar{f} : X/R \rightarrow Y$ with $f = \bar{f} \circ q$. 

54
Proof. It turns out that the right thing to do is to take $\mathcal{T}$ to be the largest topology on $X/R$ so that $q : X \to X/R$ is continuous. This amounts to setting

$$\mathcal{T} := \{ U \subseteq X/R \mid q^{-1}(U) \text{ is open in } X \}.$$  

If $\mathcal{T}$ is a topology then, of course, the function $q$ is continuous. Let’s check that $\mathcal{T}$ is a topology:

- Since $q^{-1}(\emptyset) = \emptyset$, $\emptyset \in \mathcal{T}$. Similarly, since $q^{-1}(X/R) = X$, $X/R \in \mathcal{T}$.
- Now suppose $U, V \in \mathcal{T}$. Then $q^{-1}(U \cap V) = q^{-1}(U) \cap q^{-1}(V)$ is open since $q^{-1}(U), q^{-1}(V)$ are open. Hence $U \cap V \in \mathcal{T}$.
- Finally if $\{ U_\alpha \}_{\alpha \in A} \subseteq \mathcal{T}$ then $q^{-1}(\bigcup_{\alpha \in A} U_\alpha) = \bigcup_{\alpha \in A} q^{-1}(U_\alpha)$ is open in $X$. Hence $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$. We conclude that $\mathcal{T}$ is a topology.

Suppose $f : X \to Y$ is a continuous function that is constant on the equivalence classes of the relation $R$. Then since $q : X \to X/R$ is a coequalizer of the diagram $R \xrightarrow{p_1} X \xrightarrow{p_2} X$ in $\textbf{Set}$ where $p_1, p_2$ are projections, there exists a unique function $\bar{f} : X/R \to Y$ so that $f = \bar{f} \circ q$. It remains to check that $\bar{f}$ is continuous. Let $W \subseteq Y$ be an open subset. Then

$$q^{-1}(\bar{f}^{-1}(W)) = (\bar{f} \circ q)^{-1}(W) = f^{-1}(W)$$

is open in $X$ since $f$ is continuous. By definition the topology $\mathcal{T}$, the set $\bar{f}^{-1}(W)$ is open in $X/R$. So $f$ is continuous and we are done. \[\square\]

**Definition 18.4.** The topology $\mathcal{T}$ on the quotient $X/R$ constructed above is called the **quotient topology**, $(X/R, \mathcal{T})$ is called the **quotient space** and $q : X \to X/R$ the **quotient map**.

**Proposition 18.5.** The category $\textbf{Top}$ of topological spaces has all coequalizers and therefore it is cocomplete.

**Proof.** Let $A \xrightarrow{f} B$ be a diagram in $\textbf{Top}$ and $R$ be the equivalence relation on the set underlying the space $B$ generated by $S = \{(g(a), h(a)) \in B \times B \mid a \in A\}$. Let $B/\sim$ be the corresponding quotient space and $q : B \to B/\sim$ the quotient map. Then $q : B \to B/\sim$ is the coequalizer of $A \xrightarrow{g} B$.

We conclude that $\textbf{Top}$ has all coequalizer, and consequently that $\textbf{Top}$ is cocomplete. \[\square\]

**Remark 18.6.** There are many categories that are not cocomplete. For example the category $\textbf{FinSet}$ of finite sets does not have arbitrary small coproducts. The category of finite groups does not even have binary coproducts (but this is a bit hard to prove unless you are comfortable with free groups).

**Lecture 19.** Natural transformations and functor categories.

**Last time:** Proved that the categories $\textbf{Vect}$ of real vector spaces and $\textbf{Top}$ of topological spaces are cocomplete since they have coproducts and coequalizers.

There is one more colimit that I would like to single out. It’s dual to pullbacks.

**Definition 19.1.** A **pushout** in a category $\mathcal{C}$ is the colimit of a diagram of the shape

$$\begin{array}{ccc}
A & \xrightarrow{g} & B \\
\downarrow f & & \downarrow \quad \\
\phantom{\downarrow f} & \downarrow c & \\
\phantom{\downarrow f} & \phantom{\downarrow f} & \phantom{\downarrow f}
\end{array}$$

(19.20)
That is, a pushout of (19.20) an object \( d \) of \( \mathcal{C} \) and a pair of morphisms \( q_a : a \to d, q_b : b \to d \) so that:

(a) the diagram \( \begin{array}{ccc} c & \overset{g}{\to} & b \\ \downarrow{f} & & \downarrow{q_b} \\ a & \overset{q_a}{\to} & d \end{array} \) commutes and

(b) given a commuting diagram \( \begin{array}{ccc} c & \overset{g}{\to} & b \\ \downarrow{f} & & \downarrow{q_b} \\ a & \overset{q_a}{\to} & d \\ \downarrow{h_a} & \nearrow{q} & \downarrow{h_b} \\ & & e \end{array} \) there exists a unique morphism \( d \overset{q}{\to} e \) so that

\[ \begin{array}{ccc} c & \overset{g}{\to} & b \\ \downarrow{f} & & \downarrow{q_b} \\ a & \overset{q_a}{\to} & d \\ \downarrow{h_a} & \nearrow{q} & \downarrow{h_b} \\ & & e \end{array} \]

commutes.

**Example 19.2.** Since a pushout is a colimit and since the category \( \text{Set} \) is cocomplete, pushouts exist in \( \text{Set} \). Explicitly, given three sets and two functions, \( \begin{array}{ccc} C & \overset{g}{\to} & B \\ \downarrow{f} & & \downarrow{q_b} \\ A & \overset{q_a}{\to} & \emptyset \end{array} \), the pushout of the diagram can be constructed as follows: take the quotient \((A \sqcup B)/_{\sim}\) of the disjoint union by the equivalence relation generated by

\[ S = \{(a, b) \mid a \in A \leftrightarrow A \sqcup B, b \in B \leftrightarrow A \sqcup B, a = f(c), b = g(c) \text{ for some } c \in C\}. \]

The structure maps \( \iota_A : A \hookrightarrow (A \sqcup B)/_{\sim} \) and \( \iota_B : B \hookrightarrow (A \sqcup B)/_{\sim} \) are the “inclusions” followed by the quotient map \( q : (A \sqcup B) \to (A \sqcup B)/_{\sim} \).

**Example 19.3.** Since \( \text{Top} \) is cocomplete, it has pushouts. They can be constructed as follows. Let \( \begin{array}{ccc} A & \overset{f}{\leftarrow} & B \overset{g}{\to} & C \end{array} \) be a triple of topological spaces with two continuous functions. As in the case of sets the pushout of this diagram is the quotient \((A \sqcup B)/_{\sim}\) of the coproduct \( A \sqcup B \) by the equivalence relation \( \sim \) generated by

\[ S = \{(a, b) \mid a \in A \leftrightarrow A \sqcup B, b \in B \leftrightarrow A \sqcup B, a = f(c), b = g(c) \text{ for some } c \in C\}. \]

and we give \((A \sqcup B)\) the coproduct topology and \((A \sqcup B)/_{\sim}\) the quotient topology. The structure maps \( \iota_A : A \hookrightarrow (A \sqcup B)/_{\sim} \) and \( \iota_B : B \hookrightarrow (A \sqcup B)/_{\sim} \) are again the “inclusions” followed by the quotient map \( q : (A \sqcup B) \to (A \sqcup B)/_{\sim} \).

**Natural transformations.**

**Definition 19.4.** Let \( F, G : \mathcal{C} \to \mathcal{D} \) be two functors between two categories. A **natural transformation** \( \alpha \) from \( F \) to \( G \) assigns to each object \( c \in \mathcal{C} \) a morphism \( \alpha_c : F(c) \to G(c) \) in \( \mathcal{D} \) so that for any morphism \( c \overset{\gamma}{\to} c' \), the diagram

\[ \begin{array}{ccc} F(c) & \overset{\alpha_c}{\to} & G(c) \\ \downarrow{F(\gamma)} & & \downarrow{G(\gamma)} \\ F(c') & \overset{\alpha_{c'}}{\to} & G(c') \end{array} \]

commutes.
We write \( \alpha : F \Rightarrow G \) or \( \begin{array}{c} F \\ \downarrow \alpha \\ G \end{array} \) \( \Rightarrow \) \( D \) for a natural transformation.

**Remark 19.5.** A natural transformation \( \alpha : F \Rightarrow G \) is a map from the collection of objects \( C_0 \) of \( C \) to the collection \( D_1 \) of morphisms of \( D \):

\[
\alpha : C_0 \to D_1 \quad c \mapsto \alpha_c.
\]

**Example 19.6.** (Natural transformation and cones.) Let \( C, D \) be two categories. For any object \( d \) of \( D \), we have the constant functor \( \Delta_d : C \to D \). The functor is defined by

\[
\Delta_d(c) = \gamma \mapsto d
\]

for any morphism \( \gamma \) in \( C \).

For any functor \( F : C \to D \), a natural transformation \( \alpha : \Delta_d \Rightarrow F \) assigns to each object \( c \) of \( C \) a morphism \( \alpha_c : d = \Delta_d(c) \to F(c) \) of \( D \) so that for any morphism \( c \xrightarrow{\gamma} c' \), the diagram

\[
d \xrightarrow{id_d=\Delta_d(\gamma)} d \\
F(c) \xrightarrow{\alpha_c} F(c')
\]

commutes. This is equivalent to the diagram

\[
d \xrightarrow{id} \xrightarrow{\alpha_c} \xrightarrow{\alpha_{c'}} \xrightarrow{F(\gamma)} F(c')
\]

commuting. Thus a natural transformation \( \alpha : \Delta_d \Rightarrow F \) "is" a cone on \( F \).

**Example 19.7.** Recall that we have a contravariant functor \( (\cdot)^* : \text{Vect}_\mathbb{R} \to \text{Vect}_\mathbb{R}^{\text{op}} \) defined by

\[
(V \xrightarrow{T} W) \mapsto (W^* \xrightarrow{T^*} V^*)
\]

where \( V^* = \text{Hom}(V, \mathbb{R}) \), \( W^* = \text{Hom}(W, \mathbb{R}) \), and \( T^*(l) = l \circ T \) for every \( l \in W^* \). Applying the functor \( (\cdot)^* \) again we get \( (\cdot)^* : \text{Vect}_\mathbb{R}^{\text{op}} \to (\text{Vect}_\mathbb{R}^{\text{op}})^{\text{op}} = \text{Vect} \).

On the other hand for any vector space \( V \) there is a linear evaluation map \( \text{ev}_V : V \to (V^*)^* \). It is given by

\[
(\text{ev}_V(v))(l) = l(v)
\]

for every \( v \in V \) and \( l \in V^* \). I claim that these evaluation maps assemble into a natural transformation \( \text{ev} : \text{id}_{\text{Vect}_\mathbb{R}} \Rightarrow (\cdot)^* \circ (\cdot)^* \). To prove the claim we need to check that for any linear map \( T : V \to W \), the diagram

\[
V \xrightarrow{\text{ev}_V} (V^*)^* \\
\downarrow T \\
W \xrightarrow{\text{ev}_W} (W^*)^*
\]

commutes. And the diagram does commute because for every \( l \in W^* \) and for every \( v \in V \),

\[
(T^* (\text{ev}_V(v)))(l) = \text{ev}_V(v) (T^*(l)) = (T^*(l))(v) = l(T(v)) = \text{ev}_W(T(v))(l).
\]

It is in this sense that taking double dual of a vector space is "natural."
**Definition 19.8.** Let \( F, G, H : C \to D \) be three functors between two categories. \( \alpha : F \Rightarrow G \) and \( \beta : G \Rightarrow H \) two natural transformations: \( \begin{array}{c} C \xrightarrow{\alpha} G \xrightarrow{\beta} H \end{array} \). The **vertical composition** of \( \beta \) and \( \alpha \) is the natural transformation

\[ \beta \circ \alpha : F \Rightarrow H \]

defined by

\[ (\beta \circ \alpha)_c = \beta_c \circ \alpha_c \]

for every \( c \in C \). Note that that \( \circ \) on the right hand side of the equation denotes the composition of morphisms in \( D \).

Strictly speaking we should check that \( \beta \circ \alpha \) as defined above is a natural transformation. Now for every \( c \xrightarrow{\gamma} c' \) in \( C \) in the diagram

\[
\begin{array}{ccc}
F(c) & \xrightarrow{\alpha_c} & G(c) & \xrightarrow{\beta_c} & H(c) \\
\downarrow F(\gamma) & & \downarrow G(\gamma) & & \downarrow H(\gamma) \\
F(c') & \xrightarrow{\alpha_{c'}} & G(c') & \xrightarrow{\beta_{c'}} & H(c')
\end{array}
\]

the left and right squares commute. Hence the outer square

\[
\begin{array}{ccc}
F(c) & \xrightarrow{\beta_c \circ \alpha_c} & H(c) \\
\downarrow F(\gamma) & & \downarrow H(\gamma) \\
F(c') & \xrightarrow{\beta_{c'} \circ \alpha_{c'}} & H(c')
\end{array}
\]

commutes as well, and we are done.

**Exercise 19.9.** Prove that vertical composition of natural transformation is associative. That is, given four functors \( F, G, H, K : C \to D \) and three natural transformations \( \alpha : F \Rightarrow G, \beta : G \Rightarrow H, \) and \( \gamma : H \Rightarrow K \), show that \( \gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha \).

Hint: compute the components \( (\gamma \circ (\beta \circ \alpha))_c \) and \( ((\gamma \circ \beta) \circ \alpha)_c \) at some object \( c \in C \), then use associativity of composition of morphisms in \( D \).

**Remark 19.10.** There is also the **horizontal composition** of natural transformations: given three categories, four functors, and two natural transformation like this

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C \\
\downarrow H & & \downarrow L & & \downarrow C
\end{array}
\]

there is a natural transformation

\[
\begin{array}{ccc}
A & \xrightarrow{\beta \ast \alpha} & C \\
\downarrow L \circ H & & \downarrow C
\end{array}
\]

We will define the horizontal composite \( \beta \ast \alpha \) later, in Lecture 27.
Definition 19.11. Let \( \mathcal{C} \) and \( \mathcal{D} \) be two categories. The functor category \( [\mathcal{C}, \mathcal{D}] \) is defined as follows:

- the objects of \( [\mathcal{C}, \mathcal{D}] \) are functors from \( \mathcal{C} \) to \( \mathcal{D} \);
- the morphisms of \( [\mathcal{C}, \mathcal{D}] \) are natural transformations;
- the composition of morphisms is vertical composition of natural transformation;
- the identity morphism on a functor \( F : \mathcal{C} \to \mathcal{D} \) is defined by \( (\text{id}_F)_c = \text{id}_{F(c)} \) for every \( c \in \mathcal{C} \).

Here as before \( \text{id}_{F(c)} \) denotes the identity morphism on \( F(c) \) in the target category \( \mathcal{D} \).

Remark 19.12. Another common notation for a functor category \( [\mathcal{C}, \mathcal{D}] \) is \( \mathcal{D}^\mathcal{C} \), in analogy to \( Y^X \), the notation of the set of functions from a set \( X \) to another set \( Y \).

Remark 19.13. Note that for the definition of a functor category \( [\mathcal{C}, \mathcal{D}] \) to make sense we need the associativity of vertical composition, which is guaranteed by Exercise [19.9](#).

Remark 19.14. Functor categories are important and sometimes occur somewhat unexpectedly. For example a directed graph is a functor and the category of directed graphs is a functor category. We’ll discuss graphs in the next lecture.


Last time:
- defined pushouts;
- defined natural transformations: a natural transformation \( \alpha : F \Rightarrow G \) from a functor \( F : \mathcal{C} \to \mathcal{D} \) to a functor \( G : \mathcal{C} \to \mathcal{D} \) is a map \( \alpha : \mathcal{C}_0 \to \mathcal{D}_1 \) \( c \mapsto (F(c) \xrightarrow{\alpha_c} G(c)) \), so that for any morphism \( c \xrightarrow{\gamma} c' \), the diagram \( F(\gamma) \) \( \xrightarrow{\alpha_c} \) \( G(\gamma) \) commutes;
- defined functor categories \( [\mathcal{C}, \mathcal{D}] \equiv \mathcal{D}^\mathcal{C} \): the objects are functors, morphisms are natural transformations and composition of morphisms is the vertical composition of natural transformations.

An example of a functor category is provided by the category \( \text{Graph} \) of directed graphs (sometimes also called directed multigraphs) which we presently define.

Definition 20.1. A directed graph \( \Gamma \) is a set \( \Gamma_0 \) of vertices (or nodes) a set \( \Gamma_1 \) of edges (or arcs or arrows) and a pair of functions \( s, t : \Gamma_1 \to \Gamma_0 \) that assigns to each edge its source and target.

Example 20.2. Consider the graph \( \Gamma \) given by the following data. \( \Gamma_1 = \{\alpha, \beta\} \), \( \Gamma_0 = \{x, y\} \), \( s(\alpha) = s(\beta) = x \), and \( t(\alpha) = t(\beta) = y \). This described the graph with two vertices and two parallel edges that can be drawn like this: \( \bullet \xrightarrow{\alpha} \bullet \). \( \xrightarrow{\beta} \bullet \).

Example 20.3. The data \( \Gamma_1 = \{\alpha, \beta\} \), \( \Gamma_0 = \{x, y\} \), \( s(\alpha) = s(\beta) = x \), \( t(\alpha) = x \), \( t(\beta) = y \) encodes the graph \( \xrightarrow{\alpha} \bullet \). \( \xrightarrow{\beta} \bullet \).

Example 20.4. Note that any small category \( \mathcal{C} \) has an underlying graph: forget the unit map and the composition of morphisms and you are left with a directed graph.
Remark 20.5. Every graph can be viewed as a functor from the category

\[ \mathcal{C} = 1 \xrightarrow{\sigma} \tau \xrightarrow{0}, \]

the category with two objects and two non-identity morphisms, to the category \( \text{Set} \) of sets. This can be seen as follows. Given a graph

\[ \Gamma = \Gamma_1 \xrightarrow{s} \tau \xrightarrow{t} \Gamma_0 \]

define the corresponding functor \( \Gamma : \mathcal{C} \to \text{Set} \) by \( \Gamma(1) = \Gamma_1, \Gamma(0) = \Gamma_0, \Gamma(\sigma) = s, \) and \( \Gamma(\tau) = t. \)

Conversely given a functor \( \Gamma : \mathcal{C} \to \text{Set} \) we set \( \Gamma_1 := \Gamma(1), \Gamma_0 := \Gamma(0), s := \Gamma(\sigma), t := \Gamma(\tau) \)

Viewing graphs as functors allows us to turn the collection of all graphs into a (locally small) category \( \text{Graph} \) as follows:

**Definition 20.6.** The category \( \text{Graph} \) of (directed) graphs is the functor category

\[ [1 \xrightarrow{\sigma} \tau \xrightarrow{0}, \text{Set}] \]

Let’s unpack the definition. We already know how to interpret functors as graphs. We now interpret natural transformation between functors as homomorphisms of directed graphs.

Let \( \Gamma, \Lambda : 1 \xrightarrow{\sigma} \tau \xrightarrow{0} \to \text{Set} \) be two functors/graphs and let \( f : \Lambda \Rightarrow \Gamma \) be a natural transformation. Since \( \mathcal{C} = 1 \xrightarrow{\sigma} \tau \xrightarrow{0} \) has exactly two objects, the natural transformation \( f \) consists of two functions \( f_1 : \Lambda(1) \to \Gamma(1) \) and \( f_0 : \Lambda(0) \to \Gamma(0) \) so that the following diagrams (in \( \text{Set} \)) commute. Consequently a morphism \( f : \Lambda \to \Gamma \) of graphs consists of a map \( f_1 : \Lambda_1 \to \Gamma_1 \) on edges and a map \( f_0 : \Lambda_0 \to \Gamma_0 \) on vertices that are compatible with the source and target maps: \( s_\Gamma \circ f_1 = f_0 \circ S_\Lambda; \) \( t_\Gamma \circ f_1 = f_0 \circ t_\Lambda. \) That is

\[ f \left( x \xrightarrow{\alpha} y \right) = f_0(x) \xrightarrow{f_1(\alpha)} f_0(y) \]

for any edge \( x \xrightarrow{\alpha} y \) of \( \Lambda. \)

**Natural isomorphisms.**

**Definition 20.7.** Let \( F, G : \mathcal{C} \to \mathcal{D} \) be two functors between two categories. A natural transformation \( \alpha : F \Rightarrow G \) is a natural isomorphism if it is an isomorphism in the functor category \([\mathcal{C}, \mathcal{D}]\). That is, there exists a natural transformation \( \beta : G \Rightarrow F \) such that \( \beta \circ \alpha = \text{id}_F \) and \( \alpha \circ \beta = \text{id}_G \) (where \( \circ \) denotes the vertical composition of natural transformations).

**Remark 20.8.** If \( \alpha : F \Rightarrow G \) is a natural isomorphism and \( \beta : G \Rightarrow F \) its inverse, then for every \( c \in \mathcal{C}, \alpha_c : F(c) \to G(c) \) is an isomorphism in \( \mathcal{D} \). This is because \( \beta_c : G(c) \to F(c) \) is the inverse of \( \alpha_c \) in \( \mathcal{D} \). 60
The following lemma is useful in practice.

**Lemma 20.9.** Let \( C \xrightarrow{\alpha} D \) be a natural transformation. Suppose that \( \alpha_c : F(c) \to G(c) \) is an isomorphism in \( D \) for every \( c \in C \). Then \( \alpha \) is a natural isomorphism.

**Proof.** We define the components of the (potential) inverse \( \beta \) of \( \alpha \) by

\[
\beta(c) = \beta_c = (\alpha_c)^{-1} : G(c) \to F(c)
\]

for every \( c \in C \). We need to make sure that the map \( \beta : C_0 \to D_1 \) defined by (20.21) is a natural transformation: for any morphism \( c \xrightarrow{\gamma} c' \) in \( C \) the diagram

\[
\begin{array}{ccc}
G(c) & \xrightarrow{\beta_c = (\alpha_c)^{-1}} & F(c) \\
\downarrow{G(\gamma)} & & \downarrow{F(\gamma)} \\
G(c') & \xrightarrow{\beta_{c'} = (\alpha_{c'})^{-1}} & F(c')
\end{array}
\]

(20.22)

commutes. Since \( \alpha \) is a natural transformation, the diagram

\[
\begin{array}{ccc}
F(c) & \xrightarrow{\alpha_c} & G(c) \\
\downarrow{F(\gamma)} & & \downarrow{G(\gamma)} \\
F(c') & \xrightarrow{\alpha_{c'}} & G(c')
\end{array}
\]

commutes, i.e.,

\[
\alpha_{c'} \circ F(\gamma) = G(\gamma) \circ \alpha_c.
\]

It follows that

\[
F(\gamma) \circ (\alpha_c)^{-1} = (\alpha_{c'})^{-1} \circ G(\gamma),
\]

which is exactly the commutativity of (20.22) and so we are done. \( \square \)

**Example 20.10.** Recall that for any vector space \( V \) there is a linear map \( ev_V : V \to (V^*)^* \) defined by \( (ev_V(v))(l) = l(v) \) for all \( v \in V \) and \( l \in V^* \). Recall also that these maps assemble into a natural transformation \( ev : id_{Vect} \Rightarrow (\cdot)^* \circ (\cdot)^* \).

Consider now the category \( FDVect_\mathbb{R} \) of finite-dimensional real vector spaces. It’s a subcategory of \( Vect_\mathbb{R} \). For any finite dimensional vector space \( V \) the linear map \( ev_V : V \to (V^*)^* \) is an isomorphism. (This is not entirely obvious. The key points are that (1) if \( dim V \) is finite, then \( dim V = dim V^* \), hence \( dim V = dim(V^*)^* \) and that (2) \( ev_V : V \to (V^*)^* \) is injective if \( dim V \) is finite.) By Lemma 20.9 the natural transformation \( ev : id_{FDVect_\mathbb{R}} \Rightarrow (\cdot)^* \circ (\cdot)^* \) is a natural isomorphism. That is, any finite dimensional vector space \( V \) is naturally isomorphic to its double dual \( V^{**} \).

**Definition 20.11.** Two categories \( \mathcal{C} \) and \( \mathcal{D} \) are isomorphic if there are functors \( F : \mathcal{C} \to \mathcal{D} \) and \( G : \mathcal{D} \to \mathcal{C} \) such that \( F \circ G = id_\mathcal{D} \) and \( G \circ F = id_\mathcal{C} \).

**Example 20.12.** Recall that there is the category \( Mat \) whose collection of objects is the set \( \mathbb{N} \) of natural numbers. Recall further that for any two natural numbers \( n, m \) the set \( \text{Hom}_{\text{Mat}}(n, m) \) of morphisms in \( \text{Mat} \) is the set of all \( m \times n \) matrices.

Now consider the category \( Coord \) of coordinate vector spaces (this is not a standard name nor notation). The objects of \( Coord \) are the spaces \( \mathbb{R}^n, n \geq 0 \), and for any \( n, m \geq 0 \) the set of morphisms \( \text{Hom}_{Coord}(\mathbb{R}^n, \mathbb{R}^m) \) is the set of linear maps from \( \mathbb{R}^n \) to \( \mathbb{R}^m \).
One of the basic results of linear algebra says that the categories \text{Mat} and \text{Coord} are isomorphic. The functor \( F : \text{Mat} \to \text{Coord} \) sends \( n \in \mathbb{N} \) to \( \mathbb{R}^n \) and \( m \times n \) matrix \( A = \begin{bmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{bmatrix} \) to the linear map \( \overline{A} : \mathbb{R}^n \to \mathbb{R}^m \) defined by
\[
\overline{A}(\vec{v}) = A \cdot \vec{v} \quad \text{for all} \quad \vec{v} \in \mathbb{R}^n.
\]
The inverse functor \( G : \text{Coord} \to \text{Mat} \) sends \( \mathbb{R}^n \) to the natural number \( n \) and a linear transformation \( R : \mathbb{R}^n \to \mathbb{R}^m \) to its matrix with respect to the standard basis of \( \mathbb{R}^n \) and \( \mathbb{R}^m \). Note that the matrix multiplication is defined precisely to make sure that \( F : \text{Mat} \to \text{Coord} \) is a functor: products of matrices correspond to compositions of the corresponding linear maps.

On the other hand, the categories \text{Mat} of matrices and \text{FDVect} of finite dimensional vector spaces are also “the same” categories. But they cannot be isomorphic: an isomorphism \( F : \text{Mat} \to \text{FDVect} \) would define a bijection on objects. The category \text{Mat} is countably many objects while the collection of all objects of \text{FDVect}_0 is not even a (small) set: there is a finite dimensional vector space for every finite set and the collection of all finite sets is at least as big as the collection of all (small) sets. To express the intuition that \text{Mat} and \text{FDVect} are the “same” we need a definition.

**Definition 20.13.** Two categories \( \mathcal{C} \) and \( \mathcal{D} \) are equivalent if there is a pair for functors \( F : \mathcal{C} \to \mathcal{D} \) and \( G : \mathcal{D} \to \mathcal{C} \) and a pair of natural isomorphisms \( \eta : \text{id}_\mathcal{C} \Rightarrow G \circ F \) and \( \varepsilon : \text{id}_\mathcal{D} \Rightarrow F \circ G \).

We will refer to such two functors functors \( F \) and \( G \) as weakly invertible and say that \( G \) as a weak inverse of \( F \).

In the next lecture we will prove that the categories \text{Mat} of matrices and \text{FDVect} of finite dimensional vector spaces are equivalent.

**Lecture 21. Equivalences of categories and fully faithful and essentially surjective functors.**

Last time:

- Defined directed graphs, noted that they can be viewed as functors, defined the functor category \text{Graph} of directed graphs.
- Defined natural isomorphisms.
- Defined equivalences of categories: two categories \( \mathcal{C} \) and \( \mathcal{D} \) are equivalent if there is a pair of functors \( F : \mathcal{C} \to \mathcal{D} \) and \( G : \mathcal{D} \to \mathcal{C} \) and a pair of natural isomorphisms \( \eta : \text{id}_\mathcal{C} \Rightarrow G \circ F \) and \( \varepsilon : \text{id}_\mathcal{D} \Rightarrow F \circ G \).

I also promised to prove that the categories \text{Mat} of matrices and \text{FDVect} of finite dimensional vector spaces are equivalent. We’ll do it twice. First we’ll do it directly, “by hand.” We’ll then deduce it as a consequence of a theorem.

