Last time: A topology on a set $X$ is a collection $T$ of subsets of $X$ (elements of $T$ are called "open sets") so that
1. $\emptyset, X \in T$
2. If $U, V \in T$ then $U \cap V \in T$
3. If $\{U_a\}_{a \in A} \subseteq T$ then $\bigcup_{a \in A} U_a \in T$

A pair $(X, T)$ where $T$ is a topology on $X$ is a topological space.

A function $f : (X, T_X) \to (Y, T_Y)$ is continuous if for $U \in T_Y$
\[ f^{-1}(U) \in T_X. \]

We proved: the composite of two continuous functions is continuous.

Also, for $(X, T_X)$,
\[ (X, T_X) \to (X, T_X) \] is continuous.

Notation: Topological spaces and continuous maps form a category, which we denote by $\text{Top}$. Note that $\text{Top}$ is locally small and there is a forgetful functor $U : \text{Top} \to \text{Set}$.

Object $U((X, T_X)) = X$.

Question: What's the intuition for the definition of continuity?

Recall: A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at $x_0 \in \mathbb{R}$ if for every $\varepsilon > 0$
\[ \exists \delta > 0 \text{ so that } \forall x \in \mathbb{R} \]
\[ |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon \]

A function $f : \mathbb{R} \to \mathbb{R}$ is continuous if it is continuous at every point $x_0 \in \mathbb{R}$.

This $\varepsilon, \delta$ definition of continuity generalizes to arbitrary metric spaces:

Definition: Let $(X, d_X), (Y, d_Y)$ be two metric spaces.
A function $f : X \to Y$ is continuous at $x_0 \in X$ if for $\varepsilon > 0$
There exists $\exists \delta > 0$ so that $\forall x \in X$, $d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \varepsilon$ \hspace{1cm} (9.1)

A function $f : (X, d_X) \rightarrow (Y, d_Y)$ is continuous if it is continuous at every $x_0 \in X$.

Lemma 9.2 

Equation (9.1) is equivalent to

\hspace{1cm} (9.2) \hspace{1cm} \mathcal{B}_{\varepsilon}(x_0) \subseteq f^{-1}(\mathcal{B}_{\varepsilon}(f(x_0)))

[Recall \hspace{1cm} \mathcal{B}_{\delta}(x_0) = \{ x \in X \mid d_X(x, x_0) < \delta \}]

Proof \hspace{1cm} (9.1) holds

$\iff \hspace{1cm} (x \in \mathcal{B}_{\delta}(x_0) \Rightarrow f(x) \in \mathcal{B}_{\varepsilon}(f(x)))$

$\iff \hspace{1cm} (x \in \mathcal{B}_{\delta}(x_0) \Rightarrow x \in f^{-1}(\mathcal{B}_{\varepsilon}(f(x))))$

$\iff \hspace{1cm} \mathcal{B}_{\delta}(x_0) \subseteq f^{-1}(\mathcal{B}_{\varepsilon}(f(x)))$

Lemma 9.3 

Let $(X, d_X), (Y, d_Y)$ be two metric spaces. Then $f : X \rightarrow Y$ is continuous (in the sense of def 9.1)

$\Rightarrow \hspace{1cm} \forall U \subseteq Y$ open w.r.t. $d_Y$, $f^{-1}(U) \subseteq X$ is open w.r.t. $d_X$.

(i.e. $U \in \mathcal{T}_{d_Y} \Rightarrow f^{-1}(U) \in \mathcal{T}_{d_X}$)

Proof \hspace{1cm} ($\Rightarrow$)

Suppose $U \in \mathcal{T}_{d_Y}$. If $f^{-1}(U) = \emptyset$, it's open. Suppose $f^{-1}(U) \neq \emptyset$.

Pick any $x_0 \in f^{-1}(U)$. Then $f(x_0) \in U$. Since $U$ is open, $\exists \varepsilon > 0$ st. $\mathcal{B}_{\varepsilon}(f(x_0)) \subseteq U$. Since $f$ is continuous at $x_0$, $\exists \delta > 0$ st. $\forall x$, $d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \varepsilon$.

By Lemma 9.2, $\mathcal{B}_{\varepsilon}(x_0) \subseteq f^{-1}(\mathcal{B}_{\varepsilon}(x_0)) \subseteq f^{-1}(U)$.

Therefore $\forall x_0 \in f^{-1}(U)$ $\exists \delta > 0$ st. $\mathcal{B}_{\varepsilon}(x_0) \subseteq f^{-1}(U)$.

$\Rightarrow f^{-1}(U)$ is open in $X$ w.r.t. $d_X$.

($\Leftarrow$) Suppose $\forall U \subseteq Y$ open, $f^{-1}(U)$ is open, $x_0 \in X$, $\varepsilon > 0$.

We proved last time; $\forall y \in Y$ $\forall \delta > 0$ $\mathcal{B}_{\varepsilon}(y) \subseteq Y$ is open (w.r.t. $d_Y$).

$\Rightarrow \mathcal{B}_{\varepsilon}(f(x_0)) \subseteq Y$ is open. $\Rightarrow f^{-1}(\mathcal{B}_{\varepsilon}(f(x_0)))$ is open in $X$. (by our assumption)
Since \( f(x_0) \in B_r(f(x_0)) \), \( x_0 \in f^{-1}(B_r(f(x_0))) \)

Since \( f^{-1}(B_r(f(x_0))) \) is open \( \exists \delta > 0 \) so that

\( B_\delta(x_0) \subseteq f^{-1}(B_r(f(x_0))) \)

\( \Rightarrow \) By lemma 9.2, \( f \) is continuous at \( x_0 \).

Since \( x_0 \) is arbitrary, \( f \) is continuous.

\[ \text{Conclusion} \quad \text{The definition of } f: (X, T_x) \to (Y, T_y) \text{ being continuous is a generalization of the } \varepsilon-\delta \text{ definition of continuity.} \]

**Definition** Let \( (X, T) \) be a topological space. A subset \( C \subseteq X \) is closed if \( X \setminus C \) is open, i.e., \( X \setminus C \in T \)

\[ \forall x \in X, x \notin C \implies \exists \delta > 0 \text{ such that } B_\delta(x) \subseteq X \setminus C \]

**Exercise** Let \( (X, T) \) be a topological space. Then

(i) \( X, \emptyset \) are closed.

(ii) If \( C_1, C_2 \subseteq X \) are closed then \( C_1 \cup C_2 \) is closed.

(iii) If \( \{C_\alpha\}_{\alpha \in A} \) is a collection of closed sets, then \( \bigcap_{\alpha \in A} C_\alpha \) is closed.

**Note** \( \forall n \in \mathbb{N}, n > 0, \left[ \frac{1}{n}, 1 - \frac{1}{n} \right] \subseteq \mathbb{R} \text{ is closed, but } \bigcup_{n > 0} \left[ \frac{1}{n}, 1 - \frac{1}{n} \right] = (0, 1) \)

which is not closed.

**Terminology** Let \( X \) be a set, \( T_1, T_2 \) two topologies on \( X \)

\( T_1 \) is smaller (or coarser or weaker) than \( T_2 \) if \( T_1 \subseteq T_2 \)

\( T_2 \) is bigger (or finer or stronger) than \( T_1 \) if \( T_1 \subseteq T_2 \).

**Remark** For any topological space \( (Y, T_y) \), and any set \( X \) any function \( f: (Y, T_y) \rightarrow (X, T_x) \) is continuous.

Similarly, any function \( h: (X, \mathcal{P}(X)) \rightarrow (Y, T_y) \) is continuous.
Subspace topology

Lemma 9.4 Let \((X, T)\) be a topological space, \(Y \subseteq X\) a subset. The set \(T^Y = \{ U \subseteq Y \mid \exists U' \in T \text{ s.t. } U = Y \cap U' \}\) is a topology on \(Y\). Moreover \(T^Y\) is the smallest topology so that the inclusion \(i : Y \to X, \ i(y) = y\) is continuous.

Proof \(Y = X \cap Y \in T^Y, \ \emptyset = X \cap Y \in T^Y\).

If \(U, V \in T^Y\), then \(U \cap Y, V \in T^Y\). Let \(U = \bar{U} \cap Y, \ V = \bar{V} \cap Y\).

Then \(U \cap V = \bar{U} \cap Y \cap \bar{V} \cap Y = (\bar{U} \cap \bar{V}) \cap Y, \ \text{but} \ \bar{U} \cap \bar{V} \in T \Rightarrow U \cap V \in T^Y.

Similarly, if \(\bigcup_{a \in A} U_a \in T^Y\), then \(\bigcup_{a \in A} U_a \cap Y = Y \cap \bigcup_{a \in A} U_a \Rightarrow \bigcup_{a \in A} U_a \in T^Y\).

Moreover, if \(T'\) is a topology on \(Y\) and \(i : (Y, T') \to (X, T)\) is continuous, then \(\forall \bar{U} \in T, \ T' \ni i^{-1}(\bar{U}) = \bar{U} \cap Y \in T^Y\).

It follows that \(T^Y \subseteq T'\).

Definition Let \((X, T)\) be a topological space. A subset \(B \subseteq T\) is a basis for \(T\) if any \(U \in T\) is a union of elements of \(B\).

Ex Consider \(\mathbb{R}^n\) with the standard topology \(T\). Suppose \(U \in T\) ie \(U \subseteq \mathbb{R}^n\) is open. Then \(\forall x \in U \exists r > 0\) so that \(B_{(r)}(x) \subseteq U\).

\[
\Rightarrow U = \bigcup_{x \in U} B_r(x) \quad \text{ball w.r.t. the Standard (Euclidean) Distance}
\]

\[
\Rightarrow B = \{ B_r(x) \mid x \in \mathbb{R}^n, r > 0 \} \text{ is a basis for } T.
\]

Ex Let \((X, d)\) be a metric space. Then \(B = \{ B_r(x) \mid x \in X, r > 0 \} \) is a basis for the topology \(T_d\) defined by the metric \(d\).