Recall that a choice of an ordered basis \( B = \{b_1, \ldots, b_n\} \) of an \( n \)-dimensional vector space \( V \) defines an isomorphism
\[
\psi_B : \mathbb{R}^n \to V, \quad \psi_B \left( \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right) = \sum_{i=1}^{n} x_i b_i.
\]
If $W$ is an $m$-dimensional vector space, $C = \{c_1, \ldots, c_n\}$ a basis of $W$, and $T : V \to W$ is a linear map, there is a matrix $[T]_{CB} = (t_{ij})$ defined by $T(b_j) = \sum_{i=1}^{m} t_{ij}c_j$. Then for any $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$,

$$\psi_C ([T]_{CB}\vec{x}) = T (\psi_B (\vec{x})), $$

i.e., the diagram

$$\psi_B \downarrow \quad \downarrow \psi_C$$

$$\begin{array}{ccc}
V & \xrightarrow{T} & W \\
\downarrow{\psi_B} & & \downarrow{\psi_C} \\
\mathbb{R}^n & \xrightarrow{[T]_{CB}} & \mathbb{R}^m
\end{array}$$

(21.23)

commutes. Here we are confusing the matrix $[T]_{CB}$ with the linear map $\vec{x} \mapsto [T]_{CB}\vec{x}$. Furthermore, if $S : W \to U$ is another linear map and $D$ is a basis of $U$, then

$$[S \circ T]_{DB} = [S]_{DC} \cdot [T]_{CB}$$

where $\cdot$ in the right hand side denotes matrix multiplication.

We now define a functor $F : \text{Mat} \to \text{FDVect}$: for $n \in \mathbb{N} = \text{Mat}_0$, let

$$F(n) = \mathbb{R}^n.$$

Given an $m \times n$ matrix $A \in \text{Hom}_{\text{Mat}}(n, m)$, let $F(A) : \mathbb{R}^n \to \mathbb{R}^m$ be the corresponding linear map. That is,

$$F(A) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = A \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

It is easy to check that $F$ so defined is a functor.

To construct a weak inverse $G : \text{FDVect} \to \text{Mat}$ of the functor $F$ choose a basis $B_V$ for every finite-dimensional vector space $V$. If $V$ is $\mathbb{R}^n$, we choose $B_{\mathbb{R}^n}$ to be the standard basis of $\mathbb{R}^n$. Then, for any $m \times n$ matrix $A$ and the corresponding linear map $F(A) : \mathbb{R}^n \to \mathbb{R}^m$ we have

$$[F(A)]_{B_{\mathbb{R}^m}, B_{\mathbb{R}^n}} = A.$$ 

Now, define a functor $G : \text{FDVect} \to \text{Mat}$ by

$$G \left( V \xrightarrow{T} W \right) = \dim V \xrightarrow{[T]_{B_W B_V}} \dim W$$

for any linear map $T$. Again, it is not hard to check that $G$ is a functor. By construction of $F$ and $G$,

$$G \circ F \left( n \xrightarrow{A} m \right) = n \xrightarrow{A} m.$$

Therefore, $G \circ F = \text{id}_{\text{Mat}}$. On the other hand since (21.23) commutes the diagram

$$\begin{array}{ccc}
V & \xrightarrow{T} & W \\
\downarrow{\psi_V} & & \downarrow{\psi_W} \\
(F \circ G) (V) & \xrightarrow{(F \circ G)(T)} & (F \circ G) (W)
\end{array}$$
commutes. Here the linear maps $\psi_V, \psi_W$ denote the isomorphisms defined by our choice of the bases of $V$ and $W$, respectively. It follows that

$$\psi : F \circ G \Rightarrow \text{id}_{\text{FDVect}}$$

is a natural transformation. Since for each finite-dimensional vector space $V$, $\psi_V : \mathbb{R}^{\dim V} \to V$ is an isomorphism, $\psi$ is a natural isomorphism by Lemma 20.9.

We conclude that $F : \text{Mat} \to \text{FDVect}$ and $G : \text{FDVect} \to \text{Mat}$ are weak inverses of each other and that therefore the categories $\text{Mat}$ and $\text{FDVect}$ are equivalent.

---

**Definition 21.1.** A functor $F : C \to D$ is **essentially surjective** if for every object $d \in D$, there exists an object $c \in C$ such that $F(c)$ is isomorphic to $d$.

**Example 21.2.** The functor $F : \text{Mat} \to \text{FDVect}$ constructed above is essentially surjective since any finite-dimensional vector space $V$ is isomorphic to $\mathbb{R}^{\dim V} = F(\dim V)$. Note that $F$ is not surjective on objects: not every finite dimensional vector space equals $\mathbb{R}^n$ for some $n \in \mathbb{N}$.

**Example 21.3.** The inclusion functor $\iota : \text{Ab} \to \text{Group}$ from the category of abelian groups to groups is not essentially surjective since a non-abelian group can not be isomorphic to an abelian one.

Recall that a functor $F : C \to D$ is **full** if for any two objects $a, b \in C$

$$F : \text{Hom}_C(a, b) \to \text{Hom}_D(F(a), F(b))$$

is surjective. A functor $F$ is **faithful** if for any pair of objects $a, b \in C$,

$$F : \text{Hom}_C(a, b) \to \text{Hom}_D(F(a), F(b))$$

is injective. $F$ is fully faithful if $F$ is both full and faithful.

**Lemma 21.4.** Let $F : C \to D$ be (a part of) an equivalence of categories (so there is a functor $G : D \to C$ and a pair of natural isomorphisms $\alpha : \text{id}_C \Rightarrow G \circ F$, $\beta : F \circ G \Rightarrow \text{id}_D$.) Then $F$ is fully faithful and essentially surjective.

**Proof.** We first prove that $F$ is faithful. For any morphisms $c \xrightarrow{\delta} c'$ in $C$, the diagram

$$\begin{array}{ccc}
c & \xrightarrow{\alpha_c} & GF(c) \\
\downarrow{\delta} & & \downarrow{GF(\delta)} \\
c' & \xrightarrow{\alpha_{c'}} & GF(c')
\end{array}$$

(21.24)

commutes. Suppose $\gamma, \gamma' \in \text{Hom}_C(c, c')$ such that $F(\gamma) = F(\gamma')$. Then $GF(\gamma) = GF(\gamma')$. Since the diagram (21.24) commutes for $\delta = \gamma$ and for $\delta = \gamma'$ and since $\alpha_c, \alpha_{c'}$ are isomorphisms,

$$\gamma = (\alpha_{c'})^{-1} \circ GF(\gamma) \circ \alpha_c = (\alpha_{c'})^{-1} \circ GF(\gamma') \circ \alpha_c = \gamma'$$

Therefore, $F$ is faithful. By the same argument the functor $G$ is also faithful.

To show that $F$ is full given a morphism $\mu : F(c) \to F(c')$ we need to find a morphism $\gamma : c \to c'$ so that $F(\gamma) = \mu$. Let

$$\gamma = (\alpha_{c'})^{-1} \circ G(\mu) \circ \alpha_c$$
Then the diagram

\[
\begin{array}{ccc}
 c & \xrightarrow{\alpha_c} & GF(c) \\
\downarrow{\gamma} & & \downarrow{G(\mu)} \\
 c' & \xrightarrow{\alpha_{c'}} & GF(c')
\end{array}
\]

commutes.

Since \(\alpha\) is a natural transformation the diagram

\[
\begin{array}{ccc}
 c & \xrightarrow{\alpha_c} & GF(c) \\
\downarrow{\gamma} & & \downarrow{GF(\gamma)} \\
 c' & \xrightarrow{\alpha_{c'}} & GF(c')
\end{array}
\]

commutes as well. Since the morphisms \(\alpha_c, \alpha_{c'}\) are isomorphisms, it follows that

\[
GF(\gamma) = G(\mu).
\]

Since \(G\) is faithful, \(\mu = F(\gamma)\). Therefore, \(F\) is full.

Finally we argue that the functor \(F\) is essentially surjective. For any object \(d \in D\), \(\beta_d : F(G(d)) \to d\) is an isomorphism. So for every \(d \in D\), \(c = G(d) \in C_0\) is an object in \(C\) such that \(F(c)\) is isomorphic to \(d\). Therefore, \(F\) is essentially surjective. \(\square\)

The converse to Lemma 21.4 is also true.

**Lemma 21.5.** Let \(F : C \to D\) be a fully faithful and essentially surjective functor. Then \(F\) is part of an equivalence of categories (i.e., there exists a functor \(G : D \to C\) and a pair of natural isomorphisms \(\alpha : \text{id}_C \Rightarrow G \circ F\), \(\beta : F \circ G \Rightarrow \text{id}_D\).)

We will prove Lemma 21.5 in the next lecture.

**Example 21.6.** The functor \(F : \text{Mat} \to \text{FDVect}\) is fully faithful and essentially surjective. Hence \(F\) is part of an equivalence of categories.

---

**Lecture 22. Equivalences of categories. Concrete categories.**

**Last time:**

- Prove directly (“by hand”) that the categories \(\text{Mat}\) of matrices and \(\text{FDVect}\) of finite dimensional vector spaces are equivalent.
- A functor \(F : C \to D\) is **essentially surjective** if every object \(d \in D\) is isomorphic to an object in the image of \(F\).
- Prove that if \(F : C \to D\) is part of an equivalence of categories then \(F\) is fully faithful and essentially surjective.
- Stated but didn’t prove Lemma 21.5 if \(F : C \to D\) be a fully faithful and essentially surjective functor then \(F\) is part of an equivalence of categories.

**Proof of Lemma 21.5.** We want to construct a functor \(G : D \to C\) and two natural isomorphisms \(\alpha : \text{id}_C \Rightarrow G \circ F\), \(\beta : F \circ G \Rightarrow \text{id}_D\).

Since \(F\) is essentially surjective, for every object \(d \in D\), we can choose an object \(c \in C\) (\(c\) depends on \(d\)) and an isomorphism \(\beta_d : F(c) \to d\) in \(D\). Define \(G : D \to C\) on objects by setting \(G(d)\) to be the \(c\) we chose. Then \(\beta_d\) is an isomorphism from \(F(G(d))\) to \(d\):

\[
\beta_d : (F \circ G)(d) \xrightarrow{\sim} d.
\]
We need to define $G$ on morphisms. Let $d \xrightarrow{\mu} d'$ be a morphism in $\mathcal{D}$. Since $\beta_d, \beta_{d'}$ are isomorphisms, there exists a unique morphism $\nu : FG(d) \to FG(d')$ so that

\[
\begin{array}{ccc}
FG(d) & \xrightarrow{\beta_d} & d \\
\downarrow{\nu} & & \downarrow{\mu} \\
FG(d') & \xrightarrow{\beta_{d'}} & d'
\end{array}
\]

commutes. Namely,

$$
\nu := (\beta_{d'})^{-1} \circ \mu \circ \beta_d.
$$

Since $F$ is fully faithful, there exists a unique morphism $\gamma : G(d) \to G(d)$ so that $F(\gamma) = \nu$. We define $G(\mu) = \gamma$. This defines $G$ on morphisms.

It is not hard to show that $G(id_d) = id_{G(d)}$ and that $G$ preserves composition. Therefore $G$ is a desired functor.

By construction of $G$, for any morphism $\mu : d \to d'$ in $\mathcal{D}$, the diagram

\[
\begin{array}{ccc}
FG(d) & \xrightarrow{\beta_d} & d \\
\downarrow{FG(\mu)} & & \downarrow{\mu} \\
FG(d') & \xrightarrow{\beta_{d'}} & d'
\end{array}
\]

(22.25)

commutes. Hence $\beta$ is a natural transformation from $FG$ to $id_{\mathcal{D}}$. Since the components $\beta_d$ of $\beta$ are isomorphisms for all $d \in \mathcal{D}$, $\beta$ is a natural isomorphism.

It remains to construct a natural isomorphism $\alpha : id_{\mathcal{C}} \Rightarrow GF$. For any $c \in \mathcal{C}$, we have an isomorphism $\beta_{F(c)} : FG(F(c)) \to F(c)$. Since $F$ is fully faithful, there exists a unique morphism $\alpha_c : c \to GF(c)$ such that

$$
F(\alpha_c) = (\beta_{F(c)})^{-1}.
$$

We need to check that the collection $\{\alpha_c\}_{c \in \mathcal{C}}$ of isomorphisms assembles into a natural transformation $\alpha : id_{\mathcal{C}} \Rightarrow GF$. Since (22.25) commutes for any morphism $d \xrightarrow{\mu} d'$ in $\mathcal{D}$, it commutes for $\mu = F(\gamma)$ where $c \xrightarrow{\gamma} c'$ is a morphism in $\mathcal{C}$:

\[
\begin{array}{ccc}
FG(F(c)) & \xrightarrow{\beta_{F(c)}} & F(c) \\
\downarrow{FG(\mu)} & & \downarrow{\mu} \\
FG(F(c')) & \xrightarrow{\beta_{F(c')}} & F(c')
\end{array}
\]

Since $\beta_{F(c)} = (F(\alpha_c))^{-1}$ the diagram

\[
\begin{array}{ccc}
FG(F(c)) & \xrightarrow{F(\alpha_c)} & F(c) \\
\downarrow{F(GF(\gamma))} & & \downarrow{F(\gamma)} \\
FG(F(c')) & \xrightarrow{F(\alpha_{c'})} & F(c')
\end{array}
\]
commutes as well. Since $F$ is faithful,

$$
\begin{array}{c}
G(F(c)) \\
GF(\gamma)
\end{array}
\xrightarrow{\alpha_c} c
\begin{array}{c}
\downarrow \gamma \\
G(F(d')) \\
\xleftarrow{\alpha_{d'}} d'
\end{array}
$$

commutes, and we are done.

An interesting example of two equivalent categories comes from computer science: the categories $\textbf{Set}_*$ of pointed sets and $\textbf{Par}$ of sets and partial functions are equivalent. To explain the equivalence we first need to define the two categories.

**Definition 22.1.** The objects of the category $\textbf{Set}_*$ of pointed sets are pairs $(X, x)$ where $X$ is a set and $x \in X$ is an element of $X$. A morphism $f : (X, x) \to (Y, y)$ in $\textbf{Set}_*$ is a function $f : X \to Y$ such that $f(x) = y$.

To motivate the definition of $\textbf{Par}$ note that a function $f(x) = \frac{1}{x}$ (which we often think as a function from $\mathbb{R}$ to $\mathbb{R}$) is only defined on $X' = \{ x \in \mathbb{R} \mid x \neq 0 \}$.

**Definition 22.2.** The objects of the category $\textbf{Par}$ of sets and partial functions are sets. A morphism in $\textbf{Par}$ from a set $X$ to a set $Y$ is a pair $(X, f : X' \to Y)$ where $X'$ is a subset of $X$ and $f : X' \to Y$ is a function. The composition is defined by

$$(Y, Z \overset{g}{\leftarrow} Y') \circ (X, Y \overset{f}{\leftarrow} X') = (X, Z \overset{g \circ f}{\leftarrow} f^{-1}(Y')).$$

Here $g \circ f$ is a shorthand for a more accurate $g \circ (f|_{f^{-1}(Y')})$.

**Lemma 22.3.** The categories $\textbf{Set}_*$ of pointed sets and $\textbf{Par}$ of sets and partial functions are equivalent.

**Proof.** Define a functor $F : \textbf{Set}_* \to \textbf{Par}$ by setting $F(X, x) = X \setminus \{x\}$ on objects. On morphisms define $F$ by

$$F((X, x) \overset{g}{\Rightarrow} (Y, y)) = (X \setminus \{x\}) \supseteq X \setminus g^{-1}(y) \overset{g|_{X\setminus g^{-1}(y)}}\to Y \setminus \{y\}.$$  

This makes sense since $g(x) = y$ and consequently $X \setminus g^{-1}(y) \subseteq X \setminus \{x\}$. It is not hard to check that $F$ is a functor. We are done once we show that $F$ is fully faithful and essentially surjective.

Given a set $S$, choose a one element set $*_S$. Then $F(S \sqcup \{*_S\}, *_S) = S$. Hence $F$ is surjective on objects and therefore essentially surjective. If

$$F((X, x) \overset{g}{\Rightarrow} (Y, y)) = F((X, x) \overset{h}{\Rightarrow} (Y, y)),$$

then $X \setminus g^{-1}(y) = X \setminus h^{-1}(y)$ and therefore $g^{-1}(y) = h^{-1}(y)$. So if $a \in X \setminus g^{-1}(y)$, $g(a) = h(a)$ and if $a \in g^{-1}(y)$, $g(a) = h(a) = y$. We conclude that $f = g$. Hence $F$ is faithful.

Given a partial function

$$F((X, x) \overset{k}{\Rightarrow} (Y, y)) = (X \setminus \{x\}) \supseteq X' \overset{k}{\Rightarrow} (Y \setminus \{y\}),$$

consider $\tilde{k} : (X, x) \to (Y, y)$ in $\textbf{Set}_*$ defined by $\tilde{k} = \begin{cases} k(a) & \text{if } a \in X' \\ y & \text{if } a \in X \setminus X' \end{cases}$. Then $\tilde{k}^{-1}(y) = X'$ and

$$F((X, x) \overset{\tilde{k}}{\Rightarrow} (Y, y)) = (X \setminus \{x\}) \supseteq X' \overset{\tilde{k}}{\Rightarrow} (Y \setminus \{y\}).$$

Therefore $F$ is full. □

67
Remark 22.4. If $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ are fully faithful and essentially surjective, then so is the composite $G \circ F : \mathcal{A} \to \mathcal{C}$ (prove it). Hence “being equivalent” is a transitive relation for categories. It’s easy to see that it’s reflexive and symmetric. So “being equivalent” is an equivalence relation on the collection of all categories.

Concrete categories.

The categories $\text{Vect}_\mathbb{R}$, $\text{Group}$, $\text{Mon}$, and $\text{Top}$ have a common feature: their objects are sets with some extra structure and their morphisms are functions that preserve the structure. This motivates the following definition.

Definition 22.5. A concrete category is a pair $(\mathcal{C}, U : \mathcal{C} \to \text{Set})$ where $\mathcal{C}$ is a category and $U : \mathcal{C} \to \text{Set}$ is a faithful functor.

Example 22.6. The categories of vector spaces, of groups, of abelian groups, of monoids and of topological spaces are all concrete.

Remark 22.7. Note a curious feature (or perhaps a bug?) of the definition. If $(\mathcal{C}, U : \mathcal{C} \to \text{Set})$ is a concrete category and $F : \mathcal{B} \to \mathcal{C}$ is a faithful functor, then $U \circ F : \mathcal{B} \to \text{Set}$ is also faithful. Hence $(\mathcal{B}, U \circ F : \mathcal{B} \to \text{Set})$ is a concrete category.

For example, we have a fully faithful functor $F : \text{Mat} \to \text{FDVect}_\mathbb{R}$ and a faithful underlying set functor $U : \text{FDVect} \to \text{Set}$. Consequently $(\text{Mat}, U \circ F : \text{Mat} \to \text{Set})$ is a concrete category even though the objects of $\text{Mat}$ are integers, which don’t look like sets with structure (or if they do, it is not the “right” structure).

Definition 22.8. A category $\mathcal{C}$ is concretizable if it admits a faithful functor $U : \mathcal{C} \to \text{Set}$.

It turns out that any small category is concretizable.

Lemma 22.9. For any small category $\mathcal{C}$ there is a faithful functor $U : \mathcal{C} \to \text{Set}$.

Proof. Recall that the target map $t : \mathcal{C}_0 \to \mathcal{C}_1$ assigns to each morphism $x \xrightarrow{\gamma} y$ of $\mathcal{C}$ its target $y$. Define $U : \mathcal{C} \to \text{Set}$ on objects by

$$U(y) = t^{-1}(y) = \{ \mu \in \mathcal{C}_1 \mid t(\mu) = y \}.$$ 

Given a morphism $x \xrightarrow{\gamma} y$ in $\mathcal{C}$, define

$$U(\gamma) = \gamma_* : U(x) \to U(y), \quad \gamma_*(\mu) = \gamma \circ \mu$$

for every $\mu \in U(x)$. We need to check that $U$ is a functor.

Given $x \xrightarrow{\gamma} y \xrightarrow{\mu} z$, and $\mu \in U(x)$,

$$U(\mu \circ \gamma)(\mu) = (\mu \circ \gamma) \circ \mu = \mu_*(\gamma_*(\mu)) = (U(\mu) \circ U(\gamma))(\mu).$$

Also $U \left( x \xrightarrow{\text{id}_x} x \right) = (\text{id}_x)_* = \text{id}_{U(x)}$. Hence $U$ is a functor.

Finally, we prove that $U$ is faithful. Suppose that $\gamma, \nu \in \text{Hom}_\mathcal{C}(x, y)$ and that $U(\gamma) = U(\nu)$. Then $\gamma \circ \mu = \nu \circ \mu$ for every $\mu \in U(x) = t^{-1}(x) = \{ \mu \in \mathcal{C}_1 \mid t(\mu) = x \}$. Note that in particular, $\gamma = \gamma \circ \text{id}_x = \nu \circ \text{id}_x = \nu$. Therefore, $U$ is faithful. \qed

Remark 22.10. If we accept Grothendieck’s axiom of universes, then any category $\mathcal{C}$ is $V$-small (i.e., its collections $\mathcal{C}_0$ of objects and $\mathcal{C}_1$ of morphisms are elements of $V$) for a large enough universe $V$. The proof of Lemma 22.9 then shows that there is a faithful functor to $U : \mathcal{C} \to \text{V-Set}$, where $\text{V-Set}$ is the category of $V$-small sets.
In other words Grothendieck’s axiom of universes makes any category concretizable and makes the notion of a concrete category look rather silly.

**Lecture 23. Yoneda lemma.**

**Last time:**
- Prove that a fully faithful and essentially surjective functor is part of an equivalence of categories.
- Concrete categories and concretization.

Fix a universe $V$ in the background, and let $\text{Set}$ denote the category of $V$-sets. Thus a set $X$ is an object of $\text{Set}$ if and only if $X$ is an element of the universe $V$.

Recall that a category $C$ is locally small (relative to the universe $V$) if for any pair of objects $x, y \in C$, $\text{Hom}_C(x, y) \in \text{Set}$. For any locally small category $C$ and any object $c \in C$ we have a functor $\text{Hom}_C(c, \cdot) : C \to \text{Set}$, $\text{Hom}_C(c, \cdot)(x) = \text{Hom}_C(c, x) \xrightarrow{\gamma} \text{Hom}_C(c, y)$, where

$$\gamma \left( c \xrightarrow{\mu} x \right) := c \xrightarrow{\gamma \circ \mu} y$$

is the pushforward by $\gamma$. Let $F : C \to \text{Set}$ be a functor. Denote the collection of morphisms $\text{Hom}_{[C, \text{Set}]}(\text{Hom}_C(c, \cdot), F)$ in the functor category $[C, \text{Set}]$ by $\text{Nat}(\text{Hom}_C(c, \cdot), F)$ (since they are natural transformations). Given a natural transformation $\alpha : \text{Hom}_C(c, \cdot) \Rightarrow F$, we have a morphism $\alpha_c : \text{Hom}_C(c, -)(c) \to F(c)$ in $\text{Set}$, that is, a function $\alpha_c : \text{Hom}_C(c, c) \to F(c)$. Since $\text{id}_c \in \text{Hom}_C(c, c)$,

$$\alpha_c(\text{id}_c) \in F(c),$$

i.e., $\alpha_c(\text{id}_c)$ is an element of the set $F(c)$. The Yoneda lemma says that this element of $F(c)$ uniquely determines the natural transformation $\alpha$:

**Theorem 23.1 (Yoneda Lemma).** Let $F : C \to \text{Set}$ be a functor from a locally small category to $\text{Set}$, $c \in C$ an object, and $\text{Nat}(\text{Hom}_C(c, \cdot), F)$ the set of natural transformations. The function

$$\Phi : \text{Nat}(\text{Hom}_C(c, \cdot), F) \to F(c), \quad \Phi(\alpha) = \alpha_c(\text{id}_c)$$

is a bijection. (In particular $\text{Nat}(\text{Hom}_C(c, \cdot), F)$ is a (small) set.)

**Proof.** Let’s start by spelling out what it means for $\alpha : \text{Hom}_C(c, \cdot) \Rightarrow F$ to be a natural transformation: for every object $d$ of $C$, we have a function (a morphism in $\text{Set}$)

$$\alpha_d : \text{Hom}_C(c, d) \to F(d)$$

so that for every morphism $d' \xrightarrow{\gamma} d$ in $C$ the diagram of sets and functions

$$\begin{array}{ccc}
\text{Hom}_C(c, d') & \xrightarrow{\alpha_{d'}} & F(d') \\
\downarrow_{\text{Hom}_C(c, \cdot)(\gamma) = \gamma} & & \downarrow_{F(\gamma)} \\
\text{Hom}_C(c, d) & \xrightarrow{\alpha_d} & F(d)
\end{array}$$

(23.26)

commutes.

Now comes the key observation: for any morphism $c \xrightarrow{f} d \in \text{Hom}_C(c, d)$, the function $f_* : \text{Hom}_C(c, c) \to \text{Hom}_C(c, d)$ is given by $f_*(\nu) = f \circ \nu$. Therefore
\[ f_\ast (\text{id}_c) = f \circ \text{id}_c = f. \]

Since \( f \) commutes, equation (23.26) implies that

\[ \alpha_d(c \xrightarrow{f} d) = \alpha_d(f_\ast (\text{id}_c)) = F(f)(\alpha_c(\text{id}_c)). \]

Therefore the function \( \alpha_d : \text{Hom}_C(c,d) \to F(d) \) is uniquely determined by \( \alpha_c(\text{id}_c) \in F(c) \).

We now prove that \( \Phi \) is injective. Suppose \( \alpha, \beta : \text{Hom}_C(c,-) \Rightarrow F \) are two natural transformations with \( \Phi(\alpha) = \alpha_c(\text{id}_c) = \beta_c(\text{id}_c) = \Phi(\beta) \). Then for any \( d \in \mathcal{C} \) and any \( f \in \text{Hom}_C(c,d) \)

\[ \alpha_d(f) = F(f)(\alpha_c(\text{id}_c)) = F(f)(\beta_c(\text{id}_c)) = \beta_d(f) \]

by (23.28). Therefore \( \alpha = \beta \).

To prove surjectivity of \( \Phi \) we need to show that for any element \( x \in F(c) \) there is a natural transformation \( \Psi(x) : \text{Hom}_C(c,-) \Rightarrow F \) with \( \Phi(\Psi(x)) = \Psi(x)_c(\text{id}_c) = x \). So define a collection of functions \( \{\Psi(x)_d : \text{Hom}_C(c,d) \to F(d)\}_{d \in \mathcal{C}} \) by

\[ \Psi(x)_d(c \xrightarrow{f} d) = F(f)(x) \]

for every \( f \in \text{Hom}_C(c,d) \). We need to check that these functions assemble into a natural transformation. That is, we need to check that for any morphism \( d \xrightarrow{\gamma} d' \) in \( \mathcal{C} \) the diagram

\[ \text{Hom}_C(c,d) \xrightarrow{\Psi(x)_d} F(d) \]

\[ \gamma \downarrow \quad \downarrow \Phi(\gamma) \]

\[ \text{Hom}_C(c,d') \xrightarrow{\Psi(x)_{d'}} F(d') \]

commutes. Now given \( c \xrightarrow{f} d \in \text{Hom}_C(c,d) \),

\[ \Psi(x)_{d'}(\gamma \circ f) = \Psi(x)_{d'}(\gamma)(\circ f) = F(\gamma \circ f)(x) \]

by definition of \( \Psi(x)_{d'} \). On the other hand,

\[ F(\gamma)(\Psi(x)_d(f)) = F(\gamma)(F(f)(x)) = (F(\gamma) \circ F(f))(x). \]

Since \( F \) is a functor, \( F(\gamma \circ f) = F(\gamma) \circ F(f) \). Therefore (23.30) commutes and \( \Psi(x) \) is a natural transformation. Since

\[ \Phi(\Psi(x)) = \psi(x)_c(\text{id}_c) = F(\text{id}_c)(x) = \text{id}_{F(c)}(x) = x, \]

the map \( \Phi : \text{Nat}(\text{Hom}_C(c,-), F) \to F(c) \) is surjective and we are done. \( \square \)
Example 23.2. Let $\mathcal{C} = a \xrightarrow{\sigma} v$ be a category with two objects and two non-identity morphisms. We have seen that the functor category $[\mathcal{C}, \text{Set}]$ “is” the category $\text{Graph}$ of directed graphs: a functor $F : \mathcal{C} \to \text{Set}$ corresponds to a graph with the set of arrow $F(a)$, the set of vertices $F(v)$, and the source and target maps $F(\sigma), F(\tau) : F(a) \to F(v)$. Therefore, the two functors $\text{Hom}_\mathcal{C}(a, -)$ and $\text{Hom}_\mathcal{C}(v, -)$ from $\mathcal{C}$ to $\text{Set}$ are directed graphs.

The Yoneda lemma says that there are bijections

\begin{align}
(23.33) \quad \text{Hom}_{\text{Graph}}(\text{Hom}_\mathcal{C}(a, -), F) & \xrightarrow{\sim} F(a) \\
(23.34) \quad \text{Hom}_{\text{Graph}}(\text{Hom}_\mathcal{C}(v, -), F) & \xrightarrow{\sim} F(v)
\end{align}

where $F(a)$ is the set of arrows/edges of $F$ and $F(v)$ is the set of vertices.

To understand these bijections, we need to understand what graphs the functors $\text{Hom}_\mathcal{C}(a, -)$ and $\text{Hom}_\mathcal{C}(v, -)$ actually are. We compute.

The set of vertices of $\text{Hom}_\mathcal{C}(a, -)$ is $\text{Hom}_\mathcal{C}(a, -)(v) = \text{Hom}_\mathcal{C}(a, v) = \{\sigma, \tau\}$. The set of arrows of $\text{Hom}_\mathcal{C}(a, -)$ is $\text{Hom}_\mathcal{C}(a, -)(a) = \text{Hom}_\mathcal{C}(a, a) = \{\text{id}_a\}$. Since $\sigma(\text{id}_a) = \sigma \circ \text{id}_a = \sigma$ and $\tau(\text{id}_a) = \tau \circ \text{id}_a = \tau$, $\text{Hom}_\mathcal{C}(a, -)$ “is” the graph with one arrow and two vertices:

$$\xymatrix{\sigma \ar@{-}[rr]^-{\text{id}_a} & & \tau} = \text{Hom}_\mathcal{C}(a, -).$$

A homomorphism $\varphi : \text{Hom}_\mathcal{C}(a, -) \to F$ (of graphs) is uniquely determined by the image of the edge $\bullet \xrightarrow{\text{id}_a} \bullet$, which is an edge of $F$. This is the bijection (23.33).

The set of vertices of the graph $\text{Hom}_\mathcal{C}(v, -)$ is $\text{Hom}_\mathcal{C}(v, -)(v) = \text{Hom}_\mathcal{C}(v, v) = \{\text{id}_v\}$. The set of arrows of $\text{Hom}_\mathcal{C}(a, -)$ is $\text{Hom}_\mathcal{C}(a, -)(v) = \text{Hom}_\mathcal{C}(a, v) = \emptyset$. So $\text{Hom}(v, -)$ “is” the graph with no arrows and one vertex $\{\text{id}_v\}$:

$$\text{Hom}(v, -) = \bullet.$$

A homomorphism $\varphi : \text{Hom}_\mathcal{C}(v, -) \to F$ (of graphs) is uniquely determined by its value on $\bullet$, which is a vertex of $F$. This is the bijection (23.34).

The contravariant Yoneda lemma.

Recall that if we fix the second variable of the functor $\text{Hom}_\mathcal{C} : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \text{Set}$, we get a functor $\text{Hom}_\mathcal{C}(-, c) : \mathcal{C}^{\text{op}} \to \text{Set}$ for every $c \in \mathcal{C}$. On morphisms

$$\text{Hom}_\mathcal{C}(-, c) \left( d \xrightarrow{\mu} d' \right) = \mu^* : \text{Hom}_\mathcal{C}(d', c) \to \text{Hom}_\mathcal{C}(d, c)$$

for all $\mu \in \text{Hom}_\mathcal{C}(d, d')$, where

$$\mu^* \left( d' \xrightarrow{f} c \right) = d \xrightarrow{f \circ \mu} c.$$

There is a version of the Yoneda Lemma for these functors as well:

**Theorem 23.3 (Contravariant Yoneda Lemma).** Let $F : \mathcal{C}^{\text{op}} \to \text{Set}$ be a contravariant functor from a locally small category $\mathcal{C}$ to $\text{Set}$ and $c \in \mathcal{C}$. The function

$$\Phi : \text{Nat}(\text{Hom}_\mathcal{C}(-, c), F) \to F(c), \quad \Phi(\alpha) = \alpha_c(\text{id}_c)$$
is a bijection. Here as before, \( \text{Nat}(\text{Hom}_C (-, c), F) \) denotes the set of natural transformations from \( \text{Hom}_C (-, c) \) to \( F \), that is,

\[
\text{Nat}(\text{Hom}_C (-, c), F) = \text{Hom}_{[C^{\text{op}}, \text{Set}]}(\text{Hom}_C (-, c), F).
\]

We sketch two proofs of this version of the Yoneda Lemma:

**Proof 1.** The (covariant) Yoneda Lemma (Theorem 23.1) plus duality. \( \square \)

**Proof 2.** Mimic the proof of Theorem 23.1 and note that \( \alpha : \text{Hom}_C (-, c) \Rightarrow F \) is a morphism in \([C^{\text{op}}, \text{Set}]\) if and only if for every morphism \( d \xrightarrow{\mu} d' \) in \( C \), the diagram

\[
\begin{array}{ccc}
\text{Hom}_C(d, c) & \xrightarrow{\alpha_d} & F(d) \\
\uparrow{\mu^*} & & \uparrow{F(\mu)} \\
\text{Hom}_C(d', c) & \xrightarrow{\alpha_{d'}} & F(d')
\end{array}
\]

commutes. Hence for any morphism \( d \xrightarrow{\gamma} c \in \text{Hom}_C(d, c) \) the diagram

\[
\begin{array}{ccc}
\text{Hom}_C(c, c) & \xrightarrow{\alpha_c} & F(c) \\
\downarrow{\gamma^*} & & \downarrow{F(\gamma)} \\
\text{Hom}_C(d, c) & \xrightarrow{\alpha_d} & F(d)
\end{array}
\]

commutes as well and therefore

\[
\alpha_d(\gamma) = \alpha_d(\gamma^* \text{id}_c) = F(\gamma)(\alpha_c(\text{id}_c)).
\]

Consequently the map

\[
\Phi : \text{Nat}(\text{Hom}_C (-, c), F) \to F(c), \quad \Phi(\alpha) = \alpha_c(\text{id}_c)
\]

is injective. Moreover for any element \( x \in F(c) \) there is a natural transformation

\[
\Psi(x) : \text{Hom}_C (-, c) \Rightarrow F
\]

with components

\[
\Psi(x)_d : \text{Hom}_C(d, c) \to F(d), \quad \Psi(x)_d(\gamma) = F(\gamma)(x).
\]

\( \square \)

**Lecture 24. Yoneda embeddings. Representable functors.**

**Last time:**

- Proved the Yoneda lemma: for any locally small category \( C \), any object \( c \) of \( C \) and any functor \( F : C \to \text{Set} \) the function

\[
\Phi : \text{Nat}(\text{Hom}_C(c, -), F) \to F(c), \quad \Phi(\alpha) = \alpha_c(\text{id}_c)
\]

is a bijection.
- Sketched a proof of the contravariant version: for any locally small category \( C \), any object \( c \) of \( C \) and any functor \( F : C^{\text{op}} \to \text{Set} \) the function

\[
\Phi : \text{Nat}(\text{Hom}_C( - , c), F) \to F(c), \quad \Phi(\alpha) = \alpha_c(\text{id}_c)
\]

is a bijection.
Remark 24.1. It follows from the proof of the Yoneda lemma that the inverse of the bijection
\[ \Phi : \text{Nat}(\text{Hom}_C(c, -), F) \to F(c), \quad \Phi(\alpha) = \alpha_c(\text{id}_c) \]
is a bijection
\[ \Psi : F(c) \to \text{Nat}(\text{Hom}_C(c, -), F), \quad x \mapsto \left( \text{Nat}(\text{Hom}_C(c, -) \xrightarrow{\Psi(x)} F) \right), \]
where the components \( \Psi(x)_d : \text{Hom}_C(c, d) \to F(d) \) of \( \Psi(x) \) are given by [23.29]:
\[
\Psi(x)_d \left( c \xrightarrow{f} d \right) := F(f)(x) \quad \text{for all } d \in C \text{ and all } f \in (\text{Hom}_C(c, -))(d).
\]
This is a useful formula to remember.

We now promote the assignment \( c \mapsto \text{Hom}_C(c, -) \) to a fully faithful functor \( y^* : C^{\text{op}} \to [C, \text{Set}] \), the Yoneda embedding functor.

Lemma 24.2 (Yoneda Embedding). Let \( C \) be a locally small category. The assignment
\[ C \ni c \mapsto \text{Hom}_C(c, -) \in [C, \text{Set}] \]
extends to a fully faithful functor
\[ y^* : C^{\text{op}} \to [C, \text{Set}] \]
with
\[ y^*(a \xrightarrow{\gamma} b) = \text{Hom}_C(b, -) \xrightarrow{\gamma^*} \text{Hom}_C(a, -) \]
for all morphisms \( a \xrightarrow{\gamma} b \) in \( C \).

Proof. By the Yoneda lemma for all objects \( a, b \in C \) we have a bijection
\[ \Phi : \text{Nat}(\text{Hom}_C(b, -), \text{Hom}_C(a, -)) \to \text{Hom}_C(a, -)(b) = \text{Hom}_C(a, b). \]
However, it is far from clear that the collection
\[ \{ \Phi^{-1} : \text{Hom}_C(a, b) \to \text{Nat}(\text{Hom}_C(b, -), \text{Hom}_C(a, -)) \}_{(a,b) \in C \times C} \]
of bijections define a functor from \( C^{\text{op}} \) to the functor category \([C, \text{Set}]\). For example it is not immediate that if \( b = a \) then \( \Phi^{-1}(\text{id}_b) : \text{Hom}_C(b, -) \Rightarrow \text{Hom}_C(b, -) \) is the identity natural transformation. Of course if the bijections do define a functor, then the functor has to be fully faithful. The solution to this problem is to explicitly compute \( \Phi^{-1} \).

By the Yoneda lemma the natural transformation
\[ \alpha = \Phi^{-1}(\gamma) : \text{Hom}_C(b, -) \Rightarrow \text{Hom}_C(a, -) \]
is the unique natural transformation with \( \alpha_b(\text{id}_b) = \gamma \). Since for any \( b \xrightarrow{f} d \in \text{Hom}_C(b, d) \) the diagram
\[
\begin{array}{ccc}
\text{Hom}_C(b, b) & \xrightarrow{\alpha_b} & \text{Hom}_C(a, b) \\
\text{Hom}_C(b, -)(f) = f_* & | & | \\
\text{Hom}_C(b, d) & \xrightarrow{\alpha_d} & \text{Hom}_C(a, d)
\end{array}
\]
commutes,
\[ \alpha_d(f) = \alpha_d(f_* \circ \text{id}_b) = f_* (\alpha_b(\text{id}_b)) = f_* (\alpha_b(\text{id}_b)) = f_* (\gamma) = f \circ \gamma = \gamma^*(f). \]
It follows that \( (\Phi^{-1}(\gamma))_d = \gamma^* : \text{Hom}_C(b, d) \to \text{Hom}_C(a, d) \). We abuse notation and write \( \gamma^* \) for the natural transformation \( \Phi^{-1}(\gamma) \).
It remains to check that the map
\[ y^* : \mathcal{C}^{\text{op}} \to [\mathcal{C}, \text{Set}], \quad y^* (a \xrightarrow{\gamma} b) = \text{Hom}_\mathcal{C} (b, -) \xrightarrow{\gamma^*} \text{Hom}_\mathcal{C} (a, -) \]
preserves the identities and the composition, hence is a functor. For an object \( a \) in \( \mathcal{C} \)
\[ y^* (\text{id}_a) = (\text{id}_a)^* = \text{id}_{\text{Hom}_\mathcal{C} (a, -)} = \text{id}_{y^* (a)}, \]
and for any pair of composable morphisms \( a \xrightarrow{\gamma} b \xrightarrow{\delta} c \)
\[ y^* (\delta \circ \gamma) = (\delta \circ \gamma)^* = \gamma^* \circ \delta^* = y^* (\gamma) \circ y^* (\delta). \]
Hence \( y^* \) is a functor and we are done. \( \square \)

**Definition 24.3.** A subcategory \( \mathcal{D}' \) of a category \( \mathcal{D} \) is a full subcategory if for every \( d_1, d_2 \in \mathcal{D}' \), \( \text{Hom}_{\mathcal{D}'} (d_1, d_2) = \text{Hom}_{\mathcal{D}} (d_1, d_2) \) (i.e., the inclusion functor \( \mathcal{D}' \hookrightarrow \mathcal{D} \) is full).

**Remark 24.4.** Note that if \( a \) and \( b \) are two distinct objects in a category \( \mathcal{C} \) then \( \text{Hom}_\mathcal{C} (a, -) \neq \text{Hom}_\mathcal{C} (b, -) \) (why?) hence the Yoneda embedding functor \( y^* \) is injective on objects. It follows that the image \( y^* (\mathcal{C}^{\text{op}}) \) of the Yoneda embedding is a full subcategory of \([\mathcal{C}, \text{Set}]\).

**Example 24.5.** Let \( \mathcal{C} = \begin{array}{ccc} a & \xrightarrow{\sigma} & v \\ \tau \end{array} \) be a category with two objects and two parallel non-identity morphisms. We have seen that the functor category \([\mathcal{C}, \text{Set}]\) is the category of graphs, that \( y^* (a) = \text{Hom}_\mathcal{C} (a, -) \) is the graph
\[ y^* (a) = \begin{array}{ccc} \sigma & \xrightarrow{\text{id}_a} & \tau \end{array} \]
and that \( y^* (v) = \text{Hom}_\mathcal{C} (v, -) \) is the graph with one vertex and no arrow: \( y^* (v) = \bullet \).

Since \( y^* (\sigma) (\text{id}_v) = \sigma^* (\text{id}_v) = \sigma, y^* (\sigma) : y^* (v) \to y^* (a) \) is the graph homomorphism that sends the only vertex \( \text{id}_v \) of \( y^* (v) \) to the vertex \( \sigma \) of the graph \( y^* (a) \).

Similarly, since \( y^* (\tau) (\text{id}_v) = \tau^* (\text{id}_v) = \tau, y^* (\tau) : y^* (v) \to y^* (a) \) is the graph homomorphism that sends the only vertex \( \text{id}_v \) of \( y^* (v) \) to the vertex \( \tau \) of the graph \( y^* (a) \).

Note that the image of the functor \( y^* : \mathcal{C}^{\text{op}} \to [\mathcal{C}, \text{Set}] \) is the subcategory \( \begin{array}{ccc} y^* (v) & \xrightarrow{y^* (\sigma)} & y^* (a) \\ y^* (\tau) \end{array} \) of \([\mathcal{C}, \text{Set}]\). Note also that
\[ y^* : \begin{array}{ccc} a & \xrightarrow{\sigma} & v \\ \tau \end{array} \to \begin{array}{ccc} y^* (v) & \xrightarrow{y^* (\sigma)} & y^* (a) \\ y^* (\tau) \end{array} \]
is an isomorphism of categories.

There is a covariant analogue of the Yoneda embedding.

**Lemma 24.6 (Yoneda Embedding).** Let \( \mathcal{C} \) be a locally small category. The assignment
\[ \mathcal{C} \ni c \mapsto \text{Hom}_\mathcal{C} (-, c) \in [\mathcal{C}^{\text{op}}, \text{Set}] \]
extends to a fully faithful functor
\[ y : \mathcal{C} \to [\mathcal{C}^{\text{op}}, \text{Set}] \]
with
\[ y(a \xrightarrow{\gamma} b) = \text{Hom}_\mathcal{C} (a, -) \xrightarrow{\gamma^*} \text{Hom}_\mathcal{C} (b, -) \]
for all morphisms \( a \xrightarrow{\gamma} b \) in \( \mathcal{C} \).

**Proof.** Exercise. \( \square \)
Definition 24.7. Let \( \mathcal{C} \) be a locally small category. A functor \( F : \mathcal{C} \to \text{Set} \) is representable if it is isomorphic to \( y^*(c) = \text{Hom}_\mathcal{C}(c, -) \) for some \( c \in \mathcal{C} \). Here as before \( y^* : \mathcal{C}^{\text{op}} \to [\mathcal{C}, \text{Set}] \) is the Yoneda embedding.

Example 24.8. Let \( \mathcal{C} = \{ a \xrightarrow{\sigma} v \} \). We have seen that \( y^* \) sends \( a \) to the graph \( y^*(a) = \sigma \xrightarrow{id} \tau \). So any graph of the form \( v_1 \xleftarrow{\psi} v_2 \) is a representable functor in \( [\mathcal{C}, \text{Set}] \). Similarly \( y^* \) sends \( v \) to the graph with one vertex and no arrow \( y^*(v) = *_{\text{id}_v} \). So any graph with only one vertex and no arrows is a representable functor.

Example 24.9. The underlying set functor \( U : \text{Group} \to \text{Set} \) is isomorphic to \( \text{Hom}_{\text{Group}}(\mathbb{Z}, -) \) hence is representable.

Proof. Given a group \( G \) consider the function
\[
\alpha_G : \text{Hom}_{\text{Group}}(\mathbb{Z}, G) \to U(G), \quad \alpha_G(\varphi) = U(\varphi)(1) = \varphi(1).
\]
For every \( g \in G \),
\[
\varphi_g : \mathbb{Z} \to G, \quad \varphi_g(n) = g^n
\]
is a homomorphism of groups and \( \varphi_g(1) = g \). Since \( \alpha_G(\varphi_g) = g \), \( \alpha_G \) is surjective. If \( \psi, \varphi : \mathbb{Z} \to G \) are two homomorphisms with \( \alpha_G(\varphi) = \alpha_G(\psi) \), then
\[
\varphi(n) = \varphi(1)^n = (\alpha_G(\varphi)) = (\alpha_G(\psi))^n = \psi(1)^n = \psi(n)
\]
for every \( n \in \mathbb{Z} \) and therefore \( \varphi = \psi \). Hence the function \( \alpha_G \) is injective.

It remains to check that the bijections \( \{ \alpha_G : \text{Hom}_{\text{Group}}(\mathbb{Z}, G) \to U(G) \}_{G \in \text{Group}} \) assemble into a natural isomorphism \( \alpha : \text{Hom}_{\text{Group}}(\mathbb{Z}, -) \Rightarrow U \). For a homomorphisms \( f : G \to H \) consider the diagram
\[
\begin{array}{ccc}
\text{Hom}_{\text{Group}}(\mathbb{Z}, G) & \xrightarrow{\alpha_G} & U(G) \\
f^* \downarrow & & \downarrow U(f) \\
\text{Hom}_{\text{Group}}(\mathbb{Z}, H) & \xrightarrow{\alpha_H} & U(H)
\end{array}
\]
For any \( \varphi \in \text{Hom}_{\text{Group}}(\mathbb{Z}, G) \),
\[
\alpha_H(f^*(\varphi)) = (f^*(\varphi))(1) = (f \circ \varphi)(1) = f(\varphi(1)) = U(f)(\varphi(1)) = U(f)(\alpha_G(\varphi)).
\]
Therefore, the diagram commutes. Hence \( \alpha \) is a natural transformation. Since all of the components of \( \alpha \) are bijections, \( \alpha \) is a natural isomorphism. \( \square \)

Representable functors are important and have nice properties. We end the lecture with a definition.

Definition 24.10. The essential image of a functor \( F : \mathcal{C} \to \mathcal{D} \) is the full subcategory \( \mathcal{D}' \subseteq \mathcal{D} \) so that \( d \in \mathcal{D}' \) if and only if \( d \) is isomorphic to \( F(c) \) for some \( c \in \mathcal{C} \).

Example 24.11 (Really an “example”). A functor \( F : \mathcal{C} \to \text{Set} \) is representable if and only if \( F \) is in the essential image of the Yoneda embedding \( y^* \).

Example 24.12 (Again, this is really an “example”). A functor \( F : \mathcal{C} \to \mathcal{D} \) is essentially surjective if the essential image of \( F \) is all of \( \mathcal{D} \).

Last time:
- Yoneda embeddings: for any locally small category \( \mathcal{C} \) the functors
  \[
  y^* : \mathcal{C}^{\text{op}} \to [\mathcal{C}, \text{Set}], \quad y^*(c \xrightarrow{\gamma} c') = \text{Hom}_{\mathcal{C}}(c',-) \xrightarrow{\gamma^*} \text{Hom}_{\mathcal{C}}(c, -)
  \]
  and
  \[
  y : \mathcal{C} \to [\mathcal{C}^{\text{op}}, \text{Set}], \quad y(c \xrightarrow{\gamma} c') = \text{Hom}_{\mathcal{C}}(-, c) \xrightarrow{\gamma^*} \text{Hom}_{\mathcal{C}}(-, c')
  \]
  are fully faithful functors.
- A functor \( F : \mathcal{C} \to \text{Set} \) is representable iff it is isomorphic to the functor \( \text{Hom}_{\mathcal{C}}(c, -) \) for some \( c \in \mathcal{C} \). Equivalently \( F \) is representable iff \( F \) lies in the essential image of the Yoneda embedding \( y^* : \mathcal{C}^{\text{op}} \to [\mathcal{C}, \text{Set}] \).
- Similarly, a functor \( G : \mathcal{C}^{\text{op}} \to \text{Set} \) is representable iff \( G \) lies in the essential image of the Yoneda embedding \( y : \mathcal{C} \to [\mathcal{C}^{\text{op}}, \text{Set}] \).

Example 25.1. The forgetful functor \( U : \text{Vect}_\mathbb{R} \to \text{Set} \) is isomorphic to \( \text{Hom}_{\text{Vect}_\mathbb{R}}(\mathbb{R}, -) \) hence is representable.

Proof. For any real vector space \( V \) define
\[
\alpha_V : \text{Hom}_{\text{Vect}_\mathbb{R}}(\mathbb{R}, V) \to U(V)
\]
by \( \alpha_V(T) = T(1) \) for every \( T \in \text{Hom}_{\text{Vect}_\mathbb{R}}(\mathbb{R}, V) \). As in the case of groups, it is not hard to check that

1. for each vector space \( V \), \( \alpha_V \) is a bijection;
2. the collection of function \( \{\alpha_V : \text{Hom}_{\text{Vect}_\mathbb{R}}(\mathbb{R}, V) \to U(V)\}_{V \in \text{Vect}_\mathbb{R}} \) assembles into a natural isomorphism \( \gamma : \text{Hom}_{\text{Vect}_\mathbb{R}}(\mathbb{R}, -) \Rightarrow U \).

Hence \( U : \text{Vect}_\mathbb{R} \to \text{Set} \) is representable. \( \square \)

Definition 25.2. A representation of a functor \( F : \mathcal{C} \to \text{Set} \) is a pair \((c, \alpha)\) where \( c \) is an object of \( \mathcal{C} \) and \( \alpha : \text{Hom}_{\mathcal{C}}(c, -) \Rightarrow F \) is a natural isomorphism.

Similarly, a representation of a functor \( F : \mathcal{C}^{\text{op}} \to \text{Set} \) is a pair \((c, \alpha)\) where \( c \) is an object of \( \mathcal{C} \) and \( \alpha : \text{Hom}_{\mathcal{C}}(-, c) \Rightarrow F \) is a natural isomorphism.

Lemma 25.3. Suppose \((c, \alpha), (c', \alpha')\) are two representation of the same functor \( F : \mathcal{C} \to \text{Set} \). Then \( c \) and \( c' \) are isomorphic in \( \mathcal{C} \).

Proof. Since \( \alpha : \text{Hom}_{\mathcal{C}}(c, -) \Rightarrow F \) and \( \alpha' : \text{Hom}_{\mathcal{C}}(c', -) \Rightarrow F \) are natural isomorphisms,
\[
\beta = \alpha^{-1} \circ \alpha' : \text{Hom}_{\mathcal{C}}(c', -) \Rightarrow \text{Hom}_{\mathcal{C}}(c, -)
\]
is a natural isomorphism. Since the Yoneda functor \( y^* : \mathcal{C}^{\text{op}} \to [\mathcal{C}, \text{Set}] \) is fully faithful, there exists a unique morphism \( \gamma : c \to c' \) so that \( y^*(\gamma) = \beta \). Since \( \beta \) is an isomorphism and \( y^* \) is fully faithful, \( \gamma \) has to be an isomorphism as well (check it!). \( \square \)

Remark 25.4. Recall that by Yoneda lemma for any functor \( F : \mathcal{C} \to \text{Set} \) the function
\[
\Phi : \text{Nat}(\text{Hom}_{\mathcal{C}}(c, -), F) \to F(c), \quad \Phi(\alpha) = \alpha_c(\text{id}_c)
\]
is a bijection. So we can equally define a representation of a functor \( F : \mathcal{C} \to \text{Set} \) to be a pair \((c, x)\) where \( c \in \mathcal{C} \) is an object and \( x \in F(c) \) is an element of \( F(c) \) such that \( \Psi(x) := \Phi^{-1}(x) : \text{Hom}_{\mathcal{C}}(c, -) \Rightarrow F \) is a natural isomorphism.
**Definition 25.5.** Let $\mathcal{C}$ be a locally small category and $F : \mathcal{C} \to \text{Set}$ a functor. A universal element (of the functor $F$) is a pair $(c, x)$ where $c$ is an object of $\mathcal{C}$ and $x$ is an element of the set $F(c)$ so that the corresponding natural transformation $\Psi(x) : \text{Hom}_\mathcal{C}(c, -) \Rightarrow F$ is a natural isomorphism. Here as before

$$\Phi : \text{Nat}(\text{Hom}_\mathcal{C}(c, -), F) \to F(c), \quad \Phi(\alpha) = \alpha_c(\text{id}_c),$$

is the Yoneda lemma bijection and $\Psi(x) = \Phi^{-1}(x)$.

**Example 25.6.** As we have seen in Example 24.9 the forgetful functor $U : \text{Group} \to \text{Set}$ is represented by $(\mathbb{Z}, \alpha : \text{Hom}_{\text{Group}}(\mathbb{Z}, -) \Rightarrow U)$ where $\alpha_G(\varphi) = \varphi(1)$ for every group $G$ and any $\varphi \in \text{Hom}_{\text{Group}}(\mathbb{Z}, G)$. Since $\alpha_{\mathbb{Z}}(\text{id}_{\mathbb{Z}}) = \text{id}_{\mathbb{Z}}(1) = 1 \in U(\mathbb{Z})$, the pair $(\mathbb{Z}, 1)$ is a universal element of the forgetful functor $U : \text{Group} \to \text{Set}$.

**Example 25.7.** As we have seen in Example 25.1 the forgetful functor $U : \text{Vect}_\mathbb{R} \to \text{Set}$ is represented by $(\mathbb{R}, \alpha : \text{Hom}_{\text{Vect}_\mathbb{R}}(\mathbb{R}, -) \Rightarrow U)$ where $\alpha_V(T) = T(1)$ for a vector space $V$ and any $T \in \text{Hom}_{\text{Vect}_\mathbb{R}}(\mathbb{R}, V)$. Since $\alpha_{\mathbb{R}}(\text{id}_{\mathbb{R}}) = \text{id}_{\mathbb{R}}(1) = 1 \in U(\mathbb{R})$ the pair $(\mathbb{R}, 1)$ is a universal element of the forgetful functor $U : \text{Vect}_\mathbb{R} \to \text{Set}$.

The next lemma give a useful criterion for a pair $(c, x \in F(c))$ to be a universal element of a functor $F : \mathcal{C} \to \text{Set}$.

**Lemma 25.8.** Let $\mathcal{C}$ be a locally small category, $F : \mathcal{C} \to \text{Set}$ a functor, $c \in \mathcal{C}$ an object, $x \in F(c)$ an element and $\Psi(x) = \Phi^{-1}(x) : \text{Hom}_\mathcal{C}(c, -) \Rightarrow F$ the corresponding natural transformation. Then

$$\Psi(x) \text{ is a natural isomorphism and } (c, x) \text{ is a universal element } \iff \text{for every } d \in \mathcal{C} \text{ and every } y \in F(d), \text{ there exists a unique morphism } f : c \to d \text{ so that } F(f)(x) = y.$$

**Proof.** $\Psi(x)$ is a natural isomorphism

$\iff$ For every $d \in \mathcal{C}$, $\Psi(x)_d : \text{Hom}_\mathcal{C}(c, d) \to F(d)$ is a bijection.

$\iff$ For every $d \in \mathcal{C}$ and every $y \in F(d)$, there exists unique morphism $c \overset{f}{\to} d$ so that $\Psi(x)_d(f) = y$.

By Remark 24.1

$$\Psi(x)_d(f) = y \iff F(f)x = y$$

and the result follows. $\square$

**Example 25.9.** Consider the forgetful functor $U : \text{Vect} \to \text{Set}$. Let $c \in \text{Vect}_\mathbb{R}$ be the 1-dimensional vector space $\mathbb{R}$ and $x = 1 \in U(\mathbb{R}) = \mathbb{R}$. For every vector space $V$ and every $y \in U(V)$, there exists a unique linear map $T : \mathbb{R} \to V$ that such that $U(T)(1) = T(1) = y$. Hence

$$\Psi(1) : \text{Hom}_{\text{Vect}}(\mathbb{R}, -) \Rightarrow U$$

is a natural isomorphism by Lemma 25.8 and $(\mathbb{R}, 1 \in U(\mathbb{R}) = \mathbb{R})$ is a universal element as we have seen in Example 25.1.

There is also a contravariant version of Lemma 25.8. We leave its proof as an exercise.

**Lemma 25.10.** Let $F : \mathcal{C}^{\text{op}} \to \text{Set}$ be a functor. A pair $(c \in \mathcal{C}, x \in F(c))$ is a universal element representing $F$ (and in particular, $F$ is representable) if and only if for any $d \in \mathcal{C}$ and any $y \in F(d)$, there exists a unique morphism $d \overset{h}{\to} c$ in $\mathcal{C}$ (that is, a morphism $c \overset{h^{\text{op}}}{\to} d$ in $\mathcal{C}^{\text{op}}$) so that $F(h)(x) = y$.

**Example 25.11.** Consider the contravariant powerset functor $\mathcal{P} : \text{Set}^{\text{op}} \to \text{Set}$. On objects it is defined by

$$X \mapsto \mathcal{P}(X),$$
the power set of \( X \). Given a function \( f : X \to Y \), \( \mathcal{P}(f) : \mathcal{P}(X) \to \mathcal{P}(Y) \) is defined by
\[
\mathcal{P}(f)A := f^{-1}(A)
\]
for all \( A \subseteq Y \). We claim that \( \mathcal{P} \) is representable and that \( (\{0, 1\}, \{1\} \in \mathcal{P}(\{0, 1\})) \) is a universal element.

**Proof.** For any set \( X \) and any subset \( A \subseteq X \), there exists a unique function \( \chi_A : X \to \{0, 1\} \) so that \( \chi_A^{-1}(\{1\}) = A \), namely, the characteristic function of the subset \( A \). That is, for any set \( X \), any \( A \in \mathcal{P}(X) \), there exists a unique morphism \( \chi_A \in \text{Hom}_{\text{Set}}(X, \{0, 1\}) \) so that \( \mathcal{P}(\chi)(\{1\}) = A \). Therefore the functor \( \mathcal{P} : \text{Set}^{\text{op}} \to \text{Set} \) is representable and \( (\{0, 1\}, \{1\} \in \mathcal{P}(\{0, 1\})) \) is a universal element. \( \square \)

**Lecture 26. Universal arrows. Comma categories.**

**Last time:**

- \( \mathcal{C} \) be a locally small category, \( F : \mathcal{C} \to \text{Set} \) a functor, \( c \in \mathcal{C} \), \( x \in F(c) \) and \( \Psi(x) = \Phi^{-1}(x) : \text{Hom}_{\mathcal{C}}(c, -) \Rightarrow F \) the corresponding natural transformation. Then

\[
\Psi(x) : \text{Hom}_{\mathcal{C}}(c, -) \Rightarrow F \text{ is a natural isomorphism and } (c, x) \text{ is a universal element } \iff \forall d \in \mathcal{C} \text{ and } \forall y \in F(d), \exists! \text{ morphism } f : c \to d \text{ so that } F(f)(x) = y.
\]

We also say that the pair \((c, x)\) represents the functor \( F \).

- Similarly given \( F : \mathcal{C}^{\text{op}} \to \text{Set} \), \( c \in \mathcal{C} \), \( x \in F(c) \) and \( \Psi(x) = \Phi^{-1}(x) : \text{Hom}_{\mathcal{C}}(-, c) \Rightarrow F \) the corresponding natural transformation. Then

\[
\Psi(x) : \text{Hom}_{\mathcal{C}}(-, c) \Rightarrow F \text{ is a natural isomorphism and } (c, x) \text{ is a universal element } \iff \forall d \in \mathcal{C} \text{ and } \forall y \in F(d), \exists! \text{ morphism } h : d \to c \text{ so that } F(h)(x) = y.
\]

Let \( R : \mathcal{B} \to \mathcal{A} \) be a functor between two locally small categories and \( a \in \mathcal{A} \) an object. We can follow the functor \( R \) by the functor \( \text{Hom}_{\mathcal{A}}(a, -) : \mathcal{A} \to \text{Set} \) and get a functor
\[
\text{Hom}_{\mathcal{A}}(a, -) \circ R = \text{Hom}_{\mathcal{A}}(a, R(-)) : \mathcal{B} \to \text{Set}.
\]
What does the representability of the functor \( \text{Hom}_{\mathcal{A}}(a, R(-)) : \mathcal{B} \to \text{Set} \) amount to?

**Lemma 26.1.** Let \( R : \mathcal{B} \to \mathcal{A} \) be a functor between two locally small categories and \( a \in \mathcal{A} \) an object.

The functor \( \text{Hom}_{\mathcal{A}}(a, R(-)) : \mathcal{B} \to \text{Set} \) is representable. \( \iff \)

There is an object \( b_0 \in \mathcal{B} \) and a morphism \( a \xrightarrow{\eta} R(b_0) \) so that for any \( b \in \mathcal{B} \) and any \( a \xrightarrow{h} R(b) \) there is a unique morphism \( h : b_0 \to b \) in \( \mathcal{B} \) with \( R(h) \circ \eta = h \), i.e., the diagram
\[
\begin{array}{ccc}
a & \xrightarrow{\eta} & R(b_0) \\
\downarrow{h} & & \downarrow{R(h)} \\
R(b) & \xrightarrow{=} & R(b)
\end{array}
\]
commutes.

**Proof.** By Lemma 25.8 the functor \( G := \text{Hom}_{\mathcal{A}}(a, R(-)) : \mathcal{B} \to \text{Set} \) is representable \( \iff \)
there is an object \( b_0 \in \mathcal{B} \) and an element \( \eta \in G(b_0) = \text{Hom}_{\mathcal{A}}(a, R(b_0)), \) i.e., a morphism \( a \xrightarrow{\eta} R(b_0) \)
in \( \mathcal{A} \), so that for any \( b \in \mathcal{B} \) and any \( h \in G(b) = \text{Hom}_A(a, R(b)) \) there is a unique object \( \tilde{h} \in (\text{Hom}_\mathcal{B}(b_0, -))(b) \) (i.e., a morphism \( \tilde{h} : b_0 \to b \) in \( \mathcal{B} \)) with

\[
G(\tilde{h})\eta = h.
\]

Since \( G(\tilde{h}) = \text{Hom}_A(a, -)(R(\tilde{h})) = R(\tilde{h})_* \), equation (26.36) is equivalent to

\[
R(\tilde{h}) \circ \eta = h,
\]

which is exactly the commutativity of (26.35), and we are done. \( \square \)

**Example 26.2.** Recall that for any set \( X \) there is a vector space \( F(X) \) and a function \( X \xrightarrow{\eta_X} U(F(X)) \) (where \( U : \text{Vect} \to \text{Set} \) is the forgetful functor) with the following universal property: for any vector space \( W \) and any function \( h : X \to U(W) \) there is a unique linear map \( F(X) \xrightarrow{\tilde{h}} W \) so that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\eta_X} & U(F(X)) \\
\downarrow{h} & & \downarrow{U(\tilde{h})} \\
U(W) & & 
\end{array}
\]

commutes. By Lemma [26.1] the functor \( \text{Hom}_{\text{Set}}(X, U(-)) : \text{Set} \to \text{Set} \) is representable and is represented by \( (F(X), X \xrightarrow{\eta_X} U(F(X))) \).

**Remark 26.3.** In Lemma [26.1] the object \( b_0 \in \mathcal{B} \) depends on \( a \) as does the morphism \( \eta : a \to R(b_0) \). If the functor \( \text{Hom}_A(a, R(-)) : \mathcal{B} \to \text{Set} \) are representable for every object \( a \in \mathcal{A} \) we get a function \( \mathcal{A}_0 \ni a \mapsto b_0(a) \in \mathcal{B}_0 \). We’ll see in Lecture 29 that the function on objects extends to a functor \( L : \mathcal{A} \to \mathcal{B} \). The morphisms \( \eta : a \to R(b_0) = R(L(a)) \) depend on \( a \in \mathcal{A}_0 : \eta = \eta_a \). We will see that the collection of morphisms \( \{\eta_a\}_{a \in \mathcal{A}_0} \) consists of the components of a natural transformation \( \eta : \text{id}_\mathcal{A} \Rightarrow RL \).

It will be useful to have the dual version of Lemma [26.1]

**Lemma 26.4.** Let \( L : \mathcal{A} \to \mathcal{B} \) be a functor between two locally small categories and \( b \in \mathcal{B} \) an object. The functor \( \text{Hom}_B(L(-), b) : \mathcal{B} \to \text{Set} \) is representable. \( \iff \)

There is an object \( a_0 \in \mathcal{A} \) and a morphism \( L(a_0) \xrightarrow{\varepsilon} b \) so that for any \( a \in \mathcal{A} \) and any morphism \( L(a) \xrightarrow{h} b \) there is a unique morphism \( \tilde{h} : a \to a_0 \) in \( \mathcal{A} \) with \( \varepsilon \circ L(\tilde{h}) \circ h = h \), i.e., the diagram

\[
\begin{array}{ccc}
b & \xleftarrow{\varepsilon} & L(a_0) \\
\downarrow{h} & & \downarrow{L(\tilde{h})} \\
L(a) & & 
\end{array}
\]

commutes.

**Proof.** By Lemma [25.10] the functor \( \text{Hom}_B(L(-), b) : \mathcal{B} \to \text{Set} \) is representable \( \iff \) there is \( a_0 \in \mathcal{A} \) and \( \varepsilon \in \text{Hom}_B(L(-), b)(a_0) = \text{Hom}_B(L(a_0), b) \) so that for any \( a \in \mathcal{A} \) and any \( h \in \text{Hom}_B(L(-), b)(a) = \text{Hom}_B(La, b) \) there is a unique \( a \xrightarrow{\tilde{h}} a_0 \) in \( \mathcal{A} \) so that

\[
(\text{Hom}_B(L(-), b)(\tilde{h}))(\varepsilon) = h.
\]

Since \( \text{Hom}_B(L(-), b)(\tilde{h}) = L(h)^* \) equation (26.38) amounts to

\[
\varepsilon \circ L(h) = h,
\]
which is the commutativity of \([26.37]\).

\[\square\]

**Definition 26.5.** Given a functor \(U : \mathcal{C} \to \mathcal{A}\) and an object \(a \in \mathcal{A}\), a **universal arrow** from \(a\) to \(U\) is a pair \((c, a \xrightarrow{\gamma} U(c)) \in \mathcal{C}_0 \times \mathcal{A}_1\) with the following universal property: for any pair \((c', f : a \to U(c')) \in \mathcal{C}_0 \times \mathcal{A}_1\) there exists a unique morphism \(\tilde{f} : c \to c'\) in \(\mathcal{C}\) so that the diagram

\[
\begin{array}{ccc}
a & \xrightarrow{\gamma} & U(c) \\
\downarrow{f} & & \downarrow{U(\tilde{f})} \\
U(c') & & \\
\end{array}
\]

commutes.

As we have just seen in Lemma \([26.1]\) the existence of a universal arrow \((c, a \xrightarrow{\gamma} U(c))\) for a functor \(U : \mathcal{C} \to \mathcal{A}\) is equivalent to representability of the functor \(\text{Hom}_\mathcal{A}(a, U(-)) : \mathcal{C} \to \text{Set}\) (with the arrow representing the functor). On the other hand universal arrows can also be interpreted as initial objects in the appropriate categories. To explain this we need to first define comma categories.

**Definition 26.6.** Let \(F : \mathcal{B} \to \mathcal{A}\) and \(G : \mathcal{C} \to \mathcal{A}\) be a pair or functors. The **comma category** \((F \downarrow G)\) (also denoted by \((F \Rightarrow G)\) and no longer denoted by \((F, G)\), which was Lawvere’s original notation back in 1963) is defined as follows:

The objects of \((F \downarrow G)\) are triples \((b, F(b) \xrightarrow{\gamma} G(c), c)\) where \(b \in \mathcal{B}\), \(c \in \mathcal{C}\), and \(F(b) \xrightarrow{\gamma} G(c) \in \mathcal{A}\).

A morphism in \((F \downarrow G)\) from \((b, F(b) \xrightarrow{\gamma} G(c), c)\) to \((b', F(b') \xrightarrow{\gamma'} G(c'), c')\) is a pair of morphisms \((b \xrightarrow{\beta} b', c \xrightarrow{\gamma} c') \in \mathcal{B}_1 \times \mathcal{C}_1\) so that the diagram

\[
\begin{array}{ccc}
F(b) & \xrightarrow{\gamma} & G(c) \\
F(\beta) \downarrow & & \downarrow{F(\mu)} \\
F(b') & \xrightarrow{\gamma'} & G(c') \\
\end{array}
\]

commutes.

Given two morphisms \((\beta, \mu) : (b, F(b) \xrightarrow{\gamma} G(c), c \to) (b', F(b') \xrightarrow{\gamma'} G(c'), c')\) and \((\nu, \tau) : (b', F(b') \xrightarrow{\gamma'} G(c'), c') \to (b'', F(b'') \xrightarrow{\gamma''} G(c''), c'')\) in the comma category \((F \downarrow G)\) their composition is defined “coordinate-wise”:

\[
(b' \xrightarrow{\nu \circ \beta} b'', c' \xrightarrow{\tau \circ \mu} c'') = \left(b \xrightarrow{\nu \circ \beta} b'', c \xrightarrow{\tau \circ \mu} c''\right)
\]

(It is not hard to check that \((\nu \circ \beta, \tau \circ \mu)\) is a morphism in \((F \downarrow G)\) from \((b, F(b) \xrightarrow{\gamma} G(c), c)\) to \((b'', F(b'') \xrightarrow{\gamma''} G(c''), c'')\) so the definition of composition makes sense.)

**Remark 26.7.** Let \(1\) be the category with one objects and one morphisms: \(1 = \left\{ \bullet \xrightarrow{\text{id}_\bullet} \right\}\). For any category \(\mathcal{A}\), the functor

\[
ev : [1, \mathcal{A}] \to \mathcal{A}, \quad ev \left(F \xrightarrow{\alpha} G\right) := (a \bullet : F(\bullet) \to G(\bullet))
\]

is an isomorphism of categories. In particular, we can identify an object \(a \in \mathcal{A}\) with the functor

\[
a : 1 \to \mathcal{A}, \quad a(\bullet \xrightarrow{\text{id}_\bullet} \bullet) = a \xrightarrow{\text{id}_a} a.
\]
Consequently given a functor $U : C \rightarrow A$ and an object $a \in A$ (i.e., a functor $a : 1 \rightarrow A$), we have the comma category $(a \downarrow U)$. The objects of $(a \downarrow U)$ are triples $(\bullet, a \rightarrow U(c), c)$ where $\bullet$ is the only object of $1$, $c \in C$, and $\gamma \in \text{Hom}_A(a, U(c))$. So in effect, the objects of $(a \downarrow U)$ are pairs $(a \rightarrow U(c), c)$. A morphism in $(a \downarrow U)$ from $(a \rightarrow U(c), c)$ to $(a \rightarrow U(c'), c')$ is a morphism $f : c \rightarrow c'$ in $C$ so that the diagram

\[
\begin{array}{ccc}
    a & \xrightarrow{\gamma} & U(c) \\
    \downarrow{\text{id}_a} & & \downarrow{U(f)} \\
    a & \xrightarrow{\gamma'} & U(c')
\end{array}
\]

commutes.

**Lemma 26.8.** Let $U : C \rightarrow A$ be a functor, $a \in A$ an object. The pair $(c, a \rightarrow U(c))$ is a universal arrow from $a$ to $U$ if and only if $(a \rightarrow U(c), c)$ is initial in the comma category $(a \downarrow U)$.

**Proof.** $(a \rightarrow U(c), c)$ is initial in $(a \downarrow U)$

$\iff$ For every object $(a \rightarrow U(c'), c')$ in $(a \downarrow U)$, there exists unique morphism $c \rightarrow c'$ so that the diagram

\[
\begin{array}{ccc}
    a & \xrightarrow{\gamma} & U(c) \\
    \downarrow{\text{id}_a} & & \downarrow{U(f)} \\
    a & \xrightarrow{\gamma'} & U(c')
\end{array} \hspace{1cm}
\begin{array}{ccc}
    a & \xrightarrow{\gamma} & U(c) \\
    \downarrow{\text{id}_a} & & \downarrow{U(f)} \\
    a & \xrightarrow{\gamma'} & U(c')
\end{array} =
\begin{array}{ccc}
    a & \xrightarrow{\gamma} & U(c) \\
    \downarrow{\text{id}_a} & & \downarrow{U(f)} \\
    a & \xrightarrow{\gamma'} & U(c')
\end{array}
\]

commutes.

$\iff$ The pair $(c, a \rightarrow U(c))$ is a universal arrow from $a$ to $U$. $\square$

**Remark 26.9.** Given two functors $F : B \rightarrow A$ and $G : C \rightarrow A$ the corresponding comma category $(F \downarrow G)$ is analogous to the fiber product $B \times_A C$ in the category $\text{CAT}$ of categories, but it is not exactly a fiber product. There are two projection functors

\[
p_1 : (F \downarrow G) \rightarrow B, \quad p_1(b, F(b) \rightarrow G(c), c) = b \\
p_2 : (F \downarrow G) \rightarrow C, \quad p_2(b, F(b) \rightarrow G(c), c) = c
\]

but the diagram

\[
(\begin{array}{ccc}
    (F \downarrow G) & \xrightarrow{p_2} & C \\
    p_1 & & \downarrow{G} \\
    B & \xrightarrow{F} & A
\end{array})
\]

does not commute. Instead there is a natural transformation $\alpha : F \circ p_1 \Rightarrow G \circ p_2$. Can you figure out what $\alpha$ is?

Last time:
- For a functor \( R : \mathcal{B} \to \mathcal{A} \) and an object \( a \in \mathcal{A} \) representability of \( \text{Hom}_{\mathcal{A}}(a, R(-)) : \mathcal{B} \to \text{Set} \) is equivalent to:
  There is an object \( L(a) \in \mathcal{B} \) and a morphism \( a \xrightarrow{\eta_a} R(L(a)) \) so that for any \( b \in \mathcal{B} \) and any \( a \xrightarrow{h} R(b) \) there is a unique \( \tilde{h} : L(a) \to b \) such that the diagram
  \[
  \begin{array}{ccc}
  a & \xrightarrow{\eta_a} & R(L(a)) \\
    & \searrow & \downarrow \\
    & h & \Rightarrow R(\tilde{h}) \\
    & \nearrow & b
  \end{array}
  \]
  commutes.

- For a functor \( L : \mathcal{A} \to \mathcal{B} \) and an object \( b \in \mathcal{B} \) representability of \( \text{Hom}_{\mathcal{B}}(L(-), b) : \mathcal{A} \to \text{Set} \) is equivalent to:
  There is an object \( R(b) \in \mathcal{A} \) and a morphism \( L(R(b)) \xrightarrow{\varepsilon_b} b \) so that for any \( a \in \mathcal{A} \) and any \( L(a) \xrightarrow{h} b \) there is a unique \( \tilde{h} : L(a) \to b \) such that the diagram
  \[
  \begin{array}{ccc}
  b & \xleftarrow{\varepsilon_b} & L(R(b)) \\
    & \searrow & \downarrow \\
    & h & \Rightarrow L(\tilde{h}) \\
    & \nearrow & L(a)
  \end{array}
  \]
  commutes.

Horizontal composition of natural transformations.

Let \( \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xleftarrow{\beta} \mathcal{C} \) be three categories, four functors, and two natural transformations.

Our goal is to construct a natural transformation
\[
\beta \ast \alpha : KG \Rightarrow LH,
\]
the horizontal composition of \( \alpha \) and \( \beta \).

Since \( \alpha \) is a natural transformation, given \( a \in \mathcal{A} \), we have a morphism \( KG(a) \xrightarrow{\alpha_a} K(a) \) in \( \mathcal{B} \). Since \( \beta \) is a natural transformation, the square \( \beta_{G(a)} \)
\[
\begin{array}{ccc}
KG(a) & \xrightarrow{K(\alpha_a)} & KG(a) \\
\beta_{G(a)} & \downarrow & \downarrow \beta_{H(a)} \\
LH(a') & \xleftarrow{L(\alpha_a)} & LH(a')
\end{array}
\]
commutes. Define a function \( \beta \ast \alpha : \mathcal{A}_0 \to \mathcal{C}_1 \) by
\[
(\beta \ast \alpha)_a := \beta_{H(a)} \circ K(\alpha_a) = L(\alpha_a) \circ \beta_{G(a)}
\]
for every \( a \in A \):

\[
\begin{array}{ccc}
KG(a) & \xrightarrow{K(\alpha_a)} & KG(a) \\
\beta_{G(a)} & & \beta_{H(a)} \\
LH(a') & \xrightarrow{L(\alpha_a)} & LH(a')
\end{array}
\]

We need to check that \( \beta \circ \alpha \) is a natural transformation from \( KG \) to \( LH \), that is, for every \( a \xrightarrow{\delta} a' \) in \( A \), the diagram

\[
\begin{array}{ccc}
KG(a) & \xrightarrow{KG(\delta)} & KG(a') \\
(\beta \circ \alpha)_a & & (\beta \circ \alpha)_{a'} \\
LH(a) & \xrightarrow{LH(\delta)} & LH(a')
\end{array}
\]

commutes. Since the diagram

\[
\begin{array}{ccc}
G(a) & \xrightarrow{\alpha_a} & H(a) \\
G(\delta) & & H(\delta) \\
G(a') & \xrightarrow{\alpha_{a'}} & H(a')
\end{array}
\]

commutes in \( B \), the cube

\[
\begin{array}{ccc}
KG(a) & \xrightarrow{K(\alpha_a)} & KH(a) \\
\beta_{G(a)} & & \beta_{H(a)} \\
LG(a) & \xrightarrow{L(\alpha_a)} & LH(a) \\
KG(a') & \xrightarrow{K(\alpha_{a'})} & KH(a') \\
\beta_{G(a')} & & \beta_{H(a')} \\
LG(a') & \xrightarrow{L(\alpha_{a'})} & LH(a')
\end{array}
\]

commutes in \( C \). Therefore the diagonal square commutes as well, which is what we wanted to prove. Hence \( \beta \circ \alpha : KG \Rightarrow LH \) defined by (27.39) is a natural transformation and we are done.

There are two special cases of horizontal composition which are known as whiskering. Suppose as before that we have

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & B \\
H & & L \\
\xleftarrow{\beta} & & \xrightarrow{C}
\end{array}
\]

**Special case 1:** Suppose \( G = H \) and \( \alpha = \text{id}_G \):

\[
\begin{array}{ccc}
A & \xrightarrow{\text{id}_G} & B \\
G & & L \\
\xleftarrow{\beta} & & \xrightarrow{C}
\end{array}
\]

Then \( \beta \circ \text{id}_G : KG \Rightarrow LG \) is given by

\[
(\beta \circ \text{id}_G)_a = \beta_{H(a)} \circ K((\text{id}_G)_a) = \beta_{G(a)} \circ \text{id}_{K(a)} = \beta_{G(a)}
\]
for all $a \in \mathcal{A}$. It is common to denote $\beta \ast \text{id}_G$ by $\beta_G$ or by $\beta G$: $\mathcal{A} \xymatrix{ \ar[r]^-{\beta G} & \mathcal{C} }$

**Special case 2:** Suppose $K = L$ and $\beta = \text{id}_K$: $\mathcal{A} \xymatrix{ \ar[r]^-{H} & \mathcal{B} & \ar[r]^-{\text{id}_K} & \mathcal{C} }$. Then $\text{id}_K \ast \alpha : KG \Rightarrow KH$ is given by

$$\text{id}_K \ast \alpha_a = (\text{id}_K)_{H(a)} \circ K(\alpha_a) = \text{id}_{KH(a)} \circ K(\alpha_a) = K(\alpha_a)$$

for all $a \in \mathcal{A}$. It is common to denote $\text{id}_K \ast \alpha$ by $K\alpha$: $\mathcal{A} \xymatrix{ \ar[r]^-{K\alpha} & \mathcal{C} }$

Whiskering will be important when we discuss the triangle identities for the unit and the counit of an adjunction.

**Adjoint functors**

There are three equivalent ways of defining adjunctions and adjoint functors:

1. as a natural bijections of Hom’s;
2. in terms of units, counits, and the triangle identities;
3. in terms of universal arrows.

We start with the first one. Remember that for us, all categories under discussion are ($V$-)locally small for a large enough universe $V$.

**Definition 27.1.** An adjunction is a pair of functors $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{A}$ together with a family of bijections

$$\theta = \theta_{a,b} : \text{Hom}_B(Fa, b) \to \text{Hom}_A(a, Gb), \quad (a, b) \in \mathcal{A}_0 \times \mathcal{B}_0$$

which are natural in $a$ and $b$. Here “natural” means:

(i) for every morphism $a' \xrightarrow{f} a$ in $\mathcal{A}$, the diagram

$$\begin{array}{ccc}
\text{Hom}_B(Fa, b) & \xrightarrow{\theta_{a,b}} & \text{Hom}_A(a, Gb) \\
(Ff)^* \downarrow & & \downarrow f^* \\
\text{Hom}_B(Fa', b) & \xrightarrow{\theta_{a',b}} & \text{Hom}_A(a', Gb)
\end{array}$$

commutes and

(ii) for every $b \xrightarrow{g} b'$ in $\mathcal{B}$, the diagram

$$\begin{array}{ccc}
\text{Hom}_B(Fa, b) & \xrightarrow{\theta_{a,b}} & \text{Hom}_A(a, Gb) \\
g_* \downarrow & & \downarrow (Gg)_* \\
\text{Hom}_B(Fa, b') & \xrightarrow{\theta_{a,b'}} & \text{Hom}_A(a, Gb')
\end{array}$$

commutes as well.
Similarly for any set $Y$ commutativity of the diagram $F$ commutes. This is because the linear map $T$ commutes. This is because for any are bijections. Moreover, for ever vector space $V$ are bijections. Hence, the family of functions

$\theta = \theta_{X,W} : \text{Hom}_{\text{Vect}}(F(X), W) \to \text{Hom}_{\text{Set}}(X, U(W))$

are bijections. Moreover, for ever vector space $V$ and any linear map $S : W \to V$ the diagram

$\text{Hom}_{\text{Vect}}(F(X), W) \xrightarrow{\theta_{X,W}} \text{Hom}_{\text{Set}}(X, U(W))$

$\downarrow S_* \quad \downarrow U(S)_*$

$\text{Hom}_{\text{Vect}}(F(X), V) \xrightarrow{\theta_{X,V}} \text{Hom}_{\text{Set}}(X, U(V))$

commutes. This is because for any $T : F(X) \to V$

$\theta(S_* T) = U(S_* T) \circ \eta_X = U(S \circ T) \circ \eta_X = U(S) \circ U(T) \circ \eta_X = U(S)_*(U(T) \circ \eta_X) = U(S_*(\theta(T))).$

Similarly for any set $Y$ and any function $h : Y \to X$ the diagram

$\text{Hom}_{\text{Vect}}(F(Y), W) \xrightarrow{\theta_{Y,W}} \text{Hom}_{\text{Set}}(Y, U(W))$

$\downarrow (F(h))^* \quad \downarrow h^*$

$\text{Hom}_{\text{Vect}}(F(Y), W) \xrightarrow{\theta_{Y,W}} \text{Hom}_{\text{Set}}(Y, U(W))$

commutes. This is because the linear map $F(h) : F(Y) \to F(X)$ is uniquely defined by the commutativity of the diagram

$X \xrightarrow{\eta_X} UF(X)$

$h$ \quad $UF(h)$.

$Y \xrightarrow{\eta_Y} UF(Y)$
Therefore for any object $a$ and every morphism $b: a \to b$, there exists a unique morphism $h: a \to b$.

Hence for any linear map $T: F(X) \to W$

$$h^*(\theta(T)) = (U(T) \circ \eta_X) \circ h = U(T) \circ U(F(h)) \circ \eta_Y = U(F(h)^* T) \circ \eta_Y = \theta(F(h)^* T).$$

Therefore $F: \text{Set} \rightleftharpoons \text{Vect}: U$ is an adjunction. \hfill $\square$

Lecture 28. Examples of adjunctions

Last time:

- An adjunction is a pair of functors $F: \mathcal{A} \to \mathcal{B}$, $G: \mathcal{B} \to \mathcal{A}$ together with a family of bijections

$$\{\theta = \theta_{a,b}: \text{Hom}_\mathcal{B}(Fa, b) \to \text{Hom}_\mathcal{A}(a, Gb)\}_{(a,b) \in \mathcal{A} \times \mathcal{B}}$$

which are natural in $a$ and $b$. The functor $F$ is left adjoint to $G$ and $G$ is right adjoint to $F$.

- We saw one example: $F: \text{Set} \rightleftharpoons \text{Vect} : U$. The bijections

$$\theta_{X,W}: \text{Hom}_\text{Vect}(F(X), W) \to \text{Hom}_\text{Set}(X, U(W))$$

are “restrict the function defined by the linear map to the canonical basis of $F(X)$”:

$$\theta_{X,W}(T) = U(T) \circ \eta_X.$$ 

Remark 28.1. The direction of the bijections $\theta$ in the definition of an adjunction $F \dashv G$ does not matter. We may just as well require that there are bijections

$$\text{Hom}_\mathcal{A}(a, Gb) \to \text{Hom}_\mathcal{B}(Fa, b),$$

which are natural in $a$ and $b$.

Example 28.2. Consider the forgetful functor $U: \text{Top} \to \text{Set}$. The functor $U$ has a left adjoint $D: \text{Set} \to \text{Top}$ and a right adjoint $I: \text{Set} \to \text{Top}$.

On objects the functor $D: \text{Set} \to \text{Top}$ is defined by $D(X) = (X, \mathcal{P}(X))$, so any subset of $X$ is open in the topological space $D(X)$. On morphisms, $D$ is defined by $D(f) = f$ for any function $f: X \to Z$. Consequently, given any topological space $(Y, T)$, any function from $D(X)$ to $Y$ is continuous. It is easy to check that $D$ is a functor. For any set $X$ and any topological space $(Y, T)$, we define

$$\theta_{X,Y}: \text{Hom}_\text{Set}(X, U(Y, T)) \to \text{Hom}_\text{Top}(D(X), (Y, T)), \quad \theta_{X,Y}(f) = f$$

The naturality of $\theta$ is easy to check. Hence $D \dashv U$.

On objects the functor $I: \text{Set} \to \text{Top}$ is defined by $I(X) = (X, \{\emptyset, X\})$. Note that for any topological space $(Y, T)$, any function $f: (Y, T) \to (X, \{\emptyset, X\})$ is continuous. On morphisms $I$ is defined by $I(f) = f$ for every function $f: X \to Z$. We define

$$\theta_{X,Y}: \text{Hom}_\text{Set}((Y, T)), X) \to \text{Hom}_\text{Top}((Y, T), I(X)), \quad \theta_{X,Y}(f) = f$$

Again the naturality of $\theta$ is easy to check. Hence $U \dashv I$.

Example 28.3. Let $\mathbb{1} = \{ \ast \, \circ \, \text{id}_\ast \}$ be the category with one object and one morphism. Let $\mathcal{A}$ be a nonempty category. There is a canonical functor $F: \mathcal{A} \to \mathbb{1}$ that sends every object $a \in \mathcal{A}$ to $\ast$ and every morphism $\gamma \in \mathcal{A}$ to $\text{id}_\ast$.

Suppose $F$ has a left adjoint $L: \mathbb{1} \to \mathcal{A}$. Then for every $a \in \mathcal{A}$ we have a bijection

$$\text{Hom}_\mathcal{A}(L(\ast), a) \cong \text{Hom}_\mathbb{1}(\ast, F(a)) = \{ \text{id}_\ast \}.$$ 

Therefore, for any object $a \in \mathcal{A}$, there exists a unique morphism $L(\ast) \to a$. In other words $L(\ast)$ is an initial object of $\mathcal{A}$.

Suppose that $F: \mathcal{A} \to \mathbb{1}$ has a right adjoint $R: \mathbb{1} \to \mathcal{A}$. Then for every object $a \in \mathcal{A}$, we have a bijection

$$\text{Hom}_\mathbb{1}(F(a), \ast) = \{ \text{id}_\ast \} \cong \text{Hom}_\mathcal{A}(a, R(\ast)).$$
Therefore, \( R(*) \) is a terminal object of \( A \).

To summarize, left adjoints to the unique functor \( A \to \mathbb{1} \) are the initial objects in \( A \) and right adjoints are the terminal objects in \( A \).

As we have seen in Example 27.3, the underlying set functor \( U : \text{Vect} \to \text{Set} \) has a left adjoint \( F : \text{Set} \to \text{Vect} \). Other forgetful functors such as \( U : \text{Group} \to \text{Set} \) and \( U : \text{Mon} \to \text{Set} \) have left adjoints as well. Below we show that the forgetful functor from the category \( \text{Mon} \) of monoids to \( \text{Set} \) has a left adjoint.

**Example 28.4.** The forgetful functor \( U : \text{Mon} \to \text{Set} \) has a left adjoint \( F : \text{Set} \to \text{Mon} \).

To understand what we need to construct suppose that the left adjoint \( F : \text{Set} \to \text{Mon} \) exists. Then for any set \( X \) and any monoid \( M \) we have a bijection

\[ \theta_{X,M} : \text{Hom}_{\text{Set}}(X,U(M)) \cong \text{Hom}_{\text{Mon}}(F(X),M), \]

which is natural in \( X \) and \( M \). Fix \( X \) and vary \( M \). Then we have a natural isomorphism

\[ \theta_X : \text{Hom}_{\text{Set}}(X,U(-)) \cong \text{Hom}_{\text{Mon}}(F(X),-). \]

Therefore the functor \( \text{Hom}_{\text{Set}}(X,U(-)) : \text{Mon} \to \text{Set} \) is representable and \( \text{Hom}_{\text{Mon}}(F(X),-) \) represents it. By Lemma 26.1 there is a function \( X \xrightarrow{\eta_X} UF(X) \) so that for any monoid \( M \) and any function \( X \xrightarrow{f} U(M) \), there exists a unique homomorphism \( \bar{f} : F(X) \to M \) so that the diagram

\[ \begin{array}{ccc} X & \xrightarrow{\eta_X} & UF(X) \\ f \downarrow & & \downarrow U(\bar{f}) \\ U(M) & & \end{array} \]

commutes.

**Lemma 28.5.** For any set \( X \) there is a monoid \( F(X) \) and a function \( \eta_X : X \to U(F(X)) \) such that for any monoid \( M \) and any function \( X \xrightarrow{f} U(M) \), there exists a unique homomorphism \( \bar{f} : F(X) \to M \) so that (28.41) commutes.

**Proof.** For a set \( X \), let \( F(X) \) denote the set of finite lists of elements of \( X \). That is, \( a \in F(X) \) if and only if \( a \) is the empty list () or if there exists \( n > 0 \), \( x_1, \ldots, x_n \in X \) so that \( a = x_1x_2 \ldots x_n \).

Some authors call \( a \in F(X) \) a word, \( X \) an alphabet and denote the monoid \( F(X) \) by \( X^* \). The notation \( X^* \) is fairly common in the CS textbooks.

The multiplication of elements of \( F(X) \) is concatenation:

\[(x_1x_2 \ldots x_n) \cdot (y_1y_2 \ldots y_m) = x_1x_2 \ldots x_ny_1y_2 \ldots y_m\]

The multiplication is associative. The universal arrow \( \eta_X : X \to F(X) \) is defined by sending \( x \in X \) to the list \( x \) of length one. Given a monoid \( M \) and a function \( f : X \to U(M) \), there is a homomorphism \( \bar{f} : F(X) \to M \) defined by \( \bar{f}() = e \) (\( f \) sends the empty list to the identity of \( M \)) and \( \bar{f}(x_1x_2 \ldots x_n) = f(x_1) \cdot f(x_2) \cdot \ldots \cdot f(x_n) \) where \( \cdot \) is the multiplication in \( M \). The requirement

\[ X \xrightarrow{\eta_X} UF(X) \]

that \( \begin{array}{ccc} X & \xrightarrow{\eta_X} & UF(X) \\ f \downarrow & & \downarrow U(\bar{f}) \\ U(M) & & \end{array} \) commutes makes \( \bar{f} \) unique. \( \square \)

Just as in the case of vector spaces the function \( \text{Set} \ni X \to F(X) \in \text{Mon} \) extends to a functor \( F : \text{Set} \to \text{Mon} \). Moreover the functor \( F \) is left adjoint to \( U \). There are several ways to prove it.
We can try and mimic the proof in the case of sets and vector spaces. Or we can prove a general result: see Theorem 29.2 in the next lecture.

**Remark 28.6.** The monoid \( F(X) \) constructed in Lemma 28.5 is called free (more precisely a free monoid on the set \( X \)).

We end the section with two results. The first relates adjunctions and duality. The second explains what the word “natural” in the definition of adjunction have to do with an existence of a natural isomorphism.

**Lemma 28.7.** If \( \mathcal{A} \xrightarrow{L} \mathcal{B} \) is an adjunction, then \( \mathcal{A}^{\text{op}} \xleftarrow{R^{\text{op}}} \mathcal{B}^{\text{op}} \) is also an adjunction.

**Proof.** Recall that given a functor \( F : \mathcal{C} \to \mathcal{D} \) with have a functor \( F^{\text{op}} : \mathcal{C}^{\text{op}} \to \mathcal{D}^{\text{op}} \) between the opposite categories which is defined by

\[
F^{\text{op}}(c' \xrightarrow{\gamma^{\text{op}}} c) := F^{\text{op}}(c') \xrightarrow{F(\gamma)^{\text{op}}} F^{\text{op}}(c)
\]

for all morphisms \( c' \xrightarrow{\gamma^{\text{op}}} c \) in \( \mathcal{C}^{\text{op}} \), i.e., for all morphisms \( \gamma : c \to c' \) in \( \mathcal{C} \). Since \( L : \mathcal{A} \xrightarrow{\cong} \mathcal{B} : R \) is an adjunction with \( L \dashv R \) (\( L \) is left adjoint to \( R \)), we have bijections

\[
\text{Hom}_B(La, b) \xrightarrow{\theta_{a,b}} \text{Hom}_A(a, Rb)
\]

natural in \( a \) and \( b \). But \( \text{Hom}_B(La, b) = \text{Hom}_{B^{\text{op}}}(b, L^{\text{op}}a) \) and \( \text{Hom}_A(a, Rb) = \text{Hom}_{A^{\text{op}}}(Gb, a) \). Consequently we have bijections

\[
\text{Hom}_{B^{\text{op}}}(b, L^{\text{op}}a) \xrightarrow{\bar{\theta}_{b,a}} \text{Hom}_{A^{\text{op}}}(R^{\text{op}}b, a), \quad \bar{\theta}_{b,a}(b \xrightarrow{\gamma^{\text{op}}} L^{\text{op}}a) := (a \xrightarrow{\theta_{b,a}(\gamma)} Rb)^{\text{op}}
\]

It is not hard to check that these bijections are natural in \( b \) an in \( a \). We check naturality in \( a \) and leave the other check as an exercise.

Recall that the naturality of bijections \( \{\theta_{a,b}\} \) in \( a \) means that for any morphism \( f : a' \to a \) in \( \mathcal{A} \) the diagram

\[
\begin{array}{ccc}
\text{Hom}_B(La, b) & \xrightarrow{\theta_{a,b}} & \text{Hom}_A(a, Rb) \\
((Lf)^*) \downarrow & & \downarrow f^* \\
\text{Hom}_B(La', b) & \xrightarrow{\theta_{a',b}} & \text{Hom}_A(a', Rb)
\end{array}
\]

commutes. We need to show that

\[
\text{Hom}_{B^{\text{op}}}(b, L^{\text{op}}a) \xrightarrow{(f^{\text{op}})_*} \text{Hom}_{A^{\text{op}}}(R^{\text{op}}b, a)
\]

\[
((Lf)^{\text{op}})_* \downarrow & & \downarrow (f^{\text{op}})_* \\
\text{Hom}_{B^{\text{op}}}(b, L^{\text{op}}a') \xrightarrow{\bar{\theta}_{b,a'}} \text{Hom}_{A^{\text{op}}}(R^{\text{op}}b, a')
\]

commutes. For any \( \gamma^{\text{op}} \in \text{Hom}_{B^{\text{op}}}(b, L^{\text{op}}a) \)

\[
(f^{\text{op}})_* (\bar{\theta}_{b,a}(\gamma^{\text{op}})) = f^{\text{op}} \circ (\theta_{a,b}(\gamma))^{\text{op}} = (\theta_{a,b}(\gamma) \circ f)^{\text{op}} = \theta_{a',b}(\gamma \circ Lf)
\]

\[
= \bar{\theta}_{b,a'}((\gamma \circ Lf)^{\text{op}}) = \bar{\theta}_{b,a'}((Lf)^{\text{op}} \circ \gamma^{\text{op}}) = \bar{\theta}_{b,a'}(((Lf)^{\text{op}})_*(\gamma^{\text{op}}))
\]

and we are done. \( \square \)
Proposition 28.8. The triple \((L : A \to B, R : B \to A, \{\theta_{a,b} : \text{Hom}_B(La, b) \to \text{Hom}_A(a, Rb)\})_{(a,b) \in A \times B}\) is an adjunction if and only if \(\theta : L \Rightarrow R\) is a natural isomorphism, where
\[
L := \text{Hom}_B(La, -) : A \to \text{Set} \quad \text{and} \quad R := \text{Hom}_B(-, Ra) : B \to \text{Set}.
\]

Proof. Exercise. \qed

Lecture 29. Adjunctions from the universal arrows.

Last time:
- Examples of adjunctions: \(U : \text{Top} \to \text{Set}\) has a left and a right adjoint.
- The canonical functor \(F : A \to 1\) has a left adjoint \(\iff A\) has an initial object. The canonical functor \(F : A \to 1\) has a right adjoint \(\iff A\) has a terminal object.
- Given a set \(X\) constructed of a free monoid \(X^* \equiv F(X)\) together with a universal arrow \((F(X), X \eta X \to U(F(X)))\). Waved hands that the construction leads to an adjunction \(F : \text{Set} \to \text{Mon} / \text{ob}\).

Remark 29.1. Recall that if \(R : B \to A\) is a functor then the pair \((b, a \eta R(b))\) represents the functor \(\text{Hom}_A(a, R(-)) : B \to \text{Set}\) (i.e., there is a natural isomorphism \(\alpha : \text{Hom}_B(b, -) \Rightarrow \text{Hom}_A(a, R(-))\)), \(\alpha_b(id_b) = \eta \in \text{Hom}_A(a, R(-))(b)\)

For any morphism \(a \gamma \to R(b')\) in \(A\) there is a unique morphism \(b \tilde{\gamma} \to b'\) so that the diagram
\[
\begin{array}{ccc}
a & \xrightarrow{\eta} & R(b) \\
\downarrow{\gamma} & & \downarrow{R(\tilde{\gamma})} \\
R(b') & & \\
\end{array}
\]
commutes.

Theorem 29.2. Let \(R : B \to A\) be a functor. Suppose we have a function \(L : A_0 \to B_0\) on objects so that for all \(a \in A_0\) the functor \(\text{Hom}_B(La, -)\) represents the functor \(\text{Hom}_A(a, R(-))\). Let \((La, a \eta a)\) be a representation of \(\text{Hom}_A(a, R(-))\). Then the function \(L\) extends to a functor \(L : A \to B\) which is left adjoint to the functor \(R : L \dashv R\).

Moreover the morphisms \(\{\eta_a\}_{a \in A_0}\) are components of a natural transformation \(\eta : \text{id}_A \Rightarrow RL\).

Proof. We extend the function \(L : A_0 \to B_0\) to morphisms: given \(h : a \to a'\) in \(A\), consider the diagram
\[
\begin{array}{ccc}
a & \xrightarrow{\eta_a} & RL(a) \\
h \downarrow & & \downarrow{\eta_a \circ h} \\
a' & \xrightarrow{\eta_{a'}} & RL(a') \\
\end{array}
\]
Since \((La, a \eta a)\) is a universal arrow, there exists unique \(L(a) \tilde{h} \to La'\) such that the diagram
\[
\begin{array}{ccc}
a & \xrightarrow{\eta_a} & RL(a) \\
h \downarrow & & \downarrow{R(h)} \\
a' & \xrightarrow{\eta_{a'}} & RL(a') \\
\end{array}
\]

89
commutes. Define $L(h)$ to be this $\tilde{h}$. Then by construction the diagram

\[
\begin{array}{ccc}
a & \xrightarrow{\eta_a} & RL(a) \\
h & \downarrow & \downarrow RL(h) \\
a' & \xrightarrow{\eta_{a'}} & RL(a')
\end{array}
\]

(29.42)

commutes. Note that once we show that $L$ is a functor the commutativity of (29.42) immediately implies that the collection $\{\eta_a\}_{a \in A_0}$ of morphisms is the collection of components of a natural transformation $\eta : id_A \to RL$. If $h = id_a$ then both

\[
\begin{array}{ccc}
a & \xrightarrow{\eta_a} & RL(a) \\
id_a & \downarrow & \downarrow RL(id_a) \\
a & \xrightarrow{\eta_a} & RL(a)
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
a & \xrightarrow{\eta_a} & RL(a) \\
id_a & \downarrow & \downarrow R(id_L(a)) \\
a & \xrightarrow{\eta_a} & RL(a)
\end{array}
\]

commute. But for any morphism $a \xrightarrow{h} a'$ there is only one morphism $L(a) \xrightarrow{\tilde{h}} L(a')$ so that (29.42) commutes. Hence $L(id_a) = id_{L(a)}$

Similarly given two composable morphisms $a \xrightarrow{h} a' \xrightarrow{k} a''$ in $A$ the two squares

\[
\begin{array}{ccc}
a & \xrightarrow{\eta_a} & RL(a) \\
h & \downarrow & \downarrow RL(h) \\
a' & \xrightarrow{\eta_{a'}} & RL(a') \\
k & \downarrow & \downarrow RL(k) \\
a'' & \xrightarrow{\eta_{a''}} & RL(a'')
\end{array}
\]

commute. Hence the outer square

\[
\begin{array}{ccc}
a & \xrightarrow{\eta_a} & RL(a) \\
kh & \downarrow & \downarrow RL(k) \circ RL(h) = R(L(k) \circ L(h)) \\
a'' & \xrightarrow{\eta_{a''}} & RL(a'')
\end{array}
\]

commutes as well. On the other hand by definition of $L(kh) = \tilde{kh}$ the square

\[
\begin{array}{ccc}
a & \xrightarrow{\eta_a} & RL(a) \\
kh & \downarrow & \downarrow RL(kh) \\
a'' & \xrightarrow{\eta_{a''}} & RL(a'')
\end{array}
\]

commutes. It follows that $L(k) \circ L(h) = L(kh)$.

This proves that $L$ is a functor.
It remains to construct the bijections
\( \theta_{ab} : \text{Hom}_B(La, b) \to \text{Hom}_A(a, Rb) \)
that are natural in \( a \) and \( b \). Fix \( a \in A \). Since \( \text{Hom}_B(La, -) \) represents the functor \( \text{Hom}_A(a, R(-)) \) we have a natural isomorphism
\( \rho_a : \text{Hom}_B(La, -) \to \text{Hom}_A(a, R(-)) \)
whose components \( (\rho_a)_b : \text{Hom}_B(La, b) \to \text{Hom}_A(a, R(b)) \) are given by
\[ (\rho_a)_b(La \to h \to b) = a \overset{R(h)\eta_a}{\to} R(b), \]
see Lemma \( \ref{26.1} \). Now define
\( \theta_{a,b} := (\rho_a)_b, \)
that is,
\[ \theta_{a,b}(La \to h \to b) := a \overset{R(h)\eta_a}{\to} R(b). \]
Since \( \{ (\rho_a)_b \}_{b \in B} \) is a set of components of a natural transformation, \( \theta_{a,b} \) is natural in \( b \): for any \( b \overset{k}{\to} b' \) the diagram
\[
\begin{array}{ccc}
\text{Hom}_B(La, b) & \xrightarrow{\theta_{a,b}} & \text{Hom}_A(a, Rb) \\
\downarrow{k_*} & & \downarrow{(Rk)_*} \\
\text{Hom}_B(La, b') & \xrightarrow{(\rho_a)_{b'}} & \text{Hom}_A(a, Rb')
\end{array}
\]
commutes. It remains to check naturality in \( a \): we need to show that any \( a' \overset{\ell}{\to} a \) in \( A \), the diagram
\[
\begin{array}{ccc}
\text{Hom}_B(La, b) & \xrightarrow{\theta_{a,b}} & \text{Hom}_A(a, Rb) \\
\downarrow{(F\ell)^*} & & \downarrow{\ell^*} \\
\text{Hom}_B(La', b') & \xrightarrow{(\rho_{a'})_{b'}} & \text{Hom}_A(a', Rb')
\end{array}
\]
commutes.

Since \( \eta : \text{id}_A \Rightarrow RL \) is a natural transformation, the diagram
\[
\begin{array}{ccc}
a' & \xrightarrow{\eta_{a'}} & RL(a') \\
\downarrow{\ell} & & \downarrow{RF(\ell)} \\
a & \xrightarrow{\eta_a} & RL(a)
\end{array}
\]
commutes, i.e.,
\[ RL(\ell)\eta_{a'} = \eta_a\ell. \]
Therefore, for any \( a' \overset{\ell}{\to} a \) and for any \( h \in \text{Hom}_B(La, b) \)
\[ \ell^*(\theta_{a,b}(h)) = R(h)\eta_a\ell (\ref{29.45}) = R(h)RL(\ell)\eta_{a'} = R(hL(\ell))\eta_{a'} = R(L(\ell)^*h)\eta_{a'} = \theta_{a',b}(L(\ell)^*h). \]
Hence (29.44) commutes and we are done. \( \square \)

In light of Remark \( \ref{29.1} \) Theorem \( \ref{29.2} \) can be restated as follows:

**Theorem 29.3.** Let \( R : B \to A \) be a functor. Suppose we have a function \( L : A_0 \to B_0 \) so that for any \( a \in A_0 \) there is a morphism \( a \overset{\eta_a}{\to} R(L(a)) \) with the following universal property: for any morphism \( a \overset{\gamma}{\to} R(b) \) in \( A \) there is a unique morphism \( L(a) \overset{\gamma'}{\to} b \) so that the diagram

\[
\begin{array}{ccc}
a & \overset{\eta_a}{\to} & R(L(a)) \\
\downarrow{\gamma} & & \downarrow{\gamma'} \\
b & \overset{\exists! \gamma'}{\to} & b
\end{array}
\]
adjoint to the functor $R$. The morphisms $\{\eta_a\}_{a \in A_0}$ are components of a natural transformation $\eta : \text{id}_A \Rightarrow RL$.

Example 29.4. We have seen in Example 28.4 that for any set $X$ there is a monoid $F(X)$ and a function $X \xrightarrow{\eta_X} U(F(X))$ from $X$ to the set underlying $F(X)$ so that for any monoid $M$ and any function $f : X \to U(M)$, there exists a unique homomorphism $\tilde{f} : F(X) \to M$ making the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\eta_X} & UF(X) \\
\downarrow{f} & & \downarrow{U(f)} \\
U(M) & & \\
\end{array}
$$

commute. By Theorem 29.3 the underlying set functor $U : \text{Mon} \to \text{Set}$ has a left adjoint $F : \text{Set} \to \text{Mon}$.

Lecture 30. Colimits and limits as adjoints to the diagonal functor $\Delta$.

Last time:

- **Theorem 29.3** Let $R : B \to A$ be a functor. Suppose we have a function $L : A_0 \to B_0$ so that for any $a \in A_0$ there is a morphism $a \xrightarrow{\eta_a} R(L(a))$ with the following universal property: for any morphism $a \xrightarrow{\gamma} R(b)$ in $A$ there is a unique morphism $L(a) \xrightarrow{\tilde{\gamma}} b$ so that the diagram

$$
\begin{array}{ccc}
a & \xrightarrow{\eta} & R(L(a)) \\
\downarrow{\gamma} & & \downarrow{R(\tilde{\gamma})} \\
R(b) & & \\
\end{array}
$$

commutes. Then the function $L$ extends to a functor $L : A \to B$ which is left adjoint to the functor $R$. The morphisms $\{\eta_a\}_{a \in A_0}$ are components of a natural transformation $\eta : \text{id}_A \Rightarrow RL$.

Theorem 29.3 has a dual version:

**Theorem 30.1.** Let $L : A \to B$ be a functor. Suppose that there is a function $R : B_0 \to A_0$ so that for any object $b \in B$ there is a morphism $\varepsilon_b : LR(b) \to b$ with the following universal property: for any morphism $La \xrightarrow{h} b$ in $B$ there is a unique morphism $a \xrightarrow{\tilde{h}} R(b)$ in $A$ so that the diagram

$$
\begin{array}{ccc}
b & \xleftarrow{\varepsilon_b} & LR(b) \\
\downarrow{h} & & \downarrow{L(\tilde{h})} \\
L(a) & & \\
\end{array}
$$

commutes. Then the function $R$ extends to a functor $R : B \to A$ which is right adjoint to the functor $L$. The morphisms $\{\varepsilon_b\}_{b \in B}$ are components of a natural transformation $\varepsilon : LR \Rightarrow \text{id}_B$.

**Proof.** The functor $L : A \to B$ defines a functor $L^{\text{op}} : A^{\text{op}} \to B^{\text{op}}$. By assumption for every object $b \in B$ we have a morphism $\varepsilon_b^{\text{op}} : b \to L^{\text{op}}R(b)$ in $B^{\text{op}}$ so that for any morphism $h^{\text{op}} : b \to L^{\text{op}}a$ there

\[ \varepsilon_b^{\text{op}} h^{\text{op}} = L^{\text{op}}(\tilde{h}) \]
is a unique morphism $\tilde{h}^{\text{op}} : R(b) \to a$ in $\mathcal{A}^{\text{op}}$ so that the diagram

\[
\begin{array}{ccc}
  b & \xrightarrow{\varepsilon_b^{\text{op}}} & L^{\text{op}} R(b) \\
  \downarrow \varepsilon_b^{\text{op}} & & \downarrow L^{\text{op}} (\tilde{h})^{\text{op}} \\
  R(b) & \xrightarrow{\tilde{h}^{\text{op}}} & \end{array}
\]

commutes in $\mathcal{B}^{\text{op}}$. By Theorem 29.3 the function $R : \mathcal{B}_0 \to \mathcal{A}_0$ extends to a functor $\tilde{R} : \mathcal{B}^{\text{op}} \to \mathcal{A}^{\text{op}}$ which is left adjoint to $L^{\text{op}} : \mathcal{A}^{\text{op}} \to \mathcal{B}^{\text{op}}$ and $\{\varepsilon_b^{\text{op}}\}_{b \in \mathcal{B}}$ are components of a natural transformation $\varepsilon^{\text{op}} : \text{id}_{\mathcal{B}^{\text{op}}} \Rightarrow L^{\text{op}} \tilde{R}$. By Lemma 28.7 the functor $\tilde{R} = (\tilde{R})^{\text{op}} : \mathcal{B} \to \mathcal{A}$ is right adjoint to the functor $(L^{\text{op}})^{\text{op}} = L$.

It is easy to see that the morphisms $\{\varepsilon_b\}_{b \in \mathcal{B}}$ are components of a natural transformation $\varepsilon : LR \Rightarrow \text{id}_{\mathcal{B}}$. □

Colimits and adjunctions.

Recall that given two categories $I$ and $\mathcal{C}$ we have the functor category $[I, \mathcal{C}]$. The objects of $[I, \mathcal{C}]$ are functors from $I$ to $\mathcal{C}$, which are also known as diagrams in $\mathcal{C}$ of shape $I$. The morphisms are natural transformations and the composition is the vertical composition of natural transformations.

Recall that for every object $c \in \mathcal{C}$ we have the “constant” functor $\Delta_c : I \to \mathcal{C}$ defined by

\[
\Delta_c(i) = c\quad \text{and}\quad \Delta_c(j) = c
\]

for all morphisms $i \xrightarrow{\gamma} j$ in $I$. We can extend the function $\Delta_c : I_0 \to [I, \mathcal{C}]_0, c \mapsto \Delta_c$ to a functor $\Delta : I \to [I, \mathcal{C}]$ by defining $\Delta$ on morphisms: given a morphism $c \xrightarrow{f} c'$ in $\mathcal{C}$ there is a natural transformation $\Delta_f : \Delta_c \Rightarrow \Delta_{c'}$ with components $(\Delta_f)_i = c \xrightarrow{f} c'$.

Note that $\Delta_f$ so defined is a natural transformation since for any morphism $i \xrightarrow{\gamma} j$ in $I$ the diagram

\[
\begin{array}{ccc}
  c & \xrightarrow{f} & c' \\
  \downarrow \text{id}_c & & \downarrow \text{id}_{c'} \\
  c & \xrightarrow{f} & c'
\end{array}
\]

commutes for any morphism $i \xrightarrow{\gamma} j$ in $I$. We will refer to $\Delta : \mathcal{C} \to [I, \mathcal{C}]$ as the diagonal functor.

We now assume that any functor $F : I \to \mathcal{C}$ has a colimit. We then have a function

\[
\text{colim} : [I, \mathcal{C}]_0 \to \mathcal{C}_0, \quad F \mapsto \text{colim} F,
\]

where $\text{colim} F$ denotes the vertex of the initial cocone $(\text{colim} F, \{\iota_i : F(i) \to \text{colim} F\}_{i \in I})$. Recall that the set $\{\iota_i\}_{i \in I}$ of morphisms in $\mathcal{C}$ is the set of components of a natural transformation $\iota : F \Rightarrow \Delta_{\text{colim} F}$. This natural transformation is a morphism in $[I, \mathcal{C}]$ and the morphism $\iota$ has the following universal property:
For any morphism $F \xrightarrow{\gamma} \Delta_c$ (a cocone on $F$ with the vertex $c$) there is a unique morphism $\text{colim} F \xrightarrow{\tilde{\gamma}} c$ so that the diagram

\[
\begin{array}{ccc}
F & \xrightarrow{\iota} & \Delta_{\text{colim} F} \\
\downarrow{\gamma} & & \downarrow{\Delta_\gamma} \\
& \downarrow & \\
& & \Delta_c
\end{array}
\]

commutes.

This is just a translation of the universal property of the colimits into the language of natural transformations: for any cocone $(c, \{\gamma_i : F(i) \to c\}_{i \in I})$ there is a unique morphism $\tilde{\gamma} : \text{colim} F \to c$ so that the diagrams

\[
\begin{array}{ccc}
\text{colim} F & \xrightarrow{\tilde{\gamma}} & c \\
\downarrow{\iota_i} & & \downarrow{\gamma_i} \\
F(i) & & \\
\end{array}
\]

commute for all $i \in I$. Theorem 29.3 now implies:

**Proposition 30.2.** Let $I$ and $C$ be two categories. Suppose any functor $F : I \to C$ has a colimit. Then the function $F \mapsto \text{colim} F$ extends to a functor $\text{colim} : [I, C] \to C$ which is left adjoint to the constant functor $\Delta : C \to [I, C]$:

\[
[I, C] \xrightarrow{\text{colim}} C \xleftarrow{\Delta} C
\]

Moreover the natural transformations $\{\iota_F = \iota : F \Rightarrow \Delta_{\text{colim} F}\}_{F \in [I, C]}$ assemble into a natural transformation

$\iota : \text{id}_{[I, C]} \Rightarrow \Delta \circ \text{colim}$.

**Remark 30.3.** By (29.43) the bijections $\theta_{F,c} : \text{Hom}_C(\text{colim} F, c) \to \text{Hom}_{[I,C]}(F, \Delta_c)$ are given by

\[
(\text{colim} F \xrightarrow{\tilde{\gamma}} c) \mapsto (F \xrightarrow{\iota_F} \Delta_{\text{colim} F} \xrightarrow{\Delta_\gamma} \Delta_c).
\]

**Limits and adjunctions.**

Recall that the limit of a functor $F : I \to C$ is a cone $(\lim F, \{\pi_i : \lim F \to F(i)\}_{i \in I})$ such that for any cone $(d, \{q_i : d \to F(i)\}_{i \in I})$ on $F$, there exists a unique morphism $\bar{q} : d \to L$ making the diagram

\[
\begin{array}{ccc}
d & \xrightarrow{\bar{q}} & L \\
\downarrow{q_i} & & \downarrow{\pi_i} \\
F(i) & & \\
\end{array}
\]

diagram commute for every $i \in I$. In the language of natural transformations and the diagonal functor $\Delta$ the universal property of limits translates into:
For any morphism $\Delta_d \xrightarrow{q} F$ there is a unique morphism $\Delta_d \xrightarrow{\tilde{q}} \lim F$ so that the diagram

\[
\begin{array}{ccc}
F & \xleftarrow{\pi} & \Delta_{\lim F} \\
\downarrow & \searrow & \downarrow \\
\downarrow & \Delta_{d} & \equiv \equiv \\
\end{array}
\]

commutes in the functor category $[I, C]$.

Theorem 30.1 now implies:

**Proposition 30.4.** Let $I$ and $C$ be two categories. Suppose any functor $F : I \to C$ has a limit. Then the function

$[I, C] \ni F \mapsto \lim F \in C$

extends to a functor $\lim : [I, C] \to C$. The functor $\lim$ is right adjoint to the diagonal functor $\Delta$:

$[I, C] \underleftarrow{\lim} \rightarrow C$

Moreover the natural transformations $\{\pi_F = \pi : \Delta_{\lim F} \Rightarrow F\}_{F \in [I, C]}$ assemble into a natural transformation

$\pi : \Delta \circ \lim \Rightarrow \id_{[I, C]}$.

**Remark 30.5.** The bijections $\theta_{F,c} : \Hom_C(d, \lim F) \to \Hom_{[I, C]}(\Delta_d, F)$ are given by

$(d \xrightarrow{q} \lim F) \mapsto (\Delta_d \xrightarrow{\Delta_{\tilde{q}}} \Delta_{\lim F} \xrightarrow{\pi} F)$.

**Lecture 31.** Universal arrows; units and counits. Uniqueness of left adjoints.

**Last time:**

- Suppose a category $C$ has all colimits of shape $I$. The assignment $F \mapsto \colim F$ extends to a functor $\colim$. $\colim$ is left adjoint to the diagonal functor $\Delta$:

$[I, C] \underleftarrow{\colim} \rightarrow C$

- Suppose a category $C$ has all limits of shape $I$. The assignment $F \mapsto \lim F$ extends to a functor $\lim$. $\lim$ is right adjoint to the diagonal functor $\Delta$:

$[I, C] \rightarrow C \underrightarrow{\lim}$

Recall that a universal arrow from an object $a$ of a category $A$ to a functor $R : B \to A$ is an initial object in the comma category $(a \downarrow R)$.
it is a pair \((a \overset{\eta}{\to} Rb, b)\) with \(\eta\) a morphism in \(\mathcal{A}\) and \(b\) an object of \(\mathcal{B}\) so that for \(b' \in \mathcal{B}\) and any morphism \(a \overset{h}{\to} R(b')\) there is a unique morphism \(f : b \to b'\) so that

\[
h = R(f) \circ \eta
\]
or, equivalently, the diagram

\[
\begin{array}{ccc}
a & \overset{\eta}{\to} & R(b) \\
\downarrow{h} & & \downarrow{R(f)} \\
R(b') & \overset{R(f)}{\to} & R(b')
\end{array}
\]

commutes. Dually a (co)universal arrow from a functor \(L : \mathcal{A} \to \mathcal{B}\) to \(b \in \mathcal{B}\) is a terminal object \((L(a) \overset{\varepsilon}{\to} b, a)\) in the comma category \((L \downarrow b)\):

for any morphism \(L(a') \overset{h}{\to} b\) there is a unique morphism \(a' \overset{f}{\to} a\) so that

\[
h = \varepsilon \circ L(f)
\]
or, equivalently, the diagram

\[
\begin{array}{ccc}
b & \overset{\varepsilon}{\leftarrow} & L(a) \\
\downarrow{h} & & \downarrow{L(f)} \\
L(a') & \overset{L(f)}{\to} & L(a')
\end{array}
\]

commutes. (Both terms “universal arrow” and “couniversal arrow” are used in the dual case.)

We have seen in Theorems 29.3 and 30.1 that universal arrows give rise to adjunctions. These theorems have a converse. We have glimpsed the converse in Example 28.4. We now state and prove this converse.

**Theorem 31.1.** Suppose \((L : \mathcal{A} \xrightarrow{\cong} \mathcal{B} : R, \{\theta_{a,b}\}_{(a,b) \in \mathcal{A} \times \mathcal{B}}\) is an adjunction. Then for any object \(a \in \mathcal{A}\) the pair \((La, a \overset{\eta_a}{\to} RLa)\), where \(\eta_a := \theta_{a,La}((id_{La}))\), is a universal arrow from \(a\) to the functor \(R\).

Dually for any object \(b \in \mathcal{B}\) the pair \((Rb, LRb \overset{\varepsilon_b}{\to} b)\), where \(\varepsilon_b = \theta_{Rb,b}^{-1}((id_{Rb}))\), is a universal arrow from \(L\) to \(b\).

**Proof.** Fix \(a \in \mathcal{A}\). Since the bijections \(\theta_{a,b}\) are natural in \(b\) for any morphism \(La \overset{f}{\to} b\) in \(\mathcal{B}\) the diagram

\[
\begin{array}{ccc}
\text{Hom}_\mathcal{B}(La, La) & \xrightarrow{\theta_{a,La}} & \text{Hom}_\mathcal{A}(a, RLa) \\
\downarrow{f_*} & & \downarrow{R(f)_*} \\
\text{Hom}_\mathcal{B}(La, b) & \xrightarrow{\theta_{a,b}} & \text{Hom}_\mathcal{A}(a, Rb)
\end{array}
\]

commutes. Therefore, as we have seen in the proof of the Yoneda lemma,

\[
(31.46) \quad \theta_{a,b}(f) = \theta_{a,b}(f_*id_{La}) = R(f)_*(\theta_{a,La}(id_{La})) = R(f) \circ \eta_a.
\]

Since \(\theta_{a,b} : \text{Hom}_\mathcal{B}(La, b) \to \text{Hom}_\mathcal{A}(a, Rb)\) is a bijection, for any \(h : a \to Rb\) there exists a unique \(f : La \to b\) so that

\[
h = \theta_{a,b}(f) = R(f) \circ \eta_a.
\]

Therefore \((La, a \overset{\eta_a}{\to} RLa)\) is a universal arrow from \(a\) to \(R\).

Similarly since the bijections \(\theta_{a,b}^{-1}\) are natural in \(a\) for any morphism \(a \overset{f}{\to} Rb\) in \(\mathcal{A}\) the diagram
Recall that the Yoneda Embedding \( y \) isomorphism commutes. That is, for a fixed object \( a \)
\[
\theta : \text{Hom}_A(a, Rb) \to \text{Hom}_B(La, b) \to \text{Hom}_A(Rb, Rb)
\]
commutes. Therefore
\[
\theta^{-1}_{a,b}(f) = \theta^{-1}_{a,b}(f^* \text{id}_{Rb}) = L(f)^*(\theta^{-1}_{Ra,b}(\text{id}_{Rb})) = \varepsilon_b \circ L(f).
\]
Since \( \theta^{-1}_{a,b} : \text{Hom}_A(a, Rb) \to \text{Hom}_B(La, b) \to \text{Hom}_A(Rb, Rb) \)
is a bijection, for any \( h : La \to b \) there exists a unique \( f : a \to Rb \) so that
\[
h = \theta^{-1}_{a,b}(f) = \varepsilon_b \circ L(f).
\]
Therefore \( (Rb, LRb) \) is a universal arrow from \( L \to b \).

**Remark 31.2.** It follows from Theorems 31.1, 29.3 and 30.1 that existence of an adjunction is equivalent to existence of universal arrows.

**Remark 31.3.** It follows from Theorem 29.3 and from its dual, Theorem 30.1 that given an adjunction \( (L : \mathcal{A} \rightleftarrows \mathcal{B} : R, \{\eta_a = \theta_{a,La}(\text{id}_{La})\}_{a \in \mathcal{A}} \) and \( \{\varepsilon_b = \theta_{Ra,b}^{-1}(\text{id}_{Rb})\}_{b \in \mathcal{B}} \)
are components of natural transformations \( \eta : \text{id}_A \Rightarrow RL \) and \( \varepsilon : LR \Rightarrow \text{id}_B \).

The transformation \( \eta : \text{id}_A \Rightarrow RL \) is called the unit of the adjunction \( L \dashv R \). The transformation \( \varepsilon : LR \Rightarrow \text{id}_B \) is called the counit of the adjunction.

We next prove that any two functors left adjoint to a given functor are isomorphic.

**Lemma 31.4.** Left adjoints are unique up to an isomorphism. Explicitly, suppose \( R : \mathcal{B} \to \mathcal{A}, L_1, L_2 : \mathcal{A} \to \mathcal{B} \) are functors with \( L_1 \dashv R \) and \( L_2 \dashv R \). Then there exists a natural isomorphism \( \alpha : L_1 \Rightarrow L_2 \).

**Proof.** For every \( a \in \mathcal{A}, b \in \mathcal{B} \), we have bijections
\[
\text{Hom}_B(L_1a, b) \xrightarrow{\sim} \text{Hom}_A(a, Rb) \xrightarrow{\sim} \text{Hom}_B(L_2a, b)
\]
that are natural in \( a \) and \( b \). Hence we have a family of bijections
\[
\tau_{a,b} : \text{Hom}_B(L_1a, b) \to \text{Hom}_B(L_2a, b)
\]
that are natural in \( a \) and \( b \). Naturality in \( b \) says that for every morphism \( b \xrightarrow{h} b' \) in \( \mathcal{B} \), the diagram
\[
\begin{array}{ccc}
\text{Hom}_B(L_1a, b) & \xrightarrow{\tau_{a,b}} & \text{Hom}_B(L_2a, b) \\
\downarrow h_* & & \downarrow h_* \\
\text{Hom}_B(L_1a, b') & \xrightarrow{\tau_{a,b'}} & \text{Hom}_B(L_2a, b')
\end{array}
\]
commutes. That is, for a fixed object \( a \in \mathcal{A} \), the morphisms \( \{\tau_{a,b}\}_{b \in \mathcal{B}} \)
are components of a natural isomorphism
\[
\tau_a : \text{Hom}_B(L_1a, -) \Rightarrow \text{Hom}_B(L_2a, -).
\]
Recall that the Yoneda Embedding \( y^* : \mathcal{B}^{\text{op}} \to [\mathcal{B}, \text{Set}] \) is is given by
\[
y^*(b \xrightarrow{h} b') = \text{Hom}_B(b', -) \xrightarrow{h^*} \text{Hom}_B(b, -)
\]
and that \( y^* \) is fully faithful. Hence, for every object \( a \in \mathcal{A} \), there exists a unique morphism \( \alpha_a : L_2a \to L_1a \) in \( \mathcal{B} \) so that
\[
y^*(L_2a \xrightarrow{\alpha_a} L_1a) = y^*(L_1a) \xrightarrow{\tau_a} y^*(L_2a).
\]

We need to check that \( \{\alpha_a\}_{a \in \mathcal{A}} \) are components of a natural transformation \( \alpha : L_2 \Rightarrow L_1 \), that is, that for every morphism \( a \to a' \) in \( \mathcal{A} \), the diagram
\[
\begin{array}{ccc}
L_2a & \xrightarrow{\alpha_a} & L_1a \\
\downarrow_{L_2k} & & \downarrow_{L_1k} \\
L_2a' & \xrightarrow{\alpha_{a'}} & L_1a'
\end{array}
\]
commutes. Since the Yoneda embedding \( y^* \) is fully faithful, it is enough to check that
\[
y^*(L_1a) \xrightarrow{\tau_a} y^*(L_2a)
\]
(31.48)
\[
y^*(L_1k) \xrightarrow{\tau_{a,b}} y^*(L_2k)
\]
\[
y^*(L_1a') \xrightarrow{\tau_{a',b}} y^*(L_2a)
\]
commutes. By definition of \( y^* \) commutativity of (31.48) is equivalent to the commutativity of
\[
\begin{array}{ccc}
\text{Hom}_\mathcal{B}(L_1a, b) & \xrightarrow{(L_1k)^*} & \text{Hom}_\mathcal{B}(L_2a, b) \\
\downarrow_{(L_1k)^*} & & \downarrow_{(L_2k)^*} \\
\text{Hom}_\mathcal{B}(L_1a', b) & \xrightarrow{(L_1k)^*} & \text{Hom}_\mathcal{B}(L_2a', b)
\end{array}
\]
for every object \( b \in \mathcal{B} \). But this is just naturality of \( \tau_{a,b} \) in \( a \). Hence (31.48) commutes for every morphism \( a \to a' \) in \( \mathcal{A} \). Therefore \( \alpha : L_2 \Rightarrow L_1 \) is a natural isomorphism, and in particular the two left adjoint functors \( L_2 \) and \( L_1 \) are isomorphic. \( \square \)

Lecture 32. Triangle identities.

Last time:

- An adjunction \( \mathcal{A} \xleftarrow{L} \mathcal{B} \xrightarrow{R} \mathcal{A} \) gives rise to two families of universal arrows:
  - For any \( a \in \mathcal{A} \) the pair \( (La, a \xrightarrow{\eta_a = \theta_{a,La}(\text{id}_{La})} RLa) \) is a universal arrow from \( a \) to \( R \);
  - for any \( b \in \mathcal{B} \) the pair \( (Rb, Lb \xrightarrow{\epsilon_b = \theta_{Rb,b}^{-1}(\text{id}_{Rb})} b) \) is a (co)universal arrow from \( L \) to \( b \).

- These universal arrows are components of natural transformations \( \eta : \text{id}_\mathcal{A} \Rightarrow RL \), the unit of adjunction, and \( \varepsilon : LR \Rightarrow \text{id}_\mathcal{B} \), the counit of adjunction.

- If \( L_1, L_2 : \mathcal{A} \to \mathcal{B} \) are both left adjoint to a functor \( R : \mathcal{B} \to \mathcal{A} \) then \( L_1 \) and \( L_2 \) are isomorphic.

Triangle identities.
Let \( \theta : \text{Hom}_B(L(-), \cdot) \to \text{Hom}_A(-, R(\cdot)) \) be an adjunction between two functors \( L : A \to B \) and \( R : B \to A \). We then have a pair of natural transformations

\[
\eta : \text{id}_A \Rightarrow RL, \quad \varepsilon : LR \Rightarrow \text{id}_B,
\]

the unit and the counit of the adjunction, which are defined by

\[
\eta_a = \theta_{a,La}(\text{id}_{La}) \quad \text{and} \quad \varepsilon_b = (\theta_{Rb,b})^{-1}(\text{id}_{Rb})
\]

for all \( a \in A \) and \( b \in B \). These two natural transformations satisfy a pair of equations. To state the equations we need to recall whiskering from lecture 27: given

\[
\begin{array}{ccc}
A & \xrightarrow{G} & B \\
\alpha & \downarrow & \beta \\
H & \xleftarrow{K} & C
\end{array}
\]

we denote the horizontal composition \( \beta \star \text{id}_G \) by \( \beta \circ G \) because \( (\beta \star \text{id}_G) = \beta \circ G \) for all \( a \in A \).

And given

\[
\begin{array}{ccc}
A & \xrightarrow{L} & B \\
\alpha & \downarrow & \beta \\
H & \xleftarrow{K} & C
\end{array}
\]

we denote the horizontal composition \( \text{id}_K \star \alpha \) by \( K \alpha \) because \( (\text{id}_K \star \alpha) = K \circ \alpha \) for all \( a \in A \).

**Theorem 32.1** (Triangle identities). Let \( \begin{array}{ccc}
A & \xrightarrow{L} & B \\
\alpha & \downarrow & \beta \\
\eta & \downarrow & \varepsilon
\end{array} \) be an adjunction, and \( \eta : \text{id}_A \Rightarrow RL \) and \( \varepsilon : LR \Rightarrow \text{id}_B \) the corresponding unit and counit. Then the diagrams

\[
\begin{array}{ccc}
R & \xrightarrow{\eta_R} & RLR \\
\downarrow \text{id}_R & & \downarrow \varepsilon \\
R & & L
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
L & \xrightarrow{\eta_L} & LRL \\
\downarrow \text{id}_L & & \downarrow \varepsilon_L \\
L & & R
\end{array}
\]

commute in the functor categories \([B, A]\) and \([A, B]\), respectively.

**Proof.** Equation (32.49) is equivalent to

\[
\text{id}_{R(b)} = R(\varepsilon_b) \circ \eta_R(b)
\]

for all \( b \in B \). Equation (32.50) is equivalent to

\[
\text{id}_{L(a)} = \varepsilon_{L(a)} \circ L(\eta_a)
\]

for all \( a \in A \). By (31.46)

\[
\theta_{a,b}(f) = R(f) \circ \eta_a \quad \text{for all } f \in \text{Hom}_B(La, b)
\]

and by (31.47)

\[
\theta_{a,b}^{-1}(h) = \varepsilon_b \circ L(h) \quad \text{for all } h \in \text{Hom}_B(a, Rb)
\]

Therefore

\[
\text{id}_{Rb} = \theta_{Rb,b}(\theta_{Rb,b}^{-1}(\text{id}_{Rb})) = \theta_{Rb,b}(\varepsilon_b) = R(\varepsilon_b) \circ \eta_R(b),
\]

which proves (32.49). And

\[
\text{id}_{La} = \theta_{a,La}^{-1}(\theta_{a,La}(\text{id}_{La})) = \theta_{a,La}^{-1}(\eta_a) = \varepsilon_{La} \circ L(\eta_a),
\]

which proves (32.50). \( \square \)

Theorem 32.1 has a converse:
Theorem 32.2. Suppose \( L : \mathcal{A} \rightarrow \mathcal{B} : R \) are two functors, and \( \eta : \text{id}_A \Rightarrow RL, \varepsilon : LR \Rightarrow \text{id}_B \) are two natural transformation that satisfy the triangle identities (32.49) and (32.50). Then there exists a natural isomorphism \( \theta : \text{Hom}_B(L(-), \cdot) \Rightarrow \text{Hom}_A(-, R(\cdot)) \). In other words, the quadruple \((L, R, \eta, \varepsilon)\) determines an adjunction \((L, R, \theta)\).

Proof. For all \( a \in \mathcal{A} \) and \( b \in \mathcal{B} \), consider

\[
\theta_{a,b} : \text{Hom}_B(La, b) \rightarrow \text{Hom}_A(a, Rb) \quad \theta_{a,b}(f) := R(f) \circ \eta_a
\]

and

\[
\tau_{a,b} : \text{Hom}_A(a, Rb) \rightarrow \text{Hom}_B(La, b) \quad \tau_{a,b}(h) := \varepsilon_b \circ L(h).
\]

We need to check that \( \theta_{a,b} \) and \( \tau_{a,b} \) are inverses of each other and that the collection \( \{\theta_{a,b}\}_{a \in \mathcal{A}, b \in \mathcal{B}} \) is natural in \( a \) and \( b \).

The triangle identity (32.50) tells us that the diagram

\[
\begin{array}{ccc}
La & \xrightarrow{LRf} & LRb \\
\downarrow{\varepsilon_L} & & \downarrow{\varepsilon_b} \\
La & \xrightarrow{f} & b
\end{array}
\]

commutes for any \( f \in \text{Hom}_B(La, b) \).

Therefore the diagram

\[
\begin{array}{ccc}
La & \xrightarrow{LRf} & LRb \\
\downarrow{\varepsilon_L} & & \downarrow{\varepsilon_b} \\
La & \xrightarrow{f} & b
\end{array}
\]

commutes and consequently

\[
\varepsilon_b \circ LRf \circ L\eta_a = f
\]

for any \( f \in \text{Hom}_B(La, b) \). On the other hand

\[
\tau_{a,b}(\theta_{a,b}f) = \tau_{a,b}(Rf \circ \eta_a) = \varepsilon_b \circ L(Rf \circ \eta_a) = \varepsilon_b \circ LRf \circ L\eta_a.
\]

Therefore \( \tau_{a,b}(\theta_{a,b}f) = f \) for all \( f \in \text{Hom}_B(La, b) \).

Similarly the other triangle identity and the naturality of \( \eta \) tell us that the diagram

\[
\begin{array}{ccc}
a & \xrightarrow{h} & Rb \\
\downarrow{\eta_a} & & \downarrow{\eta_{Rb}} \\
RLa & \xrightarrow{RLh} & RLRb & \xrightarrow{R\varepsilon_b} & b
\end{array}
\]

commutes. Therefore

\[
h = R\varepsilon_b \circ RLh \circ \eta_a
\]

for all \( h \in \text{Hom}_A(a, Rb) \). On the other hand

\[
\theta_{a,b}(\tau_{a,b}h) = \theta_{a,b}(\varepsilon_b \circ Lh) = R(\varepsilon_b \circ Lh) \circ \eta_a = R\varepsilon_b \circ RLh \circ \eta_a.
\]

It follows that \( \theta_{a,b}(\tau_{a,b}h) = h \) for all \( h \in \text{Hom}_A(a, Rb) \). We conclude that \( \theta_{a,b} \) and \( \tau_{a,b} \) are inverses of each other.
It remains to check the naturality of $\{\theta_{a,b}\}$. That is, we need to check that for any morphism $a' \xrightarrow{h} a$ in $\mathcal{A}$ and any morphism $b \xrightarrow{k} b'$ in $\mathcal{B}$, the diagrams

\[
\begin{align*}
\Hom_B(La, b) \xrightarrow{\theta_{a,b}} \Hom_A(a, Rb) && \Hom_B(La, b) \xrightarrow{\theta_{a,b}} \Hom_A(a, Rb) \\
\Hom_B(La', b) \xrightarrow{\theta_{a',b}} \Hom_A(a', Rb) && \Hom_B(La, b') \xrightarrow{\theta_{a,b'}} \Hom_A(a, Rb') \\
(Lh)^* && (Rk)^* 
\end{align*}
\]

commute.

Since $\eta$ is a natural transformation, the diagram $h \xrightarrow{\eta_a} RLa'$ commutes. Therefore for any $f \in \Hom_B(La, b)$

$$h^*(\theta_{a,b}(f)) = \theta_{a,b} \circ h = Rf \circ \eta_a \circ h = Rf \circ RLh \circ \eta_{a'} = R(f \circ Lh) \circ \eta_{a'} = \theta_{a',b}((Lh)^* f).$$

Finally for any $f \in \Hom_B(La, b)$

$$(Rk)_* \theta_{a,b}(f) = Rk \circ Rf \circ \eta_a = R(k \circ f) \circ \eta_a = \theta_{a,b'}(k_* f)$$

and we are done. \qed

We conclude that adjunctions between two functors $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$ can be viewed equivalently as

1. a natural isomorphism $\theta : \Hom_B(L-,-) \Rightarrow \Hom_A(-,R(-))$ between two functors;
2. a pair of natural transformations $\eta : \id_A \Rightarrow RL$, $\varepsilon : LR \Rightarrow \id_B$ which satisfy the triangle identities;
3. as a collection of universal arrows $\{(La, a \xrightarrow{\eta_a} RLa)\}_{a \in \mathcal{A}}$ from objects of $\mathcal{A}$ to the functor $R$;
4. as a collection of universal arrows $\{(Rb, LRb \xrightarrow{\varepsilon_b} b)\}_{b \in \mathcal{B}}$ from the functor $L$ to the objects of $\mathcal{B}$.

Lecture 33. Adjunctions compose. RAPL.

Last time:

- Proved that the unit $\eta : \id_A \Rightarrow RL$ and the counit $\varepsilon : LR \Rightarrow \id_B$ of an adjunction $L \dashv R$ satisfy the triangle identities:

  $$R\varepsilon \circ \eta R = \id_R \quad \text{and} \quad \varepsilon L \circ L\eta = \id_L.$$

- Proved the converse: if $L : \mathcal{A} \to \mathcal{B}$, $R : \mathcal{B} \to \mathcal{A}$ are two functors and $\eta : \id_A \Rightarrow RL$, $\varepsilon : LR \Rightarrow \id_B$ two natural transformations that satisfy the triangle identities then there is a natural isomorphism $\theta : \Hom_B(L-,-) \Rightarrow \Hom_A(-,R(-))$.

It may be useful at this point to look at more examples.

**Example 33.1.** Let’s revisit our first example of adjunction: the free/forgetful adjunction between

\[
\begin{array}{c}
\text{sets} \xrightarrow{F} \text{sets} \xleftarrow{U} \text{vector spaces}.
\end{array}
\]

What are the unit and the counit of this adjunction?
The unit of adjunction \( \eta : \text{id}_{\text{Set}} \Rightarrow UF \) consists of the universal arrows \( \eta_X : X \to U(F(X)) \subseteq \mathbb{R}^X \) (one for each set \( X \)). We have seen that \( \eta_X(x) = \delta_x \), the “Kronecker” delta: the function \( \delta_x : X \to \mathbb{R} \) which is defined by \( \delta_x(x) = 1 \) and zero otherwise.

The counit of adjunction \( \varepsilon : FU \Rightarrow \text{id}_{\text{Vect}} \) should consist of linear maps \( \varepsilon_W : F(U(W)) \to W \) (one for each vector space \( W \)) with the property that
\[
U(\varepsilon_W) \circ \eta_{U(W)} = \text{id}_{U(W)}.
\]
That is, \( U(\varepsilon_W)(\delta_w) = w \) for all elements \( w \in W \). Since \( \varepsilon_W \) is linear and since vectors in \( F(U(W)) \) are finite linear combinations of delta functions, \( \varepsilon_W \) is given by
\[
\varepsilon_W(\sum \lambda_i \delta_{w_i}) = \sum \lambda_i w_i.
\]

**Example 33.2.** Consider the free/forgetful adjunction \( \text{Set} \xrightarrow{F} \text{Mon} \) between sets and monoids.

Recall that \( F(X) \) is the free monoid of words on a set \( X \) (which is often denoted by \( X^* \)). What are the unit and the counit of this adjunction?

The unit of adjunction \( \eta \) consists of universal arrows \( \eta_X : X \to U(F(X)) \): \( \eta_X \) sends an element \( x \in X \) to a one letter word \( (x) \). What about the counit? For any monoid \( M \), \( \varepsilon_M : F(U(M)) \to M \) is the homomorphism of monoids with the property that
\[
U(\varepsilon_M) \circ \eta_{U(M)} = \text{id}_{U(M)}.
\]
That is, \( U(\varepsilon_M) : UF(M) \to M \) is the function with \( U(\varepsilon_M)((m)) = m \) for all \( m \in M \). Therefore on an arbitrary word \( (m_1 \ldots m_n) \in F(M) \) the homomorphism \( \varepsilon_M \) is given by
\[
\varepsilon_M(m_1 \ldots m_n) = \varepsilon_M((m_1) \cdot (m_2) \ldots \cdot (m_n)) = \varepsilon_M((m_1)) \ldots \cdot \varepsilon_M((m_n)) = m_1 \ldots \cdot m_n.
\]
That is, \( \varepsilon_M \) sends a word in \( F(M) \) to the product of its letters in \( M \).

**Example 33.3.** Recall that if \( I, C \) are categories and \( C \) has all limits of shape \( I \) then the assignment \( [I, C] \ni F \mapsto \text{lim} F \) extends to a functor \( \text{lim} : [I, C] \to C \) which is right adjoint to the diagonal functor \( \Delta : C \to [I, C] \). This because for each functor \( F \) the limit cone \( \pi_F : \Delta_{\text{lim} F} \Rightarrow F \) is a terminal object in the comma category \( (\Delta \downarrow F) \), that is, it’s a universal arrow from \( \Delta \) to \( F \).

The components of the counit of adjunction \( \varepsilon : \Delta \text{lim} \Rightarrow \text{id}_{[I, C]} \) are limit cones: \( \varepsilon_F : \pi_F \circ \Delta_{\text{lim} F} \Rightarrow F \) for any functor \( F : I \to C \). The components of the natural isomorphism
\[
\theta^{-1} : \text{Hom}_C(-, \lim(-)) \Rightarrow \text{Hom}_{[I, C]}(\Delta(-), -)
\]
are then given by
\[
\theta^{-1}_{q,F} : \text{Hom}_C(q, \text{lim} F) \to \text{Hom}_{[I, C]}(\Delta_q, F), \quad (q \Rightarrow \text{lim} F) \mapsto (\Delta_q \xrightarrow{\pi_F} F).
\]
Recall that we normalize the function \( F \mapsto \text{lim} F \) by requiring that \( \text{lim}(\Delta_c) = c \) for all \( c \in C \).
Consequently the components of the unit of adjunction \( \eta : \text{id}_{C} \Rightarrow \Delta \circ \text{lim} \) are \( \{\eta_c : c \to \Delta \text{lim} c\}_{c \in C} \), which are just \( \text{id}_c \).

**Example 33.4.** If a category \( C \) has all colimits of shape \( I \) then the diagonal functor \( \Delta : C \to [I, C] \) has a left adjoint
\[
\text{colim} : [I, C] \to C.
\]
The components of the unit of adjunction \( \eta : \text{id}_{[I, C]} \Rightarrow \Delta \circ \text{colim} \) are
\[
\eta_F := \iota_F : F \Rightarrow \Delta_{\text{colim} F},
\]
for all \( F : I \to C \).
Suppose Theorem 33.6. Right adjoints preserve limits (RAPL). of the counit \( \varepsilon : \text{colim} \circ \Delta \Rightarrow \mathcal{C} \) are \( \varepsilon_c = \text{id}_c : \text{colim}(\Delta_c) \rightarrow c \). The components of the natural transformation

\[
\theta_{F,c} : \text{Hom}_C(\text{colim}(\cdot), \cdot) \Rightarrow \text{Hom}_{[I,\mathcal{C}]}(\cdot, \Delta_c)
\]

are given by

\[
\theta_{F,c} : \text{Hom}_C(\text{colim}(F), c) \Rightarrow \text{Hom}_{[I,\mathcal{C}]}(F, \Delta_c), \quad (\text{colim} F \xrightarrow{\gamma} c) \mapsto (F \xrightarrow{\Delta_c \circ \gamma} \Delta_c).
\]

We now return to developing more theory.

**Lemma 33.5.** Suppose we have two adjunctions: \( \mathcal{A} \xrightarrow{\perp} \mathcal{B} \xrightarrow{\perp} \mathcal{C} \). Then \( \mathcal{A} \xrightarrow{\perp} \mathcal{C} \).

**Proof.** Let \( \theta : \text{Hom}_C(\text{colim}(-), \cdot) \Rightarrow \text{Hom}_B(-, K(\cdot)) \) and \( \tau : \text{Hom}_B(\text{colim}(-), \cdot) \Rightarrow \text{Hom}_A(-, G(\cdot)) \) denote the two corresponding natural isomorphisms. We would like to define a natural isomorphism \( \mu : \text{Hom}_C(\text{colim}(\cdot), \cdot) \Rightarrow \text{Hom}_B(-, G(\cdot)) \).

For any pair of morphism \( m : a' \rightarrow a \) in \( \mathcal{A} \) and \( \ell : c \rightarrow c' \) in \( \mathcal{C} \) the diagrams

\[
\text{Hom}_C(\text{colim} F, c) \xrightarrow{\theta_{Fa,c}} \text{Hom}_B(Fa, Kc) \quad \text{and} \quad \text{Hom}_B(Fa, Kc) \xrightarrow{\tau_{a,Kc}} \text{Hom}_A(a, GKc)
\]

commute. Hence the outer square in the diagram

\[
\text{Hom}_C(\text{colim} F, c) \xrightarrow{\theta_{Fa,c}} \text{Hom}_B(Fa, Kc) \xrightarrow{\tau_{a,Kc}} \text{Hom}_A(a, GKc)
\]

commutes. Therefore \( \{\mu_{a,c} := \tau_{a,Kc} \circ \theta_{Fa,c}\}_{a \in \mathcal{A}, c \in \mathcal{C}} \) are components of a natural isomorphism \( \mu : \text{Hom}_C(\text{colim}(\cdot), \cdot) \Rightarrow \text{Hom}_B(-, G(\cdot)) \) and we are done. \( \square \)

**Right adjoints preserve limits (RAPL).**

**Theorem 33.6.** Suppose \( \mathcal{A} \xrightarrow{\perp} \mathcal{B} \) is an adjunction. Then the functor \( R \) preserves all the limits that exist in \( \mathcal{B} \): if \( D : I \rightarrow \mathcal{B} \) is a functor with a limit cone \( (\text{lim} D, \{\lambda_j : \text{lim} D \rightarrow D(j)\}_{j \in I}) \), then \( (R(\text{lim} D), \{R(\lambda_j) : R(\text{lim} D) \rightarrow RD(j)\}_{j \in I}) \) is a limit cone in \( \mathcal{A} \).

**Corollary 33.7.** Left adjoints preserve colimits.

**Proof.** Duality. \( \square \)

We will prove Theorem 33.6 in the next lecture. We end the lecture with examples to illustrate the theorem.
Example 33.8. The forgetful functor \( U : \text{Mon} \to \text{Set} \) has a left adjoint \( F : \text{Set} \to \text{Mon} \). Since \( F \) is a left adjoint, \( F \) preserves colimits and, in particular, coproducts. Coproducts in \( \text{Set} \) are disjoint unions. Hence, for any two sets \( X, Y \), the free monoid \( F(X \sqcup Y) \) is the coproduct in the category \( \text{Mon} \) of the free monoids \( F(X) \) and \( F(Y) \). In particular, the coproduct of \( F(X) \) and \( F(Y) \) exists in \( \text{Mon} \).

On the other hand, since \( U \) is right adjoint to \( F \), \( U \) preserves limits and, in particular, products. This is why if \( M, N \) are two monoids, then the set underlying their categorical product \( M \times N \) has to be the cartesian product \( U(M) \times U(N) \) of the underlying sets.

Example 33.9. The forgetful functor \( U : \text{Mon} \to \text{Set} \) does not have a right adjoint. If it did, it would preserve all colimits and, in particular, initial objects. But the initial object in \( \text{Mon} \) is a one-element monoid \( \{e\} \) while the initial object in \( \text{Set} \) is the empty set \( \emptyset \). So \( U : \text{Vect} \to \text{Set} \) does not have a right adjoint.

Example 33.10. The forgetful functor \( U : \text{Vect} \to \text{Set} \) does not preserve colimits because it does not preserve initial objects: the initial vector space is the zero dimensional vector space \( \{0\} \) but \( U(\{0\}) \neq \emptyset \). So \( U : \text{Vect} \to \text{Set} \) does not have a right adjoint.

Example 33.11. The forgetful functor \( U : \text{Top} \to \text{Set} \) has a right adjoint and a left adjoint. Consequently, \( U \) preserves both limits and colimits, in particular, it preserves products and coproducts. This is why the set underlying the product of topological spaces is the cartesian product of the sets underlying the factors and the set underlying the disjoint union (coproduct) of topological spaces is the coproduct of the sets underlying the corresponding topological spaces.

Lecture 34. Proof that Right Adjoints Preserve Limits. Adjoint equivalences.

Last time:

- Proved that adjunctions compose.
- Stated and explored consequences of but didn’t prove Theorem 33.6 right adjoints preserve limits.

Proof of Theorem 33.6 right adjoints preserve limits. Suppose that \( A \underoverset{\perp}{L}{\overset{R}{\to}} B \) is an adjunction and \( D : I \to B \) is a functor with a limit cone \( (\lim D, \{\lambda_j : \lim D \to D(j)\}_{j \in I}) \). We’d like to show that \( (R(\lim D), \{R(\lambda_j) : R(\lim D) \to RD(j)\}_{j \in I}) \) is a limit cone.

Let \( (a, \{\alpha_i : a \to RD(i)\}_{i \in I}) \) be a cone in \( A \) over the functor \( RD \). We want to show that there is a unique morphism \( h : a \to R(\lim D) \) so that the diagram

\[
\begin{array}{ccc}
a & \xrightarrow{h} & R\lim D \\
\downarrow{\alpha_i} & & \downarrow{R\lambda_i} \\
RD(i) & & \\
\end{array}
\]

commutes for all \( i \in I \), or, equivalently

\[(34.51)\quad (R\lambda_i)_* h = \alpha_i \]

for all \( i \). Since \( (a, \{\alpha_i : a \to RD(i)\}_{i \in I}) \) is a cone, the diagram

\[
\begin{array}{ccc}
RD(i) & \xrightarrow{c} & RD(j) \\
\downarrow{\alpha_i} & & \downarrow{\alpha_j} \\
RD(i) & \xrightarrow{RD(\gamma)} & RD(j) \\
\end{array}
\]

commutes for all \( i, j \in I \). Thus, by the uniqueness property of adjoints, we have

\[
RD(\gamma)_* h = c_{RD(j)} h = \alpha_j
\]

for all \( i, j \in I \). This shows that \( h \) is a morphism in \( A \) with the desired properties.
commutes for all morphisms $i \to j$ in $I$. Hence, $RD(\gamma)_* \alpha_i = \alpha_j$. Since $R$ is right adjoint to $L$, there is a natural isomorphism $\theta : \text{Hom}_B(L-, \cdot) \Rightarrow \text{Hom}_A(\cdot, R(\cdot))$. Therefore for any $i \to j$ in $I$ the diagram

\[
\begin{array}{ccc}
\text{Hom}_A(a, R(Di)) & \xrightarrow{\theta^{-1}_{a,Di}(\alpha_i)} & \text{Hom}_B(La, Di) \\
RD(\gamma)_* & \downarrow & D(\gamma)_* \\
\text{Hom}_A(a, R(Dj)) & \xrightarrow{\theta^{-1}_{a,Dj}(\alpha_j)} & \text{Hom}_B(La, Dj)
\end{array}
\]

commutes. Hence

\[
D(\gamma)_*(\theta^{-1}_{a,D(i)}(\alpha_i)) = \theta^{-1}_{a,D(j)}(RD(\gamma)_* \alpha_i) = \theta^{-1}_{a,D(j)}(\alpha_j).
\]

Therefore $(La, \{\bar{\alpha}_i : La \to D(i)\}_{i \in I})$ is a cone on the functor $D$, where $\bar{\alpha}_i := \theta^{-1}_{a,D(i)}(\alpha_i)$. Since $(\lim D, \{\lambda_j : \lim D \to D(j)\}_{j \in I})$ is a limit cone there is a unique morphism $\beta : La \to \lim D$ so that

\[
\begin{array}{ccc}
La & \xrightarrow{\beta} & \lim D \\
\downarrow & & \downarrow \\
D(i) & \xrightarrow{\lambda_i} & \lim D
\end{array}
\]

commutes for all $i$. Applying the bijection $\theta_{a,\lim D} : \text{Hom}_B(La, \lim D) \to \text{Hom}_A(a, R(\lim D))$ to $\beta$ we get $h = \theta_{a,\lim D}(\beta) : a \to R(\lim D)$ in $A$. Since the diagram

\[
\begin{array}{ccc}
\text{Hom}_B(La, \lim D) & \xrightarrow{\theta_{a,\lim D}} & \text{Hom}_A(a, R(\lim D)) \\
\downarrow & & \downarrow \text{R(}\lambda_i) \text{,} \\
\text{Hom}_B(La, Di) & \xrightarrow{\theta_{a,Di}} & \text{Hom}_A(a, RD(i))
\end{array}
\]

commutes

\[
R(\lambda_i)_* h = R(\lambda_i)_* (\theta_{a,\lim D}(\beta)) = \theta_{a,Di}(\lambda_i)_* \beta = \theta_{a,Di}(\bar{\alpha}_i) = \theta_{a,Di}(\theta^{-1}_{a,Di}(\alpha_i)) = \alpha_i.
\]

Hence the diagram

\[
\begin{array}{ccc}
a & \xrightarrow{\alpha_i} & R(\lim D) \\
\downarrow & & \downarrow \text{R(}\lambda_i) \\
RD(i) & \xrightarrow{\beta} & \lim D
\end{array}
\]

commutes for all $i$.

Moreover if $h' : a \to R(\lim D)$ is another morphism with $R(\lambda_i) \circ h' = \alpha_i$ for all $i$ then, by essentially the same argument, $\lambda_i \circ \theta^{-1}_{a,\lim D}(h') = \bar{\alpha}_i$ for all $i$. Consequently $\theta^{-1}_{a,\lim D}(h') = \beta = \theta^{-1}_{a,\lim D}(h)$. Since $\theta^{-1}_{a,\lim D}$ is a bijection, $h' = h$. Therefore $h$ is unique and $(R(\lim D), \{R(\lambda_i)\}_{i \in I})$ is a limit cone of $R \circ D$. \hfill \Box

Equivalences and adjoint equivalences.

An adjunction $(L, R, \eta : \text{id}_A \Rightarrow RL, \varepsilon : LR \Rightarrow \text{id}_B)$ need not be an equivalence of categories since the unit $\eta$ and the counit $\varepsilon$ need not be natural isomorphisms. More generally, the categories $A$ and $B$ need not even be equivalent. For example, the functor $F : C \to \mathbb{1}$ (where $\mathbb{1}$ is the category
with one object and one morphism) has a right adjoint and a left adjoint if \( \mathcal{C} \) has terminal and initial objects. The functor \( F \) is an equivalence of categories only if for every \( c, c' \in \mathcal{C} \),
\[
F : \text{Hom}_{\mathcal{C}}(c, c') \to \text{Hom}_{\mathcal{Y}}(\ast, \ast) = \{\text{id}_\ast\}
\]
is a bijection. This happens if and only if any two objects \( c, c' \) of \( \mathcal{C} \) is uniquely isomorphic. There are of course lots of categories with initial and terminal object where many objects are not isomorphic to each other.

On the other hand, an equivalence of categories \((F, G, \alpha : \text{id}_{\mathcal{A}} \Rightarrow GF, \beta : FG \Rightarrow \text{id}_{\mathcal{B}})\) need not be an adjunction since the natural isomorphisms \( \alpha \) and \( \beta \) need not satisfy the triangle identities. It turns out we can modify either of the two natural isomorphisms and get an adjunction.

**Theorem 34.1.** Let \((F, G, \alpha : \text{id}_{\mathcal{A}} \Rightarrow GF, \beta : FG \Rightarrow \text{id}_{\mathcal{B}})\) be an equivalence of categories. There exist a natural isomorphism \( \eta : \text{id}_{\mathcal{A}} \Rightarrow GF \) so that \((F, G, \eta, \beta)\) is an adjunction.

**Proof.** We would like to find a natural isomorphism \( \eta : \text{id}_{\mathcal{A}} \Rightarrow GF \) so that the triangle identities

\[
\begin{align*}
F \eta &= (\beta F)^{-1} \\
G \eta &= (F G)^{-1}
\end{align*}
\]

hold. Since \( \beta \) is a natural isomorphism, \((\beta F)_a = \beta_{Fa}\) is an isomorphism for every \( a \in \mathcal{A} \). Hence \( \beta F : FGF \Rightarrow F \) is a natural isomorphism as well. Therefore, \((\beta F)^{-1} : F \Rightarrow FGF\) makes sense. For (34.52) to hold, we must have that

\[
F \eta = (\beta F)^{-1}.
\]

For every pair of objects \( a, a' \in \mathcal{A} \) the function
\[
\text{Hom}_{\mathcal{A}}(a, a') \xrightarrow{F} \text{Hom}_{\mathcal{B}}(Fa, Fa')
\]
is a bijection \((F \) is fully faithful by Lemma 21.4). Therefore equation (34.54) uniquely defines a natural isomorphism \( \eta \).

So let’s define \( \eta : \text{id}_{\mathcal{A}} \Rightarrow GF \) by (34.54). Then the first triangle identity, (34.52), holds by definition. We need to check that the second identity, (34.53), holds as well.

Since \( \beta : FG \Rightarrow \text{id}_{\mathcal{B}} \) is a natural transformation, for any morphism \( b' \xrightarrow{h} b \) in \( \mathcal{B} \) the diagram

\[
\begin{array}{ccc}
FG(b') & \xrightarrow{\beta_{b'}} & b' \\
| \quad | & \quad & | \\
FG(h) & \downarrow h & FG(b) \\
\quad \beta_b & \quad & \quad \\
& \beta_b & b
\end{array}
\]

commutes. Let \( h = FG(b) \xrightarrow{\beta_b} b \). Then the diagram

\[
\begin{array}{ccc}
FG(FG(b)) & \xrightarrow{\beta_{FG(b)}} & FG(b) \\
| \quad | & \quad & | \\
FG(\beta_b) & \downarrow \beta_b & FG(b) \\
\quad \beta_b & \quad & \quad \\
& \beta_b & b
\end{array}
\]

is a bijection. This happens if and only if any two objects \( c, c' \) of \( \mathcal{C} \) is uniquely isomorphic.
commutes, i.e.,

$$\beta_b \circ FG(\beta_b) = \beta_b \circ \beta_{FG(b)}$$

for every $b \in B$. Since $\beta_b$ is an isomorphism,

(34.55)  
$$\beta_{FG(b)} = FG(\beta_b)$$

for every $b \in B$. Equation (34.52) says that

$$\beta_F(a) \circ F(\eta_a) = \text{id}_{F(a)}$$

for every $a \in A$. Hence

$$\text{id}_{FG(b)} = \beta_{FG(b)} \circ F(\eta_{G(b)}) \overset{(34.55)}{=} FG(\beta_b) \circ F(\eta_{G(b)}) = F(G(\beta_b) \circ \eta_{G(b)})$$

for all $b \in B$, which is the commutativity of (34.53). So we are done. \[ \square \]

Remark 34.2. In Theorem 34.1, we started with an equivalence of categories $(F, G, \alpha : \text{id}_A \Rightarrow GF, \beta : FG \Rightarrow \text{id}_B)$, declared $\beta$ to be the counit of the adjunction that we wanted to construct, and constructed a natural isomorphism $\eta : \text{id}_A \Rightarrow GF$ so that $(F, G, \eta, \beta)$ an adjunction with $F \dashv G$, $\eta$ the unit and $\beta$ the counit.

But we also can set $\varepsilon = \alpha^{-1} : GF \Rightarrow \text{id}_A$ and then construct $\eta : \text{id}_B \Rightarrow FG$ so that $G \dashv F$. Consequently, if a functor $F : A \to B$ is fully faithful and essentially surjective, then $F$ is part of an equivalence of categories and therefore its weak inverse $G$ is both its right adjoint and left adjoint. Hence the functor $F$ preserves limits and colimits (see Theorem 33.6 and Corollary 33.7).


Last time:

- Proved that right adjoints preserve limits (RAPL).
- Proved that any equivalence of categories is an adjoint equivalence. Consequently any fully faithful and essentially surjective functor preserves both limits and colimits.

There is one more class of functors that preserve limits.

**Theorem 35.1.** Let $C$ be a (locally small) category, $c \in C$. The functor

$$\text{Hom}_C(c, -) : C \to \text{Set}$$

preserves limits.

**Proof.** Let $J$ be a small category, $D : JC$ a functor, and $(l, \{p_i : l \to D(i)\}_{i \in J})$ its limit cone. After applying $\text{Hom}_C(c, -)$ to the limit cone, we get a cone

$$(\text{Hom}_C(c, l), \{ (p_i)_* = \text{Hom}_C(c, -)(p_i) : \text{Hom}_C(c, l) \to \text{Hom}_C(c, D(i)) \}_{i \in J})$$

in the category $\text{Set}$ of sets and functions. We need to check: for any cone $(S, \{q_i : S \to \text{Hom}_C(c, D(j))\}_{j \in J})$ there exists a unique function $S \xrightarrow{\varphi} \text{Hom}_C(c, l)$ so that the diagram

(35.56)  
$$\begin{array}{ccc}
S & \xrightarrow{\varphi} & \text{Hom}_C(c, l) \\
q_i & \downarrow & \\
\text{Hom}_C(c, D(i)) & \overset{(p_i)_*}{\to} & \text{Hom}_C(c, D(i))
\end{array}$$

\[ 107 \]
commutes for every \( j \in J \). Since \( (S, \{ q_i : S \to \text{Hom}_C(c, D(j)) \}_{j \in J}) \) is a cone, the diagram

\[
\begin{array}{ccc}
S & \xrightarrow{q_i} & S \\
\downarrow{q_i} & & \downarrow{q_i} \\
\text{Hom}_C(c, D(i)) & \xrightarrow{D(\gamma)} & \text{Hom}_C(c, D(j))
\end{array}
\]

(35.57)

commutes for all morphisms \( \gamma : i \to j \) in \( J \). Now, for any \( s \in S \), \( q_j(s) \) is an element of the set \( \text{Hom}_C(c, D(j)) \), hence it’s a morphism \( q_j(s) : c \to D(j) \) in \( C \). Commutativity of (35.57) say that \( q_j(s) = D(\gamma)_*(q_i(s)) = D(\gamma) \circ q_i(s) \). That is, for every \( s \in S \) and every \( \gamma : i \to j \) in \( J \), the diagram

\[
\begin{array}{ccc}
S & \xrightarrow{q_i(s)} & S \\
\downarrow{D(\gamma)} & & \downarrow{D(\gamma)} \\
D(i) & \xrightarrow{D(\gamma)} & D(j)
\end{array}
\]

(35.58)

commutes in \( C \). Since \( (l, \{ p_i : l \to D(i) \}_{i \in J}) \) is a limit cone, for every \( s \in S \), there exists a unique morphism \( \varphi_s : c \to l \) in \( C \) so that

\[
\begin{array}{ccc}
c & \xrightarrow{\varphi_s} & l \\
\downarrow{q_j(s)} & & \downarrow{p_j} \\
D(j)
\end{array}
\]

(35.59)

commutes for all \( j \in J \). In other words, we get a map \( \varphi : S \to \text{Hom}_C(c, l) \), \( \varphi(s) = \varphi_s \), so that

\[
(p_j)_*(\varphi(s)) = p_j \circ \varphi_s = q_j(s)
\]

for every \( s \in S \) and every \( j \in J \). Hence

\[
(p_j)_* \circ \varphi = q_j
\]

for every \( j \in J \). The map \( \varphi \) is unique since each \( \varphi(s) \) is unique.

\[ \square \]

**Corollary 35.2.** For any (locally small) category \( C \) and any \( c \in C \), the functor \( \text{Hom}_C(-, c) \) preserves colimits.

**Proof.** Duality.  \( \square \)

**Theorem 35.3.** Suppose \( F, G : A \to B \) are two isomorphic functors. If the functor \( F \) preserves limits, then \( G \) preserves limits as well. Hence all (covariant) representable functors preserve limits.

**Proof.** Exercise.  \( \square \)

---

**Monads.**

“A monad is just a monoid in the category of endofunctors, what is the problem?”

James Iry paraphrasing MacLane and attributing it to Philip Wadler.


We start by revisiting the definition of a monoid. Recall that a **monoid** is a triple \((M, \mu, e)\) where \( M \) is a set, \( \mu : M \times M \to M \), \( \mu(a, b) = ab \) is a function (a multiplication) and \( e \in M \) is an element called unit so that
(i) $\mu$ is associative: $(a(bc)) = (ab)c$ for all $a, b, c \in M$, and
(ii) $e$ is a unit for $\mu$: $ea = a = ae$ for all $a \in M$.
We can rewrite (i) as $\mu(\mu(a, b), c) = \mu(a, \mu(b, c))$ for every $a, b, c \in M$, which says that the diagram

$$
M \times M \times M \xrightarrow{\mu \times \text{id}_M} M \times M
$$

commutes. We can think of $e \in M$ as a function $e : 1 \to M$, where 1 is the one element set $\{\ast\}$. Then (ii) says that

$$
\mu \circ (e \times \text{id}_M) = \text{id}_M = \mu \circ (\text{id}_M \times e),
$$
i.e., the diagrams

$$
\{\ast\} \times M \xrightarrow{e \times \text{id}_M} M \times M
$$

commute. Here $\text{id}_M : \{\ast\} \times M \to M$ is the bijection defined by $\text{id}_M(\ast, a) = a$ for all $a \in M$.

**Definition 35.4.** A **monad** on a category $C$ is a functor $T : C \to C$ (an endofunctor) together with two natural transformations: the **unit** $\eta : \text{id}_C \Rightarrow T$ and the **multiplication** $\mu : T \circ T \Rightarrow T$ so that the following three diagrams

![Monad Diagram](https://via.placeholder.com/150)

commute in the functor category $[C, C]$.

Note that since $\text{id}_C \circ T = T = T \circ \text{id}_C$, writing $\text{id}_T : \text{id}_C \circ T \Rightarrow T$ and $\text{id}_T : T \circ \text{id}_C \Rightarrow T$ makes sense.

**Remark 35.5.** The commutativity of the three diagrams above amounts to: for every object $c \in C$ the diagrams

$$
T(c) \xrightarrow{\eta_T(c)} T^2(c)
$$

commute in the category $C$. 109
Example 35.6 (the writer monad). Fix a monoid \((M, \cdot, e)\). Define a functor \(T : \text{Set} \to \text{Set}\) by
\[
T(X \xrightarrow{f} Y) = X \times M \xrightarrow{f \times \text{id}_M} Y \times M.
\]
The function \(e : \{\ast\} \to M\) defines a natural transformation \(\eta : \text{id}_{\text{Set}} \Rightarrow T\) with components
\[
\eta_X : X \xrightarrow{\sim} X \times \{\ast\} \xrightarrow{\text{id}_X \times e} X \times M,
\]
that is, \(\eta_X(x) = (x, e)\) for every \(x \in X\). \(\eta\) is a natural transformation since for any function \(f : X \to Y\) the diagram
\[
\begin{array}{ccc}
X & \xrightarrow{\eta_X} & X \times M \\
\downarrow f & & \downarrow f \times \text{id}_M \\
Y & \xrightarrow{\eta_Y} & Y \times M
\end{array}
\]
commutes.

The multiplication \(\mu : T^2 \Rightarrow T\) is defined by setting \(\mu_X : X \times M \times M \to X \times M\) to be
\[
\mu_X(x, a, b) = (x, a \cdot b).
\]
for every set \(X\). It is easy to see that \(\mu\) is a natural transformation. We now check that \((T, \eta, \mu)\) is a monad. This amounts to check the commutativity of the three diagrams above.

For any set \(X\) the function \(\mu_X \circ \eta_{TX} : TX \to TX\) is given by
\[
(x, a) \mapsto (x, a, e) \mapsto (x, ea) = (x, a).
\]
Hence \(\mu_X \circ \eta_{TX} = \text{id}_{TX}\) for any set \(X\), which is the left unit identity.

Since for any morphism \(f\), \(T(f) = f \times \text{id}_M\),
\[
T(\eta_X) = \eta_X \times \text{id}_M,
\]
i.e.,
\[
T(\eta_X)(x, a) = ((x, e), a)
\]
for all \((x, a) \in TX = X \times M\). Hence
\[
(\mu_X \circ T(\eta_X))(x, a) = \mu_X((x, e), a) = (x, ea) = (x, a) = \text{id}_{TX}(x, a),
\]
which proves the right unit identity.

Finally
\[
T(\mu_X) = \mu_X \times \text{id}_M : X \times M \times M \times M, \quad (x, a, b, c) \mapsto (x, ab, c)
\]
while
\[
\mu_{TX} : (X \times M) \times M \times M, \quad ((x, a), (b, c)) \mapsto (x, a, bc).
\]
Hence
\[
(\mu_X \circ \mu_{TX})(x, a, b, c) = (x, a(bc)
\]
while
\[
(\mu_X \circ T\mu_X) = \mu_X(x, ab, c) = (x, (ab)c).
\]
Hence the associativity of \(\mu\) follows from the associativity of the multiplication in the monoid \(M\).
Lecture 36. Monads from adjunctions.

Last time:

- Proved that for any small category $C$ and any object $c$ of $C$ the functor $\text{Hom}_C(c,-) : C \to \text{Set}$ preserves limits.
- Defined monads:

  A monad on a category $C$ is a functor $T : C \to C$ (an endofunctor) together with two natural transformations: the unit $\eta : \text{id}_C \Rightarrow T$ and the multiplication $\mu : T \circ T \Rightarrow T$ so that the following three diagrams

  $T \circ T \circ T \quad \xrightarrow{T \mu} \quad T \circ T$

  $\mu_T \downarrow \Downarrow \mu$  

  $T \circ T \quad \xrightarrow{\mu} \quad T$

  $\eta \uparrow \Downarrow \text{id}_T$

  $\text{id}_C \circ T \xrightarrow{\eta_T} T \circ T$

  $\mu \downarrow \Downarrow \mu$  

  $T \circ \text{id}_C \xrightarrow{T \eta} T \circ T$

  $\mu \downarrow \Downarrow \mu$  

  $\eta_{\text{id}_C} \downarrow \Downarrow \text{id}_T$

  commute in the functor category $[C,C]$.

There is an intimate relation between monads and adjunctions. First of all, every adjunction gives rise to a monad.

**Theorem 36.1.** Let $C \xleftarrow{\eta} \xrightarrow{\varepsilon} B$ be an adjunction with a unit $\eta : \text{id}_C \Rightarrow GF$ and a counit $\varepsilon : FG \Rightarrow \text{id}_B$. Then $(T = GF, \eta : \text{id}_C \Rightarrow T, \mu = G\varepsilon F : T^2 \Rightarrow T)$ is a monad on $C$.

**Proof.** We need to check that for every $a \in C$, the three diagrams

(36.60) $GFGF(a) \xrightarrow{(GFGF)_{\eta a}} GFGF(a)$

(36.61) $GF(a) \xrightarrow{\eta_{GF a}} GFGF(a)$

(36.62) $GF(a) \xrightarrow{\eta a} GF^2(a)$
commute. One triangle identity (namely (32.50)) tells us that the diagram

\[
F(a) \xrightarrow{\eta_a} FGF(a) \\
\downarrow{\epsilon_F a} \quad \downarrow{F a}
\]

commutes. Applying the functor \(G\) to this diagram gives us (36.61). The other triangle identity (namely namely (32.49)) tells us that for all objects \(b \in \mathcal{B}\) the diagram

\[
Gb \xrightarrow{\eta_G b} GFGb \\
\downarrow{id_G b} \quad \downarrow{G \epsilon_b} \\
Gb \xrightarrow{G \epsilon_b} Gb
\]

commutes. In particular it commutes for \(b = F a\), i.e., the diagram

\[
GFa \xrightarrow{\eta_{GF a}} GFGFa \\
\downarrow{id_{GF a}} \quad \downarrow{G \epsilon_{Fa}} \\
GFa \xrightarrow{G \epsilon_{Fa}} GFa
\]

commutes, which is (36.62).

Since \(\epsilon : FG \Rightarrow \text{id}_G\) is a natural transformation, for every morphism \(b' \xrightarrow{h} b\) in \(\mathcal{B}\), the diagram

\[
FG(b') \xrightarrow{FG b} FG(b) \\
\downarrow{\epsilon_{b'}} \quad \downarrow{h} \\
b' \xrightarrow{\epsilon_b} b
\]

commutes. Take \(b' = FG b\) and \(h = \epsilon_b\), i.e., \(h = FGb \xrightarrow{\epsilon_b} b\). Then the diagram

\[
FGFG(b) \xrightarrow{FG \epsilon_b} FG(b) \\
\downarrow{\epsilon_{FG b}} \quad \downarrow{\epsilon_b} \\
FG(b) \xrightarrow{\epsilon_b} b
\]

commutes. If we take \(b = Fa\) then the diagram

\[
FGFGF(a) \xrightarrow{FG \epsilon_{Fa}} FGF(a) \\
\downarrow{\epsilon_{FGF a}} \quad \downarrow{\epsilon_{Fa}} \\
FGF(a) \xrightarrow{\epsilon_{Fa}} F(a)
\]

commutes. Applying the functor \(G\) to the diagram above gives us the commuting diagram

\[
GFGFGF(a) \xrightarrow{GFG \epsilon_{Fa}} GFGF(a) \\
\downarrow{G \epsilon_{FGFa}} \quad \downarrow{G \epsilon_{Fa}} \\
GFGF(a) \xrightarrow{G \epsilon_{Fa}} GF(a)
\]
which is \((36.60)\). \qed

**Example 36.2** (The Finite List Monad). Consider the free/forgetful adjunction
\[
F : \text{Set} \xrightarrow{\cong} \text{Mon} : U
\]
with \(F \dashv U\). Then the functor \(T = UF : \text{Set} \to \text{Set}\) sends a set \(X\) to the set \(U(X^*)\) underlying the monoid \(X^* = F(X)\) of finite lists (words) of elements of \(X\). It will be convenient to suppress the functor \(U\) and denote by \(X^*\) both the monoid and its underlying set.

Then the components of the unit \(\eta : \text{id}_{\text{Set}} \Rightarrow UF\) of the adjunction (and of the monad) at a set \(X\) are functions \(\eta_X : X \to X^*\), \(\eta_X(x) = (x)\) where \((x)\) is the list with just one element. The components of the counit \(\varepsilon_M : FU \Rightarrow \text{id}_{\text{Mon}}\) at a monoid \(M\) are \(\varepsilon_M((m_1 \ldots m_n)) = m_1 \cdot m_2 \cdots m_n\), the product of the the element in the list \((m_1 \ldots m_n) \in M^*\).

It follows that the \(X\)-component \(\mu_X : (X^*)^* \to X^*\) of the monad multiplication \(\mu : T^2 \Rightarrow T\) is given by
\[
\mu_X((x_{11}, x_{12}, \ldots, x_{1n_1}), (x_{21}, x_{22}, \ldots, x_{2n_2}), \ldots, (x_{k1}, x_{k2}, \ldots, x_{kn_k}))
= (x_{11}, x_{12}, \ldots, x_{1n_1}, x_{21}, x_{22}, \ldots, x_{2n_2}, \ldots, x_{k1}, x_{k2}, \ldots, x_{kn_k}).
\]
That is, the components of \(\mu\) “flatten” a list of lists into one list (recall that the multiplication in \(X^*\) is concatenation).

**Example 36.3** (The Maybe Monad). Recall the category \(\text{Set}_*\) of pointed sets and base point preserving functions. The maybe monad arises from the adjunction \(F : \text{Set} \xrightarrow{\cong} \text{Set}_* : U\) where \(U\) is the forgetful functor \(U(X, x) = X\) for each \((X, x) \in \text{Set}_*\). The left adjoint \(F\) sends a set \(X\) to the pointed set \(F(X) = (X \sqcup \{*_X\}, *_X)\).

An \(X\)-component of the unit \(\eta : \text{id}_{\text{Set}} \Rightarrow T = UF\) is given by \(\eta_X : X \hookrightarrow X \sqcup \{*_X\}\), which is the inclusion. The monad multiplication \(\mu : T^2 \Rightarrow T\) is defined by sending a set with two one-element sets adjoint to the set with a one-element set adjoint:
\[
\mu_X : X \sqcup \{*_X\} \sqcup \{*_X \sqcup *_X\} \to X \sqcup \{*_X\};
\]
\(\mu_X\) is identity on \(X \sqcup \{*_X\}\) and sends the one “extra” point \(*_X \sqcup \{*_X\}\) to \(*_X\).
Lecture 37. The Kleisli category.

Last time:

- Proved that an adjunction \((F, G, \eta, \varepsilon)\) gives rise to a monad \((T = GF, \eta : \text{id} \Rightarrow GF = T, \mu = G\varepsilon F : T^2 \Rightarrow T)\).
- Looked at two examples of monads that come from adjunctions.

Remark 37.1. The notion of a monad is not just analogous to the notion of a monoid. Monads and ordinary monoids are both “monoids in a monoidal category.” Unfortunately we don’t have the time to properly state the definition of a monoidal categories. Here is a rough idea: a monoidal category \(\mathcal{C}\) is a category with a functor (a monoidal product) \(\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}\) and an object \(I \in \mathcal{C}\) which is a left and right unit for \(\otimes\).

The category \(\text{Set}\) can be made into a monoidal category by choosing \(\otimes : \text{Set} \times \text{Set} \to \text{Set}\) to be the cartesian product (there are other choices):

\[
(X \xrightarrow{f} Y) \otimes (Z \xrightarrow{g} W) = X \times Z \xrightarrow{f \times g} Y \times W
\]

The unit \(I\) is a one element set \(\{\ast\}\); for any set \(X\) we have bijections \(\lambda_X : \{\ast\} \times X \to X\) and \(\rho_X : X \times \{\ast\} \to X\) which are natural in \(X\). (More generally any category with finite products can be made into a monoidal category.)

For any category \(\mathcal{C}\), the functor category \([\mathcal{C}, \mathcal{C}]\) is monoidal. The monoidal product \(\otimes\) is given by the horizontal composition:

\[
\left( \begin{array}{c}
\mathcal{C} \\
\mathcal{C}
\end{array} \right) \otimes \left( \begin{array}{c}
\mathcal{C} \\
\mathcal{C}
\end{array} \right) = \left( \begin{array}{c}
\mathcal{C} \\
\mathcal{C}
\end{array} \right)
\]

Definition 37.2. A monoid in a monoidal category \((\mathcal{C}, \otimes, I)\) is an object \(m \in \mathcal{C}\) together with a multiplication \(\mu : m \otimes m \to m\) and a unit \(\eta : I \to m\) so that the diagrams

\[
\begin{array}{ccc}
m \otimes m & \xrightarrow{\mu \otimes \text{id}_m} & m \otimes m \\
\text{id}_m \otimes \mu & \downarrow & \downarrow \mu \\
m \otimes m & \xrightarrow{\mu} & m
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
I \otimes m & \xrightarrow{\eta \otimes \text{id}_m} & m \otimes m \\
\text{id}_m \otimes \eta & \downarrow & \downarrow \mu \\
m & \xrightarrow{\mu} & m
\end{array}
\]

commute.

Now back to monads and adjunctions. We have seen that any adjunction gives rise to a monad. The converse is true as well: every monad comes from an adjunction and there are many choices. In fact one can organize these choices into a category.
Definition 37.3. Given a monad \( T = (T, \eta : id \Rightarrow T, \mu : T^2 \Rightarrow T) \) on a category \( C \), the category \( \text{Adj}(T) \) of adjunctions inducing the monad \( T \) is defined as follows. The objects of \( \text{Adj}(T) \) are adjunctions inducing \( T \). A morphism in \( \text{Adj}(T) \) from \( F \downarrow G \) to \( F' \downarrow G' \) is a functor \( H : D \to D' \) so that the diagrams commute.

It turns out that the category \( \text{Adj}(T) \) has an initial object \( C_T \), the Kleisli category of \( T \) and a terminal object \( C^T \), the Eilenberg-Moore category of \( T \). We will discuss the Kleisli category.

The Kleisli category.

Given an adjunction \( C \rightleftarrows D \) inducing a monad \( T \), there is a full subcategory \( D' \subseteq D \) whose objects are \( \{F(a)\}_{a \in C} \). That is, for every pair of objects \( Fa, Fa' \in D' \),

\[
\text{Hom}_{D'}(Fa, Fa') := \text{Hom}_D(Fa, Fa').
\]

Since we have bijections \( \text{Hom}_D(Fa, d) \xrightarrow{\sim} \text{Hom}_C(a, Gd) \), there is a bijection

\[
\text{Hom}_D(Fa, Fa') \xrightarrow{\sim} \text{Hom}_C(a, GFa') = \text{Hom}_C(a, Ta),
\]

where \( T = GF \). This suggest an idea for construction the “smallest” adjunction inducing the monad \( T \).

Definition/proposition 37.4. Let \( T = (T, \eta : id \Rightarrow T, \mu : T^2 \Rightarrow T) \) be a monad on a category \( C \). The Kleisli category \( C_T \) is defined as follows:

1. The objects of \( C_T \) are the same as the objects of \( C \).
2. A morphism \( a \rightsquigarrow b \) in \( C_T \) is a morphism \( a \to Tb \) in \( C \). That is, \( \text{Hom}_{C_T}(a, b) := \text{Hom}_C(a, Tb) \).
3. The composition \( \circ_{kl} \) in \( C_T \) of \( f : a \rightsquigarrow b \) and \( g : b \rightsquigarrow c \) is

\[
g \circ_{kl} f = a \xrightarrow{f} Tb \xrightarrow{T(g)} TTc \xrightarrow{\mu_c} Tc, \text{ i.e.,}
\]

\[
g \circ_{kl} f := \mu_c \circ T(g) \circ f.
\]
4. The identity morphism \( \text{id}_a : a \rightsquigarrow a \) in \( C_T \) is \( \eta_a : a \to Ta \).

Proof. We check that \( C_T \) a category.

Given a morphism \( f : a \rightsquigarrow b \) and \( \text{id}_b : b \rightsquigarrow b = \eta_b : b \to Tb \),

\[
\begin{align*}
 b \rightsquigarrow b \rightsquigarrow a & = Tb \xleftarrow{\mu_b} TTb \xleftarrow{T(\eta_b)} Tb \xrightarrow{f} Tb \xleftarrow{\text{id}_Tb} Tb \xrightarrow{f} b = f.
\end{align*}
\]

Hence \( \text{id}_b \circ_{kl} f = f \). Similarly,

\[
\begin{align*}
 b \rightsquigarrow a \rightsquigarrow a & = Tb \xleftarrow{\mu_Tb} T Tb \xleftarrow{Tf} Ta \xleftarrow{\eta_a} a.
\end{align*}
\]

\[
\begin{split}
 (i) & \quad f \\
 (ii) & \quad \text{id}_{Tb} \\
 (i) & \quad \eta_{Tb} \\
 (i) & \quad f \\
 & \quad Tb
\end{split}
\]

115
The square (i) commutes because \( \eta \) is a natural transformation: from \( \text{id}_C \) to \( T \). The triangle (ii) commutes because it is the left unit in the definition of the monad. Hence \( f \circ_{kl} \text{id}_a = f \).

Finally we check associativity of the Kleisli composition. The trick is to consider the following diagram:

Commutativity of (i) is the associativity of the multiplication \( \mu \) of the monad \( T \); commutativity of (ii) is the naturality of \( \mu \); commutativity of (iii) is the definition of \( g \circ_{kl} f \). Note that

\[
T(\mu_d) \circ TT(h) \circ T(g) = T(\mu_d \circ Th \circ g) = T(h \circ_{kl} g).
\]

The rest of the proof follows from the contemplation of the diagram below:

The red path is \( \mu_d \circ T(h \circ_{kl} g) \circ f = (h \circ_{kl} g) \circ_{kl} f \), while the blue path is \( h \circ_{kl} (g \circ_{kl} f) \).

\[\square\]

**Example 37.5.** Let \( \mathcal{C} = \text{Set} \), \( M \) a monoid and \( T = (T, \eta, \mu) \) the corresponding writer monad: \( T(X \xrightarrow{f} Y) = X \times M \xrightarrow{f \times \text{id}_M} Y \times M \). The unit \( \eta : \text{id}_\text{Set} \Rightarrow T \) has components \( \eta_X : X \rightarrow X \times M \) with \( \eta_X(x) = (x, e) \) for every \( x \in X \). The multiplication \( \mu : T^2 \Rightarrow T \) is defined by setting \( \mu_X : X \times M \times M \rightarrow X \times M \) to be

\[
\mu_X(x, a, b) = (x, a \cdot b)
\]

for every \( (x, a, b) \in X \times M \times M \).

A Kleisli morphism for this monad, that is, a morphism in the Kleisli category \( \text{Set}_T \), is a function \( k : X \rightarrow Y \times M \) for some pair of sets \( X, Y \). For example, if \( M \) is a finite list monoid, a morphism \( k : X \xrightarrow{\sim} Y \) would model a process that computes a certain value and, at the same time, write a string of symbols to a log file.

The Kleisli composition of \( k : X \xrightarrow{\sim} Y \) with \( h : Y \xrightarrow{\sim} Z \) is

\[
\begin{align*}
X & \xrightarrow{k} Y \times M \xrightarrow{h \times \text{id}_M} Z \times M \times M \xrightarrow{\text{id}_X \times (\cdot)} Z \times M \\
(x) & \mapsto (k_1(x), k_2(x)) \mapsto (h_1 k_1(x), h_2 k_1(x), k_2(x)) \mapsto (h_1 k_1(x), (h_2 k_1(x)) \cdot k_2(x))
\end{align*}
\]

Again, if \( M \) is a finite list monoid and \( k, h \) are processes that compute values and write to the same log file, then their Kleisli composite would concatenate the strings they wrote.
Lecture 38. The Kleisli adjunction.

Last time:

- Proved that an adjunction \((F : C \rightleftarrows D : G, \eta : \text{id}_C \Rightarrow GF, \varepsilon : FG \Rightarrow \text{id}_D)\) gives rise to a monad \(T = (T = GF, \eta, \mu = G\varepsilon F)\).
- Defined the Kleisli category \(C_T\) of a monad \(T\) on a category \(C\): the objects of \(C_T\) are the same as the objects of \(C\); a morphism \(a \xrightarrow{f} b\) in \(C_T\) is a morphism \(a \xrightarrow{f} Tb\) in \(C\); the composition \(\circ_{kl}\) is defined by

\[
(c \xrightarrow{g} b) \circ_{kl} (b \xrightarrow{f} a) := Tc \xrightarrow{T\eta} TTc \xleftarrow{\mu} Tb \xleftarrow{f} a,
\]

where on the right the composition is in \(C\). The identity morphisms \(a \xRightarrow{id_a} a\) in \(C_T\) are \(a \xrightarrow{\eta_a} Ta\).

Today given a monad \(T\) we define an adjunction, the Kleisli adjunction \(F_T : C \rightleftarrows C_T : G_T\), where \(C_T\) is the Kleisli category of \(T\), and prove that the corresponding monad is the monad \(T\) we started out with.

Let \(T = (T, \eta, \mu)\) be a monad on a category \(C\). We define a functor \(F_T : C \to C_T\) by setting

\[F_T(a) = a\]

for all objects \(a \in C\) (remember that the categories \(C\) and \(C_T\) have exactly the same objects). On morphisms we define \(F_T\) by

\[F_T(a \xrightarrow{f} b) := a \xrightarrow{f} b \xrightarrow{\eta_b} Tb.\]

Since \(F_T(a \xrightarrow{\text{id}_a} a) = a \xrightarrow{\eta_a} Ta\), the purported functor \(F_T\) does preserve identities. We need to check that \(F_T\) preserves composition. Given two composable arrows \(c \xrightarrow{g} b \xleftarrow{f} a\) in \(C\) we need to show that

\[F_T(g) \circ_{kl} F_T(f) = F_T(g \circ f).\]

By definition of the (purported) functor \(F_T\) and of Kleisli composition

(38.63) \[F_T(g) \circ_{kl} F_T(f) = \mu_c \circ T(F_T(g)) \circ F_T(f) = \mu_c \circ T(\eta_c) \circ T(g) \circ \eta_b \circ f,\]

while

(38.64) \[F_T(g \circ f) = \eta_c \circ g \circ f.\]

Consider the diagram

\[
\begin{array}{ccc}
  a & \xrightarrow{f} & b & \xrightarrow{\eta_b} & Tb \\
  & \downarrow{g} & & \downarrow{T(g)} & \\
  c & \xleftarrow{\eta_c} & Tc & \xrightarrow{T(\eta_c)} & Tc \\
  & \downarrow{id_{Tc}} & & \uparrow{\mu_c} & \\
  & & TTc & & \\
\end{array}
\]

By (38.63) the red path in the diagram gives us \(F_T(g) \circ_{kl} F_T(f)\). By (38.64) the blue path gives us \(F_T(g \circ f)\). So we are done once we show that the diagram commutes. The upper left triangle commutes by definition of \(g \circ f\). The middle square commutes by naturality of \(\eta\). The bottom right triangle is the right unit identity of the monad \(T\). This proves that \(F_T\) is a functor.
Next, we construct the functor $G_T : C_T \to C$ in the other direction. On objects we define $G_T$ by

$$G_T(a) = T(a)$$

for every object $a \in C_T$. Given a morphism $a \xrightarrow{f} b$ in $C_T$ (i.e., $a \xrightarrow{T} T b$ in $C$), we define

$$G_T\left(b \xrightarrow{\sim} a\right) = T b \xleftarrow{\mu_b} TTb \xleftarrow{T(f)} T a,$$

i.e.,

$$G_T\left( a \xrightarrow{f} b \right) := T a \xrightarrow{\mu_T \circ Tf} T b.$$

Since $a \xrightarrow{\sim} a = a \xrightarrow{\eta_a} T a$,

$$G_T\left(a \xrightarrow{\sim} a\right) = \mu_a \circ T(\eta_a) = id_{Ta}$$

by the right unit identity for the monad: $TTa \xrightarrow{T \eta_a} Ta \xrightarrow{id_{Ta}} T a$. It remains to check that the (purported) functor $G_T$ preserves composition. Consider a pair of composable arrows $c \xleftarrow{g} b \xleftarrow{f} a$ in the Kleisli category $C_T$. The by definition of the Kleisli composition $\circ_{kl}$

$$g \circ_{kl} f = \mu_c \circ T(g) \circ f.$$

Hence

$$(38.65) \quad G_T(g \circ_{kl} f) = G_T(\mu_c \circ Tg \circ f) = \mu_c \circ T \mu_c \circ TTg \circ Tf.$$

Consider the diagram

$$
\begin{array}{ccc}
TTc & \xleftarrow{T \mu_c} & TTTc \\
\mu_c & & \mu_{TC} \\
Tc & \xleftarrow{\mu_c} & TTe \\
\end{array}
\begin{array}{ccc}
TTb & \xleftarrow{T \mu_b} & TTa \\
\mu_b & & \mu_{Tb} \\
Tb & \xrightarrow{Tg} & Ta \\
\end{array}
\begin{array}{ccc}
G_Tf & & \\
& & \\
& & \\
\end{array}
$$

The commutativity of the left square is the associativity of $\mu$. The square in the middle of the diagram commutes by naturality of $\mu$. The right triangle commutes by definition of $G_T(f)$. Hence the whole diagram commutes and therefore

$$G_T(g \circ_{kl} f) = \mu_c \circ T \mu_c \circ TTg \circ Tf = \mu_c \circ Tg \circ G_T(f) = G_T(g) \circ G_T(f).$$

Therefore $G_T$ is a functor.

**Proposition 38.1.** Let $T = (T, \eta : id_C \Rightarrow T, \mu : T^2 \Rightarrow T)$ be a monad on a category $C$, $C_T$ be the corresponding Kleisli Category, and $F_T : C \to C_T$, $G_T : C_T \to C$ the two functors constructed above. Then

(i) $G_T \circ F_T = T$;
(ii) $F_T \dashv G_T$;
(iii) the unit of the adjunction $F_T \dashv G_T$ is $\eta$, the unit of the monad $T$;
(iv) the counit of the adjunction $\varepsilon : F_T \circ G_T \Rightarrow id_{C_T}$ has components $Ta \xrightarrow{\varepsilon_a} a = Ta \xrightarrow{id_{Ta}} Ta$;
(v) $G_T \varepsilon F_T = \mu$ and consequently, the adjunction $F_T \dashv G_T$ recovers the monad $T$. 

118
Proof. (i) On objects $G_T(F_T(a)) = G_T(a) = T(a)$ for all $a \in \mathcal{C}$. By the right unit axiom of the monad $T$

$$\text{id}_{Tb} = \mu_b \circ T\eta_b$$

for any object $b$ of $\mathcal{C}$. Hence

$$G_T(F_T(a \xrightarrow{f} b)) = G_T(a \xrightarrow{f} b \xrightarrow{\eta_b} Tb) = Ta \xrightarrow{Tf} Tb \xrightarrow{T\eta_b} TTb \xrightarrow{\mu_b} Tb$$

for a morphism $a \xrightarrow{f} b$ in $\mathcal{C}$.

(ii) For any two objects $a, b \in \mathcal{C}$, a Kleisli morphism $a \xRightarrow{\eta} b$ is a morphism $a \xrightarrow{f} Tb$ in $\mathcal{C}$. Hence

$$\text{Hom}_{\mathcal{C}_T}(F_T(a), b) = \text{Hom}_{\mathcal{C}_T}(a, b) = \text{Hom}_{\mathcal{C}}(a, Tb) = \text{Hom}_{\mathcal{C}}(a, G_T(b))$$

since $F_T(a) = a$ for any $a \in \mathcal{C}$ and $G_T(b) = Tb$ for any $b \in \mathcal{C}_T$. We need to check that the bijections

$$\theta_{a,b} = \text{id} : \text{Hom}_{\mathcal{C}_T}(F_T(a), b) \rightarrow \text{Hom}_{\mathcal{C}_T}(a, G_T(b))$$

are natural in $a$ and $b$. Given $a' \xrightarrow{g} a$ in $\mathcal{C}$, we check that

$$\text{Hom}_{\mathcal{C}_T}(F_T(a), b) \xrightarrow{F_T(g)^*} \text{Hom}_{\mathcal{C}_T}(a', b)$$

commutes. For any $a \xRightarrow{k} b$ in $\mathcal{C}_T$,

$$F_T(g)^*(k) = k \circ_k F_T(g) = \mu_b \circ T(k) \circ (\eta_a \circ g).$$

Since the diagram

$$\begin{array}{ccc}
a' & \xrightarrow{g} & a \\
\downarrow{\eta_a} & & \downarrow{k} \\
T a & \xrightarrow{\eta_T b} & Tb \\
\mu_b & \downarrow{T(k)} & \downarrow{\text{id}_{Tb}} \\
Tb & \xrightarrow{T(k)} & TTb
\end{array}$$

commutes (the commutativity of the square is naturality of $\eta$, the commutativity of the triangle is the left unit identity),

$$F_T(g)^*(k) = \mu_b \circ T(k) \circ (\eta_a \circ g) = \text{id}_{Tb} \circ k \circ g = g^*(k).$$

Hence $\theta_{a,b}$ is natural in $a$.

Given $b \xRightarrow{h} b' = b \xrightarrow{h'} T b'$ in $\mathcal{C}_T$, we check that the diagram

$$\begin{array}{ccc}
\text{Hom}_{\mathcal{C}_T}(F_T(a), b) & \xrightarrow{h_*} & \text{Hom}_{\mathcal{C}_T}(a, G_T(b)) \\
\downarrow{G_T(h)_*} & & \downarrow{G_T(h)_*} \\
\text{Hom}_{\mathcal{C}_T}(F_T(a), b') & \xrightarrow{h'_*} & \text{Hom}_{\mathcal{C}_T}(a, G_T(b'))
\end{array}$$

commutes. For any $a \xRightarrow{k} b \in \text{Hom}_{\mathcal{C}_T}(F_T(a), b) = \text{Hom}_{\mathcal{C}}(a, Tb) = \text{Hom}_{\mathcal{C}}(a, G_T(b))$

$$h_*(k) = h \circ_k k = \mu_b \circ T(h) \circ k$$
while
\[ G_T(h)_* (k) = (\mu_y \circ T(h)) \circ k \]
as well.

(iii) By definition of the Kleisli category \(\mathcal{C}_T\) the identity morphism \(F_T(a) \overset{\eta_a}{\longrightarrow} F_T(a)\) in \(\mathcal{C}_T\) is the morphism \(\eta_a : a \rightarrow Ta\) in \(\mathcal{C}\). Hence the unit of adjunction \(\text{id}\) \(\Rightarrow\) \(G_T F_T\) is precisely \(\eta\).

(iv) Recall that \(G_T(a) = Ta\). By definition of the counit, \(G_T(a) \overset{\text{idG}_T(a)}{\longrightarrow} G_T(a) = Ta \overset{\text{id}_Ta}{\longrightarrow} Ta\) corresponds exactly to \(F_T G_T(a) \overset{\epsilon_a}{\longrightarrow} a\) in \(\mathcal{C}_T\), which is \(Ta \overset{\epsilon_a}{\longrightarrow} Ta\) in \(\mathcal{C}\). On the other hand the correspondence is the identity map. Hence \(\epsilon_a = Ta \overset{\text{id}_Ta}{\longrightarrow} Ta \equiv T(a) \overset{\text{id}_T(a)}{\longrightarrow} a\).

(v) Recall that \(G_T(a \overset{f}{\longrightarrow} b) = G_T(a \overset{f}{\rightarrow} Tb) = Ta \overset{\mu_b \circ f}{\longrightarrow} Tb\) for all morphisms \(a \overset{f}{\longrightarrow} b\) in \(\mathcal{C}_T\). Hence
\[ (G_T \epsilon F_T)_a = G_T(\epsilon F_T(a)) = G_T(T(a) \overset{\text{id}_T(a)}{\longrightarrow} a) = \mu_a \circ T(\text{id}_Ta) = \mu_a. \]

\[\square\